Assignment 3 = Exam 1: The Heat Equation (separation of variables) Due Tuesday October 12, 2021

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Problem 1 *(scaling in space and time) Consider the initial/boundary value problems for the 1-D heat equation:*

$$\begin{cases} u_t = ku_{xx} & \text{on } (0,L)\times(0,T) \\ u(0,t) = u_0(t), & 0<t<T \\ u(L,t) = u_1(t), & 0<t<T \\ u(x,0) = g(x), & 0<x<L \end{cases} \tag{1}$$

and

$$\begin{cases} U_t = U_{xx} & \text{on } (0,M)\times(0,S) \\ U(0,t) = U_0(t), & 0<t<S \\ U(M,t) = U_1(t), & 0<t<S \\ U(x,0) = G(x), & 0<x<M. \end{cases} \tag{2}$$

This problem considers the **equivalence** *of these two problems under various scalings in space and time. The term equivalence will be explained below.*

(a) *Since scaling is a kind of change of variables, things may be clearer in your mind if you use different names for the variables in one of the problems. Rewrite the second problem (2) in terms of spatial and time variables ξ and τ.*

(b) *(scaling in time) Choose an appropriate scaling of time $t = \beta\tau$ and leave $x = \xi$ for some $\beta > 0$ to show that given a solution $u = u(x,t)$ of (1), there are appropriate choices of the functions U_0, U_1, G (and the positive constants M and S) such that $U = U(x,\tau)$ solves (2). (You need to find β, U_0, U_1, G, M, and S, and then show the function U is a solution of (2).)*

Notice that when I define a change of variables by $t = \beta\tau$ here, this is a change of variables $(x,t) \mapsto (\xi,\tau) = (x,\tau)$, i.e., a mapping of the given domain $(0,L) \times (0,T) = \{(x,t) : 0 < x < L, 0 < t < T\}$ to the secondary domain $(0,M) \times (0,S) = \{(\xi,\tau) : 0 < \xi < M, 0 < \tau < S\}$.

(c) *Note that your choices of β, U_0, U_1, G, M, and S in part (b) above (in terms of relations with u_0, u_1, g, L and T) determine the initial/boundary value problem (2). Show that given a solution $U = U(\xi,\tau)$ of (2), those same relations can be used to define a solution $u = u(x,t)$ of (1). (Here you can find u_0, u_1, g, L and T and note that they are determined by precisely the same relaitons used in part (b). Then show the same change of variables (or more precisely the inverse $(\xi,\tau) \mapsto (x,t)$) determines a solution $u = u(x,t)$ of the first problem (1).*

The relations involved in parts (b) and (c) above, both those that were defined and those that were verified, constitute what we mean by saying the problems (1) and (2) are **equivalent** *under the given change of variables.*

(d) *(scaling in space) Use a scaling in space $x = \alpha\xi$, $t = \tau$ to find a problem (2) equivalent to (1).*

(e) *(scaling in both space and time) Determine a family of problems (2) for the equation $U_\tau = U_{\xi\xi}$ which are all equivalent to (1), and hence equivalent to each other, determined by scaling in both space and time.*

Problem 2 *(heat flow out of a rod)*

(a) *Solve the initial/boundary value problem for the heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\pi) \times (0,\infty) \\ u(0,t) = 0 = u(\pi,t), & t > 0 \\ u(x,0) = \sin x, & 0 < x < \pi. \end{cases}$$

(b) *Use mathematical software, i.e., Matlab, Mathematica, Maple, Octave, or something similar, to plot the graph*

$$\mathcal{G} = \{(x,t,u(x,t)) : (x,t) \in (0,\pi) \times (0,T)\}$$

of your solution from part (a).

