Math 4431, Final Exam (practice) Name/Section: ____________

1. (product space with finitely many factors) Let X_1 and X_2 be topological spaces and for $A_j \subset X_j$, $j = 1, 2$ define

$$A_1 \times A_2 = \{(x_1, x_2) : x_j \in A_j, j = 1, 2\}.$$

Let

$$\mathcal{B} = \{U_1 \times U_2 : U_j \text{ is an open set in } X_j\}.$$

(a) (5 points) Show that

$$\bigcup_{B \in \mathcal{B}} B = X \times Y \quad \text{and} \quad \bigcap_{j=1}^{k} B_j \in \mathcal{B} \quad \text{whenever } B_j \in \mathcal{B},\ j = 1, \ldots, k.$$

(b) (5 points) Show that

$$\mathcal{P} = \left\{ \bigcup_{\alpha \in \Gamma} B_\alpha : B_\alpha \in \mathcal{B} \text{ for } \alpha \text{ in any index set } \Gamma \right\}$$

is a topology on $X_1 \times X_2$. ($\mathcal{P}$ is, of course, called the **product topology**.

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(c) (5 points) (Theorem 3.12) Consider $p_j : X_1 \times X_2 \to X_j$ for $j = 1, 2$ by $p_j(x_1, x_2) = x_j$. Show p_1 and p_2 are continuous.

(d) (5 points) Show that if $\mathcal{T}$ is a topology on $X_1 \times X_2$ (not necessarily the product topology $\mathcal{P}$) and p_1 and p_2 are continuous with respect to $\mathcal{T}$, then $\mathcal{P} \subset \mathcal{T}$.

(e) (5 points) Give an example of a topology on $\mathbb{R} \times \mathbb{R}$ with respect to which $p_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is **not** continuous.

(f) (5 points) Give an example of a topology $\mathcal{T}$ which is on $\mathbb{R} \times \mathbb{R}$ which is different from the Euclidean topology but with respect to which $p_j : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous for $j = 1, 2$.

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2. Let X_1 and X_2 be topological spaces.

   (a) (10 points) (Theorem 3.14) If X_1 and X_2 are Hausdorff, then show $X_1 \times X_2$ is Hausdorff.

   (b) (10 points) (Theorem 3.15) If $X_1 \times X_2$ is compact, then show X_1 and X_2 are compact.

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3. (Theorem 3.20) Let us take Armstrong's definition of a connected space:

X is **connected** *if whenever $X = X_1 \cup X_2$ and $X_1, X_2 \neq \phi$, then either*

$$\overline{X}_1 \cap X_2 \neq \phi \quad \text{or} \quad X_1 \cap \overline{X}_2 \neq \phi.$$

(a) (5 points) Show that if $A \subset X$ is connected, then whenever $A \subset A_1 \cup A_2$ and $A \cap A_j \neq \phi$, $j = 1, 2$, then either

$$\overline{A}_1 \cap A_2 \neq \phi \quad \text{or} \quad A_1 \cap \overline{A}_2 \neq \phi.$$

(b) (5 points) Show that if whenever $A \subset A_1 \cup A_2$ and $A \cap A_j \neq \phi$, $j = 1, 2$, then either

$$\overline{A}_1 \cap A_2 \neq \phi \quad \text{or} \quad A_1 \cap \overline{A}_2 \neq \phi,$$

then A is connected.

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(c) (5 points) Show that if A is a connected subset of X and $A \subset U_1 \cup U_2$ where U_1 and U_2 are disjoint open sets, then either

$$A \subset U_1 \quad \text{or} \quad A \subset U_2.$$

(d) (5 points) Show that if A is a connected subspace of X and

$$A \subset S \subset \overline{A},$$

then S is connected.

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4. A topological space X is **locally connected** if for each $x \in X$ and each open set U with $x \in U$, there is some open set U_0 and some connected set C with

$$x \in U_0 \subset C \subset U.$$

   (a) (10 points) Show that the homeomorphic image of a locally connected space is locally connected.

