Assignment 2: The Heat Equation (separation of variables) Due Tuesday September 28, 2021

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Problem 1 **(a)** *Use Fourier's law to determine an appropriate condition at the end-points $x = 0$ and $x = L$ for the model of one-dimensional heat conduction with* **insulated boundary**.

(b) *Use Fourier's law to determine an appropriate boundary condition on an n-dimensional region R corresponding to heat conduction in R with* **insulated boundary**.

Problem 2 *Interpret the flux integral*

$$\int_{\partial R} \vec{\phi} \cdot \vec{n}$$

in one space dimension.

Solution: There are various ways this problem can be approached. What I had in mind is that it would be approached using the general concept of integration via Riemann sums I introduced in the lecture. From this point of view, you're looking at an integral over a set, which generally is an object denoted by

$$\int_A f$$

where $f : A \to \mathbb{R}$ is a real valued function defined on the set A. Such an integral may be thought of as a limit of Riemann sums of the form

$$\int_A f = \lim_{\|\mathcal{P}\| \to 0} \sum_j f(x_j^*)\mu(A_j)$$

where

1. $\mathcal{P} = \{A_j\}$ is a partition of the set A into pieces A_j,
2. $\mu(A_j)$ is the measure of the piece A_j,
3. x_j^* is some point in the piece A_j, so that $f(x_j^*)$ is an evaluation of the function f at a point in A_j, and
4. $\|\mathcal{P}\| \to 0$ indicates the size of the partition pieces tends to zero in the limit. (The size of the partitition pieces will probably have to be "measured" with something other than the measure μ, like some kind of "diameter" measure.)

It will be noted that there are rather a lot of moving parts in this view of integration, but they simplify rather dramatically in the case of one space dimension under consideration.

More precisely, the set A is taken in our example to be the boundary ∂R of a set R in one-dimensional Euclidean space $\mathbb{R}^1$. The natural choice for R is an open interval (a, b), and then $\partial R = \{a, b\}$ is a set containing only two points. So, first of all, there are only two partitions of ∂R, namely, $\{\{a, b\}\}$ with a single piece[1] (which is basically pretty uninteresting) and the important partition

$$\{ \ \{a\}, \ \{b\} \ \}.$$

This partition has two pieces. The limit is trivial. The evaluation points are trivial. There is essentially only one possible (interesting) Riemann sum:

$$\int_{\partial R} \vec{\phi} \cdot \vec{n} = \vec{\phi}(a) \cdot \vec{n}(a) \ \mu(\{a\}) + \vec{\phi}(b) \cdot \vec{n}(b) \ \mu(\{b\}). \tag{1}$$

The next question, perhaps, is: What is the measure? We're wanting to measure the boundary of an open interval (a, b) consisting of exactly two points a and b, or more precisely, we're wanting to measure the **pieces** of the boundary of this interval. Each piece is a set containing only one point in $\mathbb{R}$.

Let's think about how such things work in other contexts and in other dimensions.

> If you want to measure a set in $\mathbb{R}^1$, use one-dimensional measure, or **length**. In this case $\mu = \ell$ (length).
> If you want to measure a set in $\mathbb{R}^2$, use two-dimensional measure, or **area**.

[1]Yep, that's right: The partition is the set containing the set containing the points a and b.

In this case μ = area measure.

If you want to measure a set in $\mathbb{R}^3$, use three-dimensional measure, or **volume**. In this case μ = vol (volume).

You can keep going for sets in $\mathbb{R}^n$ with $n > 3$. This is called n-dimensional measure.

The boundary of a set in $\mathbb{R}^n$, if it's a nice set, is $(n-1)$-dimensional. For example, if you take a ball in $\mathbb{R}^3$ with volume $4\pi r^3/3$, then the boundary is a sphere, which is two-dimensional. So the appropriate measure for the boundary of a ball in $\mathbb{R}^3$ is **area measure**. Of course, you need to know how to measure the area of a surface/set in $\mathbb{R}^3$, which is a little different from measuring the area of a subset of $\mathbb{R}^2$, but still it's reasonable to call such a thing area measure.

Another example: If you take a disk in the plane, then the boundary is one-dimensional. It's a circle, and you should use length measure. Taking this back one step, if we start with an interval $(a, b) \subset \mathbb{R}$, then we have to ask ourself/ourselves:

> What (on earth) might be the natural **zero dimensional** Euclidean measure?