(c) *Use mathematical software to produce an animation of the graph*

$$\mathcal{G}_t = \{(x, u(x,t)) : 0 < x < \pi\}$$

with time as the animation variable. This is called the **time animation** *or* **time evolution** *of the temperature.*

Problem 3 *(laminar heat flow out of a rectangle)*

(a) *Solve the initial/boundary value problem for the heat equation*

$$\begin{cases} u_t = \Delta u & \text{on } \mathcal{U} \times (0, \infty) \\ u(x,y,t) = 0 & \text{on } \partial\mathcal{U} \text{ for } t > 0 \\ u(x,y,0) = \sin(\pi x/2)\sin(2\pi y) & \text{for } (x,y) \in \mathcal{U} \end{cases}$$

where $\mathcal{U} = (0,2) \times (0,1) \subset \mathbb{R}^2$

(b) *Use mathematical software to produce a time animation of the graph*

$$\mathcal{G}_t = \{(x, y, u(x,y,t)) : (x,y) \in \mathcal{U}\}.$$

Problem 4 *(superposition, Haberman 2.4.1)*

(a) *Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,2) \times (0,\infty) \\ u(0,t) = 0 = u(2,t), & t > 0 \\ u(x,0) = 1 - |x-1|, & 0 < x < 2 \end{cases}$$

using separation of variables and superposition. (Your solution should be given as an appropriate Fourier series.)

(b) *Use mathematical software to plot the graph of your solution. (What you should do here is plot a partial sum of your Fourier series solution with enough terms so that the graphic representation* **stabilizes***. That is, the picture you get does not change visibly in any noticeable way if you add additional terms. This is the procedure you will always need to use when you are plotting (something involved with) a Fourier series.)*

(c) *Use mathematical software to produce a time animation of your solution. Again, check for visual/graphic stabilization.*

Problem 5 *(time dependent boundary values) Consider the initial/boundary value problem*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u(0,t) = t, & t > 0 \\ u(L,t) = -t, & t > 0 \\ u(x,0) = 0, & 0 < x < L \end{cases} \tag{3}$$

for the 1-D heat equation.

(a) *Describe/discuss a physical heat conduction problem modeled by this problem.*

(b) *Find an initial/boundary value problem*

$$\begin{cases} v_t = v_{xx} + f(x) & \text{on } (0,L)\times(0,\infty) \\ v(0,t) = 0 = v(L,t), & t > 0 \\ v(x,0) = 0, & 0 < x < L. \end{cases} \tag{4}$$

equivalent to (3). Hint: Set $v = u - w$ for an appropriate function $w = w(x,t)$.

(c) *Describe/discuss a physical heat conduction problem modeled by the problem (4).*

Problem 6 *(Haberman 2.4.2)*

(a) *Solve the initial/boundary value problem for the heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u_x(0,t) = 0 = u(L,t), & t > 0 \\ u(x,0) = L^2/4 - (x - L/2)^2, & 0 < x < L. \end{cases}$$

(b) *Choose a positive value of L, and use mathematical software to plot your solution from part (a).*

(c) *With your choice of $L > 0$ from part (b), use mathematical software to produce a time animation of your solution.*

Problem 7 *(Haberman 2.5.1) Let $R = (0,L)\times(0,M)$ be a fixed rectangle in the plane modeling a heat conducting plate. Solve the boundary value problem for Laplace's equation (equilibrium solution of the heat equation):*

$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = Lx - x^2, & 0 < x < L \\ u(x,M) = 0, & 0 < x < L \\ u(0,y) = 0, & 0 < y < M \\ u(L,y) = 0, & 0 < y < M. \end{cases} \tag{5}$$

Solution: Setting $u = A(x)B(y)$, we get for the PDE

$$-\frac{A''}{A} = \frac{B''}{B} = \lambda.$$

In this case, I saw the nice boundary conditions $A(0) = 0 = A(L)$ were going to go with the equation for A, so that's why I put the negative sign there. This way we get

$$\lambda_j = j^2\pi^2/L^2 \qquad \text{and} \qquad A_j = \sin(j\pi x/L).$$

For the equation for B_j, I have a decent boundary value at $y = M$. For this reason I'll "recenter" at $y = M$ and solve $B'' = j^2\pi^2 B/L^2$ as

$$B_j(y) = a\cosh\left(\frac{j\pi}{L}(y-M)\right) + b\sinh\left(\frac{j\pi}{L}(y-M)\right).$$

