Assignment 1: The 1-D Heat Equation (Intro)
Due Tuesday September 12, 2023

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Problem 1 *(Haberman 1.2.1-3)*

(a) *Determine the dimensions of* **lineal heat energy density** $\theta = \theta(x, t)$ *for which*

$$\int_a^b \theta(x, t)\, dx$$

models the total heat energy in a thin rod found between positions $x = a$ *and* $x = b$.

(b) *Consider a (more complicated) thin rod with varying cross-sectional area* $A = A(x)$ *and* **volumetric heat energy density** *also called* $\theta = \theta(x, t)$. *Determine the expression for the total heat energy between positions* $x = a$ *and* $x = b$.

(c) *Derive a heat equation for the temperature* $u = u(x, t)$ *in the rod of varying cross-sectional area from the previous part under the assumption of constant specific heat capacity* c, *constant volumetric density* ρ, *and constant heat conductivity* K.

Problem 2 *(Haberman 1.2.8) Give an expression for the total thermal energy in a rod modeled on an interval* $0 \le x \le \ell$ *in terms of the temperature* $u = u(x, t)$.

Problem 3 *If* f *is a continuous function defined on the interval* $[0, \ell]$ *with* $\ell > 0$ *and*

$$\int_a^b f(x)\, dx = 0 \qquad \text{whenever } 0 < a < b < \ell$$

then prove $f(x) \ge 0$ *for every* x *with* $0 < x < \ell$.

Problem 4 *Find a solution* $u = u(x,t)$ *of the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell \\ u(0,t) = T_1, & t > 0 \\ u(\ell,t) = T_2, & t > 0 \end{cases}$$

where T_1 *and* T_2 *are given constants.*

Problem 5 *Find as many solutions* $u = u(x,t)$ *as you can to the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell \\ u_x(0,t) = 1 = u_x(\ell,t), & t > 0. \end{cases}$$

Do you think you have found all solutions?

Problem 6 *Let* $u = u(x,t)$ *be a solution to the problem*

$$\begin{cases} u_t = u_{xx}, & 0 < x < \ell \\ u_x(0,t) = 0 = u_x(\ell,t), & t > 0. \end{cases}$$

Show the quantity (essentially the total thermal energy)

$$\int_0^\ell u(x,t)\,dx$$

is conserved, i.e., does not change with time.

Problem 7 *(Haberman Exercise 1.3.1) One version of Newton's law of cooling states that the heat flux at the end of a thin metal rod (conducting heat) is proportional to the difference between the temperature* $u(\ell,t)$ *at the end and the* **external temperature** $T = T(t)$ *adjacent to the end. Use Fourier's law of heat conduction (with Newton's law of cooling) to derive an appropriate boundary flux condition for the 1-D heat equation.*

Problem 8 *(Haberman 1.4.1) Find the equilibrium solution associated with the problem*

$$\begin{cases} u_t = u_{xx} + x^2, & 0 < x < \ell \\ u(0,t) = T_1, & t > 0 \\ u_x(\ell,t) = 0, & t > 0 \\ u(x,0) = u_0(x), & 0 \le x \le \ell. \end{cases}$$

Here T_1 *is a given constant and* u_0 *is a given function.*

Problem 9 *(Haberman 1.4.2) Consider the equilibrium/steady state solution U of the one-dimensional heat equation on the interval $0 \leq x \leq \ell$ with constant conductivity K, fixed boundary temperatures $U(0) = 0 = U(\ell)$, and internal thermal energy rate-density generation/forcing modeled by $Q(x) = x$.*

(a) *Find an expression for the heat energy generated per unit time along the entire rod.*

(b) *Find an expression for the rate of heat energy flowing out of the rod at the ends $x = 0$ and at $x = \ell$.*

(c) *What relation should hold between your answers to the first two parts?*

Problem 10 *(Haberman 1.4.3) Determine the equilibrium temperature distribution for a one-dimensional rod consisting of two different materials in perfect thermal contact at $x = 1$ and satisfying the following conditions:*

(i) *The material modeled on $0 \leq x < 1$ has $c\rho = 1$ and $K = 1$ (where c is the specific heat capacity, ρ is the density, and K is the conductivity). Also on $0 \leq x < 1$ there is an internal unit heat source with constant density per time given by $Q = 1$.*

(ii) *The material modeled on $1 < x \leq 2$ has $c\rho = 2$ and $K = 2$ and $Q = 0$.*

(iii) $u(0) = 0 = u(2)$.

Perfect thermal contact *means the temperature $u = u(x)$ is continuous at $x = 1$ and the thermal energy exiting the portion of the rod modeled by $0 < x < 1$ is equal to the thermal energy entering the portion of the rod modeled by $1 < x < 2$. Be careful: This does not mean $U'(1^-) = U'(1^+)$. You need to use Fourier's law. See also Haberman 1.3.2.*