It turns out there is sort of a standard answer for this, and the answer is **cardinality** or **counting measure**. In this case,

$$\mu(A) = \#(A) \quad \text{is the number of elements in the set } A.$$

This does turn out to be a measure, and it turns out to give a reasonable answer. Look back at our Rieman sum (1): If we take $\mu(\{a\}) = 1 = \mu(\{b\})$, using counting measure, then

$$\int_{\partial R} \vec{\phi} \cdot \vec{n} = \vec{\phi}(a) \cdot \vec{n}(a) + \vec{\phi}(b) \cdot \vec{n}(b),$$

and it remains to interpret (only) the values of $\vec{\phi}$ and $\vec{n}$. But this is pretty easy too. There aren't many vectors in $\mathbb{R}^1$. A basis is $\{\mathbf{e}_1 = 1\}$, so vectors are just essentially scalars $\vec{v} = v \cdot 1$ with v a real number.[2] This does allow us to determine the direction of a vector $\vec{v}$ in one dimension by the sign of v with $v > 0$ meaning $\vec{v}$ points to the right and $v < 0$ meaning $\vec{v}$ points to the left.

From this point of view, it is natural to take the **outward unit vector** $\vec{n}$ to the boundary of the open interval (a, b) to have the value $\vec{n}(a) = -1$ and $\vec{n}(b) = 1$. We can also just write $\vec{\phi}$ as $\phi \cdot 1$, or even just $\vec{\phi} = \phi$ for our flux vector with $\phi > 0$

[2] I'm using the "dot" here to simply denote multiplication.

indicating motion to the right (increasing) and $\phi < 0$ indicating motion to the left (decreasing).

Now we're done: The one and only Riemann sum for this flux integral has the (interpreted) value

$$\int_{\partial R} \vec{\phi} \cdot \vec{n} = \phi(a)(-1) + \phi(b)(1) = \phi(b) - \phi(a).$$

Just to put a little cherry on top, the divergence theorem in one-dimension now reads

$$\int_R \operatorname{div} \phi = \phi(b) - \phi(a) \qquad \text{where } R = (a, b).$$

Can you give an interpretation of the integral on the left?

Problem 3 *(Heat Transfer; Haberman 1.5.2) In this problem consider* $\mathbf{v} : \mathbb{R}^n \to \mathbb{R}^n$ *as a* **velocity field***, i.e., a field having dimensions*

$$[\mathbf{v}] = \frac{L}{T}$$

and representing the motion of the medium under consideration.

(a) *If* $n = 2$ *and* $\mathbf{v} = v(\cos\theta, \sin\theta)$ *for some positive constant* v *and constant angle* θ *with* $0 < \theta < \pi/2$*, determine the rate of mass flow across the segment*

$$\{(0, y) : 0 \le y \le L\}.$$

Express your answer as a flux integral.

(b) *If* $n = 3$ *and* $\mathbf{v} = v(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$ *for some positive constant* v *and constant angles* ϕ *and* θ *with* $0 < \theta < \pi/2$ *and* $0 < \phi < \pi$*, determine the rate of mass flow across the lamina*

$$\{(0, y, z) : 0 \le y \le L, 0 \le z \le M\}.$$

Express your answer as a flux integral.

(c) *Assume heat energy transport via the flux field*

$$\vec{\phi}_\tau = \theta\mathbf{v} = c\rho u\ \mathbf{v}$$

when $n = 2$ *and* $\mathbf{v} = (v_0 + at)(\cos\theta, \sin\theta)$ *with* $c, v_0 > 0$ *and* a *constants and* **no heat diffusion***. See my notes on* **transport equations***. Write down the transport equation for the temperature* u *in terms of a given density* $\rho = \rho(x, y, t)$*. What happens when* ρ *is assumed constant? Does this make sense?*

(d) *Describe a physical system in which it might be reasonable to model a medium as having spatially dependent velocity* $\mathbf{v} = \mathbf{v}(x)$ *but constant mass density* ρ.

(e) *Given the vectorial nature of flux, it makes good sense to model the* **simultaneous diffusion and transport** *of heat energy by adding flux fields. Write down, for arbitrary spatial dimensions, the resulting partial differential equation for the temperature* u *under heat transport and diffusion with constant specific heat capacity* c *and constant conductivity* K *(assuming a given velocity field* $\mathbf{v}$*).*

Problem 4 *Find a solution* $u = u(x,t)$ *of the problem of the standard heat/diffusion equation* $u_t = u_{xx}$ *having the form*

$$u(x,t) = f(t)\sin x$$

for some real valued function $f = f(t)$.

Assuming specific heat capacity $c = 1$ *and conductivity* $K = 1$ *and that your solution models heat flow on a rod corresponding to* $0 \le x \le L$*, determine the rate and direction of heat flow at each boundary point* $x = 0$ *and* $x = L$.