The good boundary value gives $a = 0$. Thus, I attempt a superposition

$$u(x,y) = \sum_{j=1}^{\infty} a_j \sin\left(\frac{j\pi}{L}x\right)\sinh\left(\frac{j\pi}{L}(y-M)\right).$$

The last boundary condition requires

$$u(x,0) = -\sum_{j=1}^{\infty} a_j \sin\left(\frac{j\pi}{L}x\right)\sinh\left(\frac{j\pi}{L}M\right) = Lx - x^2.$$

This means

$$-a_j \sinh\left(\frac{j\pi}{L}M\right)\ \frac{L}{2} = \int_0^L (Lx - x^2)\sin\left(\frac{j\pi}{L}x\right)\,dx = \frac{2L^3}{j^3\pi^3}(1-(-1)^j).$$

That is,

$$a_j = -\frac{1}{\sinh\left(\frac{j\pi}{L}M\right)}\ \frac{4L^2}{j^3\pi^3}(1-(-1)^j).$$

This gives the solution,

$$u_j(x,y) = -\frac{8L^2}{\pi^3}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^3\sinh\left(\frac{j\pi}{L}M\right)}\sin\left(\frac{j\pi}{L}x\right)\sinh\left(\frac{j\pi}{L}(y-M)\right).$$

but I can go ahead and graph the solution for the choices $L = 2$ and $M = 1$ in anticipation of the next problem.

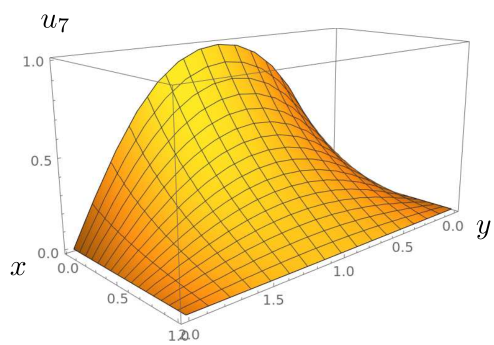

Figure 1: The sum of the first two nonzero terms $k = 0$ and $k = 1$ in the Fourier expansion of u_7.

Problem 8 *(Haberman 2.5.2) Let $R = (0, 2) \times (0, 1)$ be a fixed rectangle in the plane modeling a heat conducting plate. Consider the boundary value problem for Laplace's equation (equilibrium solution of the heat equation):*

$$\begin{cases} \Delta u = 0, & (x, y) \in R \\ u_y(x, 0) = 0, & 0 < x < 2 \\ u_y(x, 1) = g(x), & 0 < x < 2 \\ u_x(0, y) = 0, & 0 < y < 1 \\ u_x(2, y) = 0, & 0 < y < 1. \end{cases} \tag{6}$$

(a) *Under what conditions does there exist a separated variables solution of this problem. Hint(s): Plug in a function u of the form $u(x, y) = A(x)B(y)$.*

(b) *Your answer in part (a) should have told you that you can solve the problem (with a separated variables solution) only in situations when the function g in the initial condition has one of a countable collection*

$$g_1, g_2, g_3, \ldots$$

of particular special forms. Reasoning physically, what condition must g satisfy in general for there to be a solution to this equilibrium problem? Do the functions $g_1, g_2, g_3, \ldots$ all satisfy this condition?

(c) *Make a specific choice of g satisfying the general condition of part (b) but which is not one of the functions $g_1, g_2, g_3, \ldots$. Solve the problem for this choice of* *g. Hint: If you want to make things easy on yourself, you might choose g to be a certain* **discontinuous** *function. (Such a choice might be kind of interesting/exciting anyway, don't you think?)*

(d) *Note that with the choice $L = 2$ and $M = 1$, your solution U_7 of problem (5) is given by an explicit Fourier series, the partial sums of which could be plotted using mathematical software. Let U_8 be the solution of (6) you found with your choice of g. Write down the boundary value problem for Laplace's equation satisfied by $u = U_7 + U_8$. Use mathematical software to plot the solution $u = U_7 + U_8$.*

