Assignment 3 = Exam 1: The Heat Equation (separation of variables) Due Tuesday October 12, 2021

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Problem 1 *(scaling in space and time) Consider the initial/boundary value problems for the 1-D heat equation:*

$$\begin{cases} u_t = ku_{xx} & \text{on } (0,L)\times(0,T) \\ u(0,t) = u_0(t), & 0<t<T \\ u(L,t) = u_1(t), & 0<t<T \\ u(x,0) = g(x), & 0<x<L \end{cases} \tag{1}$$

and

$$\begin{cases} U_t = U_{xx} & \text{on } (0,M)\times(0,S) \\ U(0,t) = U_0(t), & 0<t<S \\ U(M,t) = U_1(t), & 0<t<S \\ U(x,0) = G(x), & 0<x<M. \end{cases} \tag{2}$$

This problem considers the **equivalence** *of these two problems under various scalings in space and time. The term equivalence will be explained below.*

(a) *Since scaling is a kind of change of variables, things may be clearer in your mind if you use different names for the variables in one of the problems. Rewrite the second problem (2) in terms of spatial and time variables ξ and τ.*

(b) *(scaling in time) Choose an appropriate scaling of time $t = \beta\tau$ and leave $x = \xi$ for some $\beta > 0$ to show that given a solution $u = u(x,t)$ of (1), there are appropriate choices of the functions U_0, U_1, G (and the positive constants M and S) such that $U = U(x,\tau)$ solves (2). (You need to find β, U_0, U_1, G, M, and S, and then show the function U is a solution of (2).)*

Notice that when I define a change of variables by $t = \beta\tau$ here, this is a change of variables $(x,t) \mapsto (\xi,\tau) = (x,\tau)$, i.e., a mapping of the given domain $(0,L) \times (0,T) = \{(x,t) : 0 < x < L, 0 < t < T\}$ to the secondary domain $(0,M) \times (0,S) = \{(\xi,\tau) : 0 < \xi < M, 0 < \tau < S\}$.

(c) *Note that your choices of β, U_0, U_1, G, M, and S in part (b) above (in terms of relations with u_0, u_1, g, L and T) determine the initial/boundary value problem (2). Show that given a solution $U = U(\xi,\tau)$ of (2), those same relations can be used to define a solution $u = u(x,t)$ of (1). (Here you can find u_0, u_1, g, L and T and note that they are determined by precisely the same relaitons used in part (b). Then show the same change of variables (or more precisely the inverse $(\xi,\tau) \mapsto (x,t)$) determines a solution $u = u(x,t)$ of the first problem (1).*

The relations involved in parts (b) and (c) above, both those that were defined and those that were verified, constitute what we mean by saying the problems (1) and (2) are **equivalent** *under the given change of variables.*

(d) *(scaling in space) Use a scaling in space $x = \alpha\xi$, $t = \tau$ to find a problem (2) equivalent to (1).*

(e) *(scaling in both space and time) Determine a family of problems (2) for the equation $U_\tau = U_{\xi\xi}$ which are all equivalent to (1), and hence equivalent to each other, determined by scaling in both space and time.*

Problem 2 *(heat flow out of a rod)*

(a) *Solve the initial/boundary value problem for the heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\pi) \times (0,\infty) \\ u(0,t) = 0 = u(\pi,t), & t > 0 \\ u(x,0) = \sin x, & 0 < x < \pi. \end{cases}$$

(b) *Use mathematical software, i.e., Matlab, Mathematica, Maple, Octave, or something similar, to plot the graph*

$$\mathcal{G} = \{(x,t,u(x,t)) : (x,t) \in (0,\pi) \times (0,T)\}$$

of your solution from part (a).

(c) *Use mathematical software to produce an animation of the graph*

$$\mathcal{G}_t = \{(x, u(x,t)) : 0 < x < \pi\}$$

with time as the animation variable. This is called the **time animation** *or* **time evolution** *of the temperature.*

Problem 3 *(laminar heat flow out of a rectangle)*

(a) *Solve the initial/boundary value problem for the heat equation*

$$\begin{cases} u_t = \Delta u & \text{on } \mathcal{U} \times (0, \infty) \\ u(x,y,t) = 0 & \text{on } \partial\mathcal{U} \text{ for } t > 0 \\ u(x,y,0) = \sin(\pi x/2)\sin(2\pi y) & \text{for } (x,y) \in \mathcal{U} \end{cases}$$

where $\mathcal{U} = (0,2) \times (0,1) \subset \mathbb{R}^2$

(b) *Use mathematical software to produce a time animation of the graph*

$$\mathcal{G}_t = \{(x, y, u(x,y,t)) : (x,y) \in \mathcal{U}\}.$$

Problem 4 *(superposition, Haberman 2.4.1)*

(a) *Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,2) \times (0, \infty) \\ u(0,t) = 0 = u(2,t), & t > 0 \\ u(x,0) = 1 - |x - 1|, & 0 < x < 2 \end{cases}$$

using separation of variables and superposition. (Your solution should be given as an appropriate Fourier series.)

(b) *Use mathematical software to plot the graph of your solution. (What you should do here is plot a partial sum of your Fourier series solution with enough terms so that the graphic representation* **stabilizes***. That is, the picture you get does not change visibly in any noticeable way if you add additional terms. This is the procedure you will always need to use when you are plotting (something involved with) a Fourier series.)*

(c) *Use mathematical software to produce a time animation of your solution. Again, check for visual/graphic stabilization.*

Problem 5 *(time dependent boundary values) Consider the initial/boundary value problem*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u(0,t) = t, & t>0 \\ u(L,t) = -t, & t>0 \\ u(x,0) = 0, & 0<x<L \end{cases} \tag{3}$$

for the 1-D heat equation.