Problem 5 *(polar coordinates; Haberman 1.5.3-9) This problem is about differential operators (and modeling heat diffusion in particular) in two dimensions* $n = 2$. *It involves the* **polar coordinates map** $\psi : [0,\infty) \times \mathbb{R} \to \mathbb{R}^2$ *given by*

$$\psi(r,\theta) = (r\cos\theta, r\sin\theta).$$

(a) *Discuss the invertibility and inverse of the polar coordinates map.*

(b) *Given a temperature function* $u = u(x,y,t)$ *in spatial rectangular coordinates satisfying*

$$u_t = k\Delta u,$$

assume the value of u *is constant along each circle* $\{(x,y) : x^2 + y^2 = r^2\}$. *Use the chain rule to find the partial differential equation satisfied by the corresponding function* $w = w(r,t) = u(x,y,t)$ *in polar coordinates given by*

$$w(r,t) = u(\psi(r,\theta),t) = u(r\cos\theta, r\sin\theta, t). \tag{2}$$

Hint(s): Applly the chain rule to (2) to compute w_r, w_θ, w_{rr} *and* $w_{\theta\theta}$. *Then consider* $w_{rr} + w_{\theta\theta}/r^2$. *(From the expressions for* w_r *and* w_θ*, you can express the first order derivatives of* u *in terms of first order derivatives of* θ.

(c) *Use the equation you found in the previous part to determine the equilibrium temperature distribution in the annular/ring region $R = \{(x,y) : r_1^2 < x^2+y^2 < r_2^2\}$ subject to the boundary conditions*

$$w(r_1,t) = T_1 \qquad \text{and} \qquad w(r_2,t) = T_2$$

are given constant temperatures.

Problem 6 *(Haberman 1.5.11) Using the same axially symmetric equation for $w = w(r,t)$ from part (b) and the same region fron part (c) of the previous problem, what must be true of the constant β in the boundary conditions*

$$\frac{\partial w}{\partial r}(r_1,t) = \beta \qquad \text{and} \qquad \frac{\partial w}{\partial r}(r_2,t) = 1$$

in order for there to exist an equilibrium temperature distribution?

Problem 7 *(Haberman 1.5.16) Let R be a fixed region in space modeling medium in which heat diffuses according to*

$$\begin{cases} c\rho u_t = K\Delta u, & (x,t) \in R \times (0,\infty) \\ u(x,0) = u_0, & x \in R \\ Du(x,t) \cdot \vec{n} = g(x), & (x,t) \in \partial R \times (0,\infty) \end{cases}$$

where c, ρ, and K are given positive constants and $u_0 = u_0(x)$ and $g = g(x)$ are given positive functions. Assuming the law of specific heat $\theta = c\rho u$ for the thermal energy density θ find an expression for the total thermal energy in R at time t in terms of u_0 and g. Hint(s): Differentiate under the integral sign, use the equation. Then use the divergence theorem and, finally, integrate with respect to time.

Problem 8 *(Haberman 2.3.7) Consider the problem*

$$\begin{cases} u_t = ku_{xx}, & 0 < x < L \\ u_x(0,t) = 0, & t > 0 \\ u_x(L,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 \le x \le L. \end{cases} \tag{3}$$

Here g is a given function. A **separated variables** *solution is a solution of the form $u(x,t) = A(x)B(t)$ where u is a product of a function A only of the spatial variable x and B is a function only of time.*

(a) *Under what conditions does there exist a separated variables solution of this problem. Hint(s): Plug in a function u of the given form and algebraically manipulate the PDE so that it takes the form*

$$\alpha(x) = \beta(t).$$

What does this tell you about the functions α and β? Then see what the boundary and initial conditions tell you/require of A and B.

(b) *Your answer in part (a) should have told you that you can solve the problem (with a separated variables solution) only in situations when the function g in the initial condition has one of a countable collection*

$$g_0, g_1, g_2, g_3, \ldots$$

of particular special forms. Assume the function g in the initial condition does have one of these special forms, and find all the associated separated variables solutions $u_j = A_j(x)B_j(t)$ for $j = 0, 1, 2, 3, \ldots$.

(c) *Attempt to find a series solution of the full problem having the form of a* **superposition**

$$u(x,t) = \sum_{j=0}^{\infty} a_j u_j(x,t) \tag{4}$$

where $a_0, a_1, a_2, a_3, \ldots$ are some constants. Do not worry about the convergence of this series, but treat it as a finite sum for your calculations. Which parts of the problem will such a function definitely satisfy (assuming convergence and that your formal calculations make sense)? What may not be satisfied?

NOTE: *You may or may not be able to identify the coefficients $a_0, a_1, a_2, a_3, \ldots$ in this problem. The main point of the problem as it is composed here is not about identifying or finding Fourier coefficients. But you may know how to do those things and want to do them. Soon (say by the time you complete Assignment 3) all of you should be able to do those things, so if you can't do them now, you might want to look back at this problem later.*

Problem 9 *(Haberman 2.3.7) Give an explanation/description of a physical problem that might be modeled by the initial/boundary value problem (3).*

Problem 10 *(Haberman 2.3.7) Determine the equilibrium temperature distribution associated with the initial/boundary value problem (3). Can you see a relation between the superposition (4) and the equilibrium? Can you use conservation of energy to determine the first coefficient* a_0*?*