Solution:

(a) We again get $-A''/A = B''/B = \lambda$ for $u = A(x)B(y)$. This time the good boundary conditions for the A problem are $A'(0) = 0 = A'(2)$. This means we are going to get a constant and cosines:

$$A_j = \cos\left(\frac{j\pi x}{2}\right) \qquad \text{for} \qquad j = \frac{j^2\pi^2}{4}, \qquad j = 0, 1, 2, 3, \dots.$$

Also B_j for $j > 0$ is given by

$$B_j(y) = a \sinh\left(\frac{j\pi y}{2}\right) + b \cosh\left(\frac{j\pi y}{2}\right)$$

so that

$$B_j' = \frac{j\pi}{2}\left[a \cosh\left(\frac{j\pi y}{2}\right) + b \sinh\left(\frac{j\pi y}{2}\right)\right]$$

and the first boundary condition tells us $a = 0$. We also have B_0 is a constant. We have obtained separated variables solutions

$$u_0 = a_0 \qquad \text{(constant)}$$

and

$$u_j(x, y) = a_j \cos\left(\frac{j\pi x}{2}\right) \cosh\left(\frac{j\pi y}{2}\right)$$

for $j = 1, 2, 3, \dots$. In order for u_0 to be a constant solution, we must have $(u_0)_y = u_0' = 0$, so that means we must have $g(x) \equiv 0$. In order for u_j for $j > 1$ to be a solution of the problem, we must have

$$(u_j)_y(x, 1) = \frac{ja_j\pi}{2} \cos\left(\frac{j\pi x}{2}\right) \sinh\left(\frac{j\pi}{2}\right) = g(x).$$

That is, g is a constant multiple of $\cos(j\pi x/2)$. If g has any of these prescribed forms:

$$0 \qquad \text{or} \qquad c\cos\left(\frac{j\pi x}{2}\right) \qquad \text{for some constant } c,$$

then we can solve the problem with a separated variables solution.

(b) Physically, the thermal energy entering the rectangle must equal that exiting; there must be no net thermal flux across the boundary. Three sides have no flux, i.e., we have insulated boundaries $x = 0$, $x = 2$, and $y = 0$. This means the remaining boundary component must have zero flux or

$$0 = \int_{y=1} Du \cdot (0,1) = \int_0^2 (u_x(x,1), u_y(x,1)) \cdot (0,1)\, dx = \int_0^2 g(x)\, dx.$$

That is,

$$\int_0^2 g(x)\, dx = 0. \tag{7}$$

This is certainly true for $g_0 = 0$. For

$$g_j(x) = c\cos\left(\frac{j\pi x}{2}\right)$$

we also have

$$\int_0^2 g_j(x)\, dx = \frac{2c}{j\pi}\sin\left(\frac{j\pi x}{2}\right)\Big|_{x=0}^{2} = 0.$$

Thus, all the boundary value functions g_j allowing separated variables solutions do satisfy the zero flux condition (7).

(c) The simplest function $g : (0,2) \to \mathbb{R}$ I can think of with $\int g = 0$ is given by

$$g(x) = \begin{cases} -1, & 0 < x < 1 \\ 1, & 1 < x < 2. \end{cases}$$

It was nice imagining you would choose the same function and make grading for me easy. (It was nice while it lasted at any rate.)

Our superposition here is

$$u(x,y) = \sum_{j=0}^{\infty} a_j \cos\left(\frac{j\pi x}{2}\right)\cosh\left(\frac{j\pi y}{2}\right),$$

and we want

$$u_y(x,1) = g(x).$$

Assuming u is continuous though u_y is not, we should have no trouble with termwise differentiation. You can see this because the coefficients become smaller in the series for u than that for u_y. What we are essentially doing is integrating termwise to get u. Explicitly, we have

$$u_y(x,1) = \sum_{j=1}^{\infty} \frac{ja_j\pi}{2} \cos\left(\frac{j\pi x}{2}\right) \sinh\left(\frac{j\pi}{2}\right).$$