(a) *Describe/discuss a physical heat conduction problem modeled by this problem.*

(b) *Find an initial/boundary value problem*

$$\begin{cases} v_t = v_{xx} + f(x) & \text{on } (0,L)\times(0,\infty) \\ v(0,t) = 0 = v(L,t), & t>0 \\ v(x,0) = 0, & 0<x<L. \end{cases} \tag{4}$$

equivalent to (3). Hint: Set $v = u - w$ for an appropriate function $w = w(x,t)$.

(c) *Describe/discuss a physical heat conduction problem modeled by the problem (4).*

Problem 6 *(Haberman 2.4.2)*

(a) *Solve the initial/boundary value problem for the heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u_x(0,t) = 0 = u(L,t), & t>0 \\ u(x,0) = L^2/4 - (x-L/2)^2, & 0<x<L. \end{cases}$$

(b) *Choose a positive value of L, and use mathematical software to plot your solution from part (a).*

(c) *With your choice of $L>0$ from part (b), use mathematical software to produce a time animation of your solution.*

Problem 7 *(Haberman 2.5.1) Let $R = (0,L)\times(0,M)$ be a fixed rectangle in the plane modeling a heat conducting plate. Solve the boundary value problem for Laplace's equation (equilibrium solution of the heat equation):*

$$\begin{cases} \Delta u = 0, & (x,y)\in R \\ u(x,0) = Lx - x^2, & 0<x<L \\ u(x,M) = 0, & 0<x<L \\ u(0,y) = 0, & 0<y<M \\ u(L,y) = 0, & 0<y<M. \end{cases} \tag{5}$$

Problem 8 *(Haberman 2.5.2) Let $R = (0,2) \times (0,1)$ be a fixed rectangle in the plane modeling a heat conducting plate. Consider the boundary value problem for Laplace's equation (equilibrium solution of the heat equation):*

$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u_y(x,0) = 0, & 0 < x < 2 \\ u_y(x,1) = g(x), & 0 < x < 2 \\ u_x(0,y) = 0, & 0 < y < 1 \\ u_x(2,y) = 0, & 0 < y < 1. \end{cases} \tag{6}$$

(a) *Under what conditions does there exist a separated variables solution of this problem. Hint(s): Plug in a function u of the form $u(x,y) = A(x)B(y)$.*

(b) *Your answer in part (a) should have told you that you can solve the problem (with a separated variables solution) only in situations when the function g in the initial condition has one of a countable collection*

$$g_1, g_2, g_3, \dots$$

of particular special forms. Reasoning physically, what condition must g satisfy in general for there to be a solution to this equilibrium problem? Do the functions $g_1, g_2, g_3, \dots$ all satisfy this condition?

(c) *Make a specific choice of g satisfying the general condition of part (b) but which is not one of the functions $g_1, g_2, g_3, \dots$. Solve the problem for this choice of g. Hint: If you want to make things easy on yourself, you might choose g to be a certain* **discontinuous** *function. (Such a choice might be kind of interesting/exciting anyway, don't you think?)*

(d) *Note that with the choice $L = 2$ and $M = 1$, your solution U_7 of problem (5) is given by an explicit Fourier series, the partial sums of which could be plotted using mathematical software. Let U_8 be the solution of (6) you found with your choice of g. Write down the boundary value problem for Laplace's equation satisfied by $u = U_7 + U_8$. Use mathematical software to plot the solution $u = U_7 + U_8$.*

Problem 9 *(Haberman 2.5.3) Solve Laplace's equation (in polar coordinates, see Assignment 2 Problem 5(b)) for an equilibrium temperature distribution $w = w(r,\theta)$* **outside** *the disk $B_a(\mathbf{0}) = \{(x,y) : x^2 + y^2 < a^2\}$, that is, on the exterior domain*

$$\mathcal{U} = \{(x,y) : x^2 + y^2 > a^2\},$$

subject to the boundary condition $w(a, \theta) = \ln 2 + 4\cos 3\theta$.

Problem 10 *(Uniqueness of solutions for the Dirichlet problem, Haberman 2.5.12)*

(a) *Use the coordinate expression*

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$$

for the divergence of a vector field $\mathbf{v} = (v_1, v_2, \dots, v_n)$ *defined on a region* $\mathcal{U} \subset \mathbb{R}^n$ *to derive the product formula*

$$\operatorname{div}(\phi \mathbf{v}) = D\phi \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v}$$

for the scaled field $\phi\mathbf{v}$ *where* $\phi : \mathcal{U} \to \mathbb{R}$ *is a scalar function.*

(b) *Obtain an identity for*

$$\int_{\mathcal{U}} w \Delta w.$$

Hint(s): Use part (a) and remember $\Delta w = \operatorname{div} Dw$.

(c) *Prove the boundary value problem*

$$\begin{cases} \Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial\mathcal{U}} = g \end{cases} \tag{7}$$

for Poisson's equation has a unique solution. The boundary problem (7) with prescribed boundary values is called the **Dirichlet problem** *for Poisson's equation. Hint(s): Note that your identity in (b) holds for any function. Take* $w = u - v$ *where* u *and* v *are two solutions of (7).*