Assignment 3: The Heat Equation
Due Tuesday September 26, 2023

John McCuan

Problem 1 (Haberman Exercise 1.4.10) Assume heat conduction is modeled in a thin metal rod by

$$c\rho u_t = (Ku_x)_x + 4 \qquad \text{on} \qquad (0, \ell) \times (0, \infty)$$

where $c\rho = K = 1$. If $u(x, 0) = f(x)$ and

$$\frac{\partial u}{\partial x}(0, t) = 5 = \frac{\partial u}{\partial x}(\ell, t) - 1,$$

Calculate the total thermal energy modeled between $x = 0$ and $x = \ell$ as a function of time.

Problem 2 (Haberman 1.5) State Fourier's law of heat conduction in n dimensions and determine from it an appropriate boundary condition to model heat conduction in an n-dimensional region (modeled by) R with **insulated boundary**.

Problem 3 (Haberman 1.5.16) Let R be a fixed region in space modeling a medium in which heat diffuses according to

$$\begin{cases} c\rho u_t = K\Delta u, & (x, t) \in R \times (0, \infty) \\ u(x, 0) = u_0, & x \in R \\ Du(x, t) \cdot \vec{n} = g(x), & (x, t) \in \partial R \times (0, \infty) \end{cases}$$

where c, ρ, and K are given positive constants and $u_0 = u_0(x)$ and $g = g(x)$ are given functions. Assuming the law of specific heat $\theta = c\rho u$ for the thermal energy density θ find an expression for the total thermal energy modeled in R at time t in terms of u_0 and g. Hint(s): Differentiate under the integral sign, use the equation. Then use the divergence theorem and, finally, integrate with respect to time.

Problems 4-8 are about the initial/boundary value problem

$$\begin{cases} u_t = k u_{xx}, & 0 < x < \ell \\ u_x(0,t) = 0, & t > 0 \\ u_x(\ell,t) = 0, & t > 0 \\ u(x,0) = g(x), & 0 \le x \le \ell. \end{cases} \tag{1}$$

Here g is a given initial function.

Problem 4 (initial condition)

(a) Find an initial function g compatible (to first order) with the insulated boundary conditions of (1).

(b) Find an initial function g satisfying $g(0) = 0$ but $g'(0) \neq 0$ and $g'(\ell) = 0$.

Problem 5 (separated variables solution) A **separated variables** solution is a solution of the PDE in (1) having the form $u(x,t) = A(x)B(t)$. That is, u is a product of two functions A and B where the function A depends only on the spatial variable x and B is a function only of time.

(a) Substitute a function u of the given form into the PDE and algebraically manipulate what you get so it takes the form

$$\alpha(x) = \beta(t). \tag{2}$$

(b) What assumption(s) did you need to make on the functions A and B in order to obtain the form (2)?

(c) What does the condition (2) tell you about the functions α and β? Hint: Differentiate with respect to x.

Problem 6 (Sturm-Liouville problem)

(a) Derive from the condition (2) and part **(c)** of Problem 5 an ordinary differential equation for the function $A = A(x)$. Hint: Your ODE should have a free/unknown constant c.

(b) Derive boundary conditions for your ODE for $A = A(x)$ from the boundary conditions in (1).

(c) What assumption(s) do you need to make on the function $B = B(t)$ in order to get the boundary conditions in part **(b)**?

Problem 7 (Sturm-Liouville spectral sequence) Recall that the ODE you obtained in part **(a)** of Problem 6 above and the associated boundary value problem for $A = A(x)$ had in it an unknown parameter c.

(a) Consider the case where the parameter c satisfies $c = 0$. Find the resulting separated variables solution in this case and determine all special cases of (1) under which this leads to a complete solution.

(b) Show there is a countable sequence of nonzero values for the parameter c for which you can solve the boundary value problem for your ODE from parts **(a)** and **(b)** of Problem 6. Hint: Consider cases $c < 0$ and $c > 0$. In each case, find the general solution of your ODE for $A = A(x)$.

(c) For which special cases of (1) can you find a separated variables solution?

Problem 8 (superposition)

(a) Determine two distinct non-constant linearly independent[1] initial functions g_1 and g_2 for which you can find a separated variables solution of the entire initial/boundary value problem (1) when $g = g_j$ for $j = 1, 2$.

(b) Animate the evolution of your solution for $g = g_1$.

(c) Solve the initial/boundary value problem (1) when $g = ag_1 + bg_2$ for some constants $a, b \in \mathbb{R}$.

Problem 9 (Green's theorem; Haberman Exercise 1.5.7)

(a) State Green's theorem. (Look it up and be sure you understand what it says if necessary.)

(b) Derive the heat equation in two dimensions using Green's theorem. Hint: Rotate your vector fields on ∂R by an angle $\pi/2$.

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[1] In this case, **linearly independent** simply means that if $ag_1 + bg_2$ is the zero function, then $a = b = 0$, that is, neither of the functions is a multiple of the other.

Problem 10 (Haberman Exercise 1.5.1) Let $\alpha = \alpha(x, y, z, t)$ model the concentration of ink particles in a volume V of still water. Derive an equation for the evolution of a given initial concentration $g : V \to \mathbb{R}$ within V under the assumption that the flux of ink particles is proportional to the spatial gradient $D\alpha$ of the concentration. Introduce appropriate proportionality constants and give physical dimensions for all quantities. Hint: Start with a conservation law for the evolution of particles on an arbitrary subregion R of V. Use the divergence theorem.

Afterthought: Go back and think carefully about the assumptions of Problem 5 part **(b)** and Problem 6 part **(c)**. What is the consequence if these assumptions fail? (Is it possible to find other non-trivial separated variables solutions you might have missed?)