That is, we need

$$\frac{2}{j\pi \sinh\left(\frac{j\pi}{2}\right)} \int_0^2 g(x) \cos\left(\frac{j\pi x}{2}\right)\, dx = a_j \tag{8}$$

for $j = 1, 2, 3, \ldots$ and no condition on a_0. So for my choice of g we get

$$\begin{aligned}
\int_0^2 g(x) \cos\left(\frac{j\pi x}{2}\right)\, dx &= -\int_0^1 \cos\left(\frac{j\pi x}{2}\right)\, dx + \int_1^2 \cos\left(\frac{j\pi x}{2}\right)\, dx \\
&= \frac{2}{j\pi}\left[-\sin\left(\frac{j\pi}{2}\right) - \sin\left(\frac{j\pi}{2}\right)\right] \\
&= -\frac{4}{j\pi}\sin\left(\frac{j\pi}{2}\right) \\
&= -\frac{4(-1)^k}{(2k+1)\pi}
\end{aligned}$$

where $j = 2k+1$ is odd, and the even coefficients all vanish.

To repeat: a_0 is arbitrary, $a_{2k} = 0$ for $k = 1, 2, 3, \ldots$, and

$$a_{2k+1} = \frac{8(-1)^{k+1}}{(2k+1)^2\pi^2 \sinh\left(\frac{(2k+1)\pi}{2}\right)} \qquad \text{for} \qquad k = 0, 1, 2, \ldots.$$

Up to an additive constant (or assuming $a_0 = 0$) this solution is

$$u_8(x,y) = \frac{8}{\pi^2} \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{(2k+1)^2 \sinh\left(\frac{(2k+1)\pi}{2}\right)} \cos\left(\frac{(2k+1)\pi x}{2}\right) \cosh\left(\frac{(2k+1)\pi y}{2}\right).$$

And the result looks just about right:

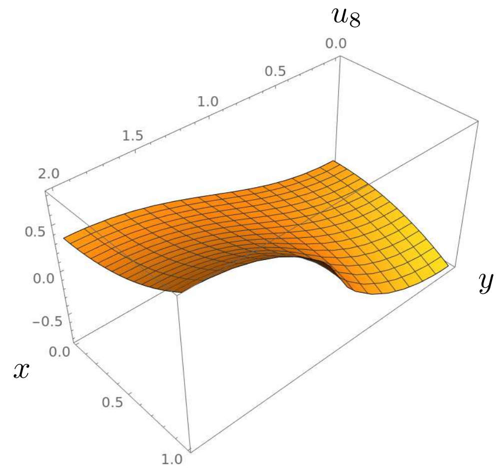

Figure 2: The first twenty-one nonzero terms in the Fourier expansion of u_8.

Julian had a different idea. He chose

$$g(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2x - 4, & 1 < x < 2. \end{cases}$$

This seems like a valid choice satisfying $\int g = 0$, so let's see what we get. For $j > 1$, Julian should get

$$\begin{aligned} \int_0^2 g(x) \cos\left(\frac{j\pi x}{2}\right) dx &= \int_0^1 2x \cos\left(\frac{j\pi x}{2}\right) dx + \int_1^2 (2x-4)\cos\left(\frac{j\pi x}{2}\right) dx \\ &= \frac{8}{j^2\pi^2}\left[-1 + (-1)^j + j\pi \sin\left(\frac{j\pi}{2}\right)\right] \\ &= -\frac{8}{j^2\pi^2}\left[2 - j\pi(-1)^k\right] \\ &= -\frac{8}{(2k+1)^2\pi^2}\left[2 + (2k+1)\pi(-1)^{k+1}\right] \end{aligned}$$

where $j = 2k+1$ is odd, and the even coefficients all vanish.

