Armstrong's problem 1.6.21
a solution

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We are asked to find a homeomorphism/automorphism of $\overline{B_1(0)} \subset \mathbb{C}$ which interchanges two points in $B_1(0)$ and leaves the boundary fixed. The formulation of the problem as one in the complex plane initially suggested to me the use of Möbius transformations

$$f(z) = \frac{az+b}{cz+d}, \qquad \text{with } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0.$$

These can be used to move points around inside a disk and leave the boundary *invariant*. They do not, in general, leave the boundary *fixed*, so I'm not sure they lead to an easy solution, though I think there can be a related solution as I'll describe at the end. Even with that, I seem to run into the rut of using something like the (non-conformal) transformation of Armstrong's problem 5.1.3. This may be just because that is what I'm thinking about at the moment, but it does seem to work.

Problem 5.1.3 gives a homeomorphism of the disk which leaves the boundary fixed. To be precise, one considers $h : \mathbb{R}^2 \to \mathbb{R}^2$ by $h(x) = r(\cos(\theta + 2\pi r), \sin(\theta+2\pi r))$ where $r = |x| = \sqrt{x_1^2 + x_2^2}$ and $x = (x_1, x_2) = r(\cos\theta, \sin\theta)$. The image of a diameter under this map is indicated in Figure 1. Notice that h restricted to the circle of radius $1/2$ is the antipodal map. In particular, all points on $\partial B_{1/2}(0)$ are interchanged. This homeomorphism can, of course, be adapted to any disk of radius r and center c by translation and dilation. To be precise, if I want a homeomorphism of the closure of $B_r(c)$ which fixes the boundary and interchanges all points on $\partial B_{r/2}(c)$, then I can take $m : B_r(c) \to B_1(0)$ by $m(x) = (x - c)/r$ and $\sigma = m^{-1} \circ h \circ m$ does the trick. Let's call $\sigma = \sigma_{r,c}$ the swapping homeomorphism.

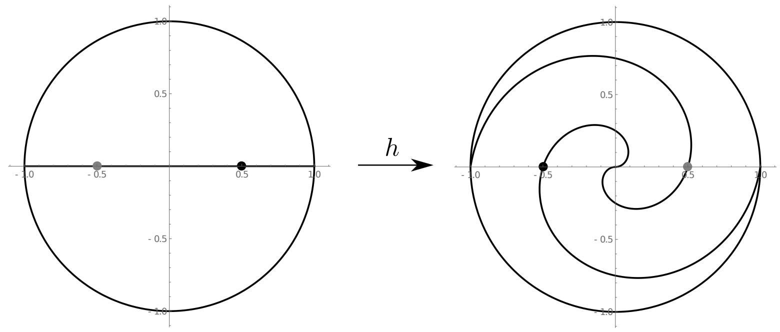

Figure 1: The mapping $h : B_1(0) \to B_1(0)$

The swapping homeomorphism can be used to interchange p and q if we take

$$r = |p - q| \quad \text{and} \quad c = \frac{p+q}{2}. \tag{1}$$

Notice that c in this case is the midpoint of the segment between p and q and $|p - c| = |q - c| = |p - q|/2$. The problem is that σ fixes $\partial B_r(c)$ rather than $\partial B_1(0)$. Nevertheless, we can swap some pairs of points in $B_1(0)$ at this point. Namely, if we make the assignment (1) and it turns out that $B_r(c) \subset\subset B_1(0)$, then we can take

$$\phi(x) = \begin{cases} \sigma(x), & x \in \overline{B_r(c)} \\ \mathrm{id}(x), & x \in \overline{B_1(0)} \backslash B_r(c). \end{cases}$$

The symbol $\subset\subset$ is a shorthand way to say $\overline{B_r(c)} \subset B_1(0)$; notice you've got the closure on the left and the interior of $B_1(0)$ on the right. When this happens, you've got $C = \overline{B_1(0)} \backslash B_r(c)$ is a closed set, and ϕ is continuous by the gluing lemma. In fact, you also get ϕ^{-1}, which is just

$$\phi^{-1}(x) = \begin{cases} \sigma^{-1}(x), & x \in \overline{B_r(c)} \\ \mathrm{id}(x), & x \in \overline{B_1(0)} \backslash B_r(c), \end{cases}$$

is continuous by the gluing lemma as well.

This brings us first to the question of when we have $B_r(c) \subset\subset B_1(0)$ under the assignment (1). The basic answer is when $r + |c| < 1$. This means

$$|p - q| + \left| \frac{p+q}{2} \right| < 1.$$

This looks like a fairly complicated relation between the points p and q, but one thing we can notice is that if $|p|$ and $|q|$ are small enough, then it will always hold. In fact, by the triangle inequality

$$|p-q| + \left|\frac{p+q}{2}\right| \leq \frac{3}{2}(|p|+|q|),$$

so as long as $|p|+|q| < 2/3$ we are in good shape to apply ϕ on $\overline{B_1(0)}$ and swap our points.

But what if $|p|+|q| \geq 2/3$? In that case, we can use a preliminary homeomorphism to "suck" the outer portion of the disk toward the origin. For this we want a homeomorphism f of $[0,1]$ (the interval of the radii) which leaves the endpoints fixed and is increasing but has $f(r) << r$ at least for most of the interval. You should recall that $f(r) = r^n$ for large integers n does this. Thus, our preliminary "sucky" homeomorphism can be $\nu(x) = r^n(\cos\theta, \sin\theta)$ where r and θ are defined near the beginning of this solution. If $|p|^n + |q|^n < 2/3$, then n is large enough. Then we can take $r = |\nu(p) - \nu(q)|$ and $c = (\nu(p)+\nu(q))/2$ in our definition of the swapping homeomorphism σ and

$$\psi : \overline{B_1(0)} \to \overline{B_1(0)} \quad \text{by} \quad \psi(x) = \begin{cases} \nu^{-1} \circ \sigma \circ \nu(x), & x \in \nu^{-1}(\overline{B_r(c)}) \\ \mathrm{id}(x), & x \in \overline{B_1(0)} \backslash \nu^{-1}(B_r(c)). \end{cases}$$

Since ν is a continuous (and open and closed) map, essentially the same gluing lemma argument that applied to ϕ shows that ψ is a homeomorphism fixing $\partial B_1(0)$ and interchanging p and q.

Another approach is the following: Take a preliminary homeomorphism h_1 sending $B_1(0)$ to $\mathbb{R}^2$. This will take the boundary out to "∞." Now move $h_1(p)$ and $h_1(q)$ to points symmetric about the origin using a Möbius transformation μ that fixes ∞. This will also be a homeomorphism (and a conformal one) of the plane. Then you can swap the points $\mu \circ h_1(p)$ and $\mu \circ h_1(q)$ with a variant of Armstrong's spiraling transformation that still leaves ∞ fixed. This will be an interesting part. Then using the inverses of h_1 and μ, you should get a different kind of swapping homeomorphism.