   (b) (10 points) Show that if X is locally connected, then for each $x \in X$ and each open set U with $x \in U$, there is an open connected set U_0 with

$$x \in U_0 \subset U.$$

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5. Let $X = (0, 1]$ and consider $\phi : X \to \mathbb{R}^2$ by

$$\phi(t) = \begin{cases} (t, \sin(1/t)), & 0 < t \leq 2/\pi \\ (6/\pi - 2t, 1), & 2/\pi \leq 3/\pi \\ (0, 7 - 2\pi t), & 3/\pi \leq t \leq 1. \end{cases}$$

Let $Y = \phi(X)$.

(a) (5 points) Show X is locally path connected.

(b) (5 points) Show ϕ is continuous so that Y is the continuous image of a locally path connected space.

(c) (5 points) Show Y is **not** locally path connected.

(d) (5 points) Show the homeomorphic image of a locally path connected space is locally path connected.

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6. Let $q : \mathbb{R} \to \{-1, 0, 1\} = Y$ by

$$p(x) = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0. \end{cases}$$

Recall that the quotient topology on Y is defined by

$$\mathcal{Q}_Y = \{V \subset Y : q^{-1}(V) \text{ is open in } \mathbb{R}\}.$$

(a) (10 points) What is $\mathcal{Q}_Y$?

(b) (10 points) (True or False) If $q : X \to Y$ is an identification map and X is Hausdorff, then Y is Hausdorff.

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7. Let $\mathbb{Z} \times \mathbb{Z} = \{(m, n) : m, n \text{ are integers}\} \subset \mathbb{R}^2$.

   (a) (5 points) Show $G = \mathbb{Z} \times \mathbb{Z}$ is a group (under addition).

   (b) (5 points) Show G acts on $\mathbb{R}^2$ by $(n, m, x, y) \mapsto (x + n, y + m)$.

   (c) (5 points) Identify the quotient space $\mathbb{R}^2/G$.

   (d) (5 points) Consider $A = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R}^2$. Show that the quotient spaces $\mathbb{R}^2/G$ and $\mathbb{R}^2/A$ are **not** homeomorphic.

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8. Consider the (universal) covering map $\phi : \mathbb{R}^2 \to \mathbb{T}^2$ by

$$\phi(x, y) = \left(1 + \frac{\cos y}{2}\right)(\cos x, \sin x, 0) + \frac{\sin y}{2}(0, 0, 1).$$

Also, consider the loops $\gamma[0, 1] \to \mathbb{T}^2$ and $\eta : [0, 1] \to \mathbb{T}^2$ by $\gamma(t) = 3(\cos 2\pi t, \sin 2\pi t, 0)/2$ and

$$\eta(t) = \left(1 + \frac{\cos 2\pi t}{2}\right)(1, 0, 0) + \frac{\sin 2\pi t}{2}(0, 0, 1)$$

respectively both of which start and end at $p = (3/2, 0, 0)$.

(a) (5 points) Draw the (image sets of the) loops γ and η on $\mathbb{T}^2$.

(b) (5 points) Recall the **concatenation** $\eta \triangleleft \gamma : [0, 1] \to \mathbb{T}^2$ is a loop given by

$$\eta \triangleleft \gamma(t) = \begin{cases} \gamma(2t), & 0 \le t \le 1/2 \\ \eta(2t - 1), & 1/2 \le t \le 1. \end{cases}$$

Find explicit formulas for the **liftings** of γ, η, $\eta \triangleleft \gamma$, and $(-\gamma) \triangleleft \eta \triangleleft \gamma$ starting at $(0, 0) \in \mathbb{R}^2$, and draw these paths.

(c) (5 points) Find a fixed endpoint homotopy of the lifting

$$\widehat{(-\gamma) \triangleleft \eta} \triangleleft \gamma$$

to a path along a straight line. To which of the four liftings of part (c) above is this straight line path homotopic?