Substituting this value in (8) we get a_0 is arbitrary, $a_{2k} = 0$ for $k = 1, 2, 3, \ldots$, and

$$a_{2k+1} = -\frac{16}{(2k+1)^3\pi^3 \sinh\left(\frac{(2k+1)\pi}{2}\right)}\left[2 + (2k+1)\pi(-1)^{k+1}\right]$$

for $k = 0, 1, 2, \ldots$. Up to an additive constant (or assuming $a_0 = 0$) this solution is

$$u_J(x,y) = -\frac{16}{\pi^3}\sum_{k=0}^{\infty} \frac{2 + (2k+1)\pi(-1)^{k+1}}{(2k+1)^3 \sinh\left(\frac{(2k+1)\pi}{2}\right)} \cos\left(\frac{(2k+1)\pi x}{2}\right)\cosh\left(\frac{(2k+1)\pi y}{2}\right).$$

This should also have no problems with convergence; notice the $1/j^3$ in the denominator. And the result also looks just about right:

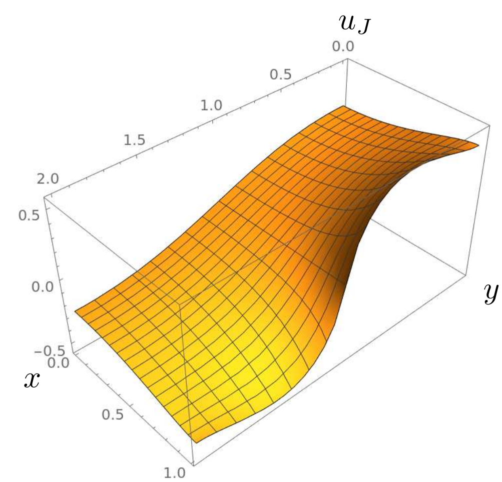

Figure 3: The first fifty nonzero terms in the Fourier expansion of u_J (Julian's solution).

(d) If the previous parts are done correctly, this should be relatively easy. Recall that (5) and (6) look like this:

$$\begin{cases} \Delta u_7 = 0, & (x,y) \in R \\ U_7(x,0) = Lx - x^2, & 0 < x < L \\ U_7(x,M) = 0, & 0 < x < L \\ U_7(0,y) = 0, & 0 < y < M \\ U_7(L,y) = 0, & 0 < y < M. \end{cases}$$

and

$$\begin{cases} \Delta U_8 = 0, & (x,y) \in R \\ (U_8)_y(x,0) = 0, & 0 < x < 2 \\ (U_8)_y(x,1) = g(x), & 0 < x < 2 \\ (U_8)_x(0,y) = 0, & 0 < y < 1 \\ (U_8)_x(2,y) = 0, & 0 < y < 1. \end{cases}$$

The boundary value problem satisfied by $u = U_7 + U_8$ is

$$\begin{cases} \Delta u = 0, & (x,y) \in R \\ u(x,0) = Lx - x^2 + U_8(x,0), & 0 < x < 2 \\ u(x,1) = U_8(x,1), & 0 < x < 2 \\ u(0,y) = U_8(0,y), & 0 < y < 1 \\ u(2,y) = U_8(2,y), & 0 < y < 1. \end{cases} \tag{9}$$

Note that the Dirichlet problem (9) for the Laplace operator on the rectangle $R = (0,2) \times (0,1)$ has a unique solution, so this is the boundary value problem satisfied by $u = U_7 + U_8$ uniquely.

NOTE: A **Dirichlet problem** is one where the actual values of the function are prescribed. Problems (5) and (9) are Dirichlet problems, and they have unique solutions. Problem (6) considered in Problem 8 is not a Dirichlet problem, but a **Neumann problem** in which only derivatives are prescribed on the boundary. This is why the solution is only unique up to an additive constant.

The only tricky bit here is that we need to use four functions $a : (0,2) \to \mathbb{R}$, $b : (0,2) \to \mathbb{R}$, $\alpha : (0,1) \to \mathbb{R}$ and $\beta : (0,1) \to \mathbb{R}$ for the Dirichlet boundary values, and we only have these functions in terms of Fourier series. In fact, we only know these functions up to a choice of an additive constant since they come from a Neumann problem. It's not a big deal really, but let us write down how the values of these functions are given:

We have $a(x) = U_8(x,0)$, that is,

$$a(x) = a_0 + \sum_{j=1}^{\infty} a_j \cos\left(\frac{j\pi x}{2}\right)$$

with coefficients

$$a_j = \frac{2}{j\pi \sinh\left(\frac{j\pi}{2}\right)} \int_0^2 g(x) \cos\left(\frac{j\pi x}{2}\right)\, dx$$

depending in turn on the choice of g. Similarly, $b(x) = U_8(x, 1)$, that is,

$$b(x) = a_0 + \sum_{j=1}^{\infty} a_j \cosh\left(\frac{j\pi}{2}\right) \cos\left(\frac{j\pi x}{2}\right),$$

$$\alpha(y) = U_8(0, y) = a_0 + \sum_{j=1}^{\infty} a_j \cosh\left(\frac{j\pi y}{2}\right),$$

and

$$\beta(y) = U_8(2, y) = a_0 + \sum_{j=1}^{\infty} (-1)^j a_j \cosh\left(\frac{j\pi y}{2}\right).$$

This means we do not actually get a unique **problem** for u. We get a family of problems corresponding to the functions a, b, α and β which are determined in turn by one real parameter a_0 and a funciton $g \in L^2(0, 2)$. Each of these problems (given the constant a_0 and the function g) has a unique solution.

Taking $a_0 = 0$ and my choice of g the solution u of (9) has graph that looks like the one on the left in Figure 4. With $a_0 = c = 0$ and Julian's choice of g the solution of (9) has graph indicated on the right.

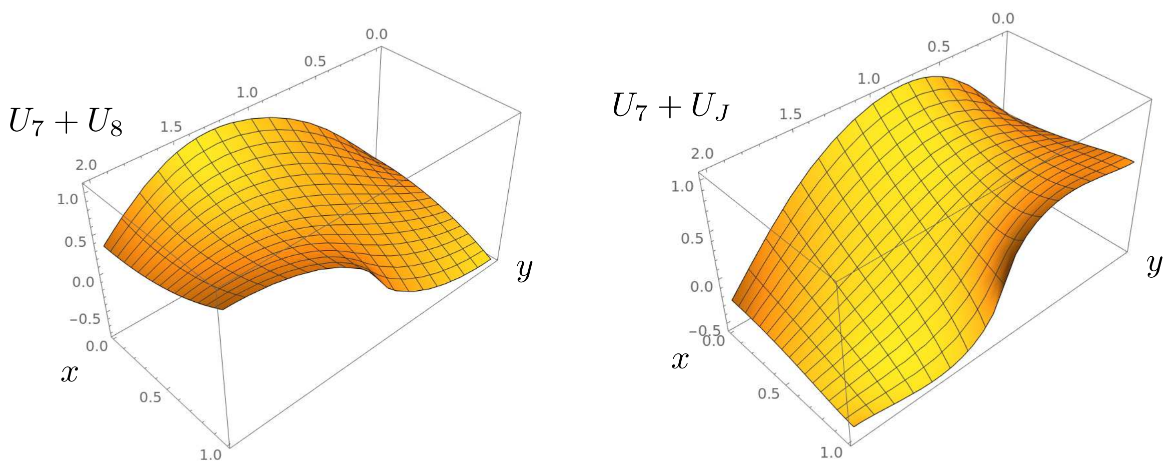

Figure 4: The graph of $u = U_7 + U_8$ when g is taken piecewise constant (left) or piecewise linear (right).

What happens if you change the constant a_0? Anything interesting?

Problem 9 *(Haberman 2.5.3) Solve Laplace's equation (in polar coordinates, see Assignment 2 Problem 5(b)) for an equilibrium temperature distribution* $w = w(r, \theta)$ **outside** *the disk* $B_a(\mathbf{0}) = \{(x, y) : x^2 + y^2 < a^2\}$*, that is, on the exterior domain*

$$\mathcal{U} = \{(x, y) : x^2 + y^2 > a^2\},$$

subject to the boundary condition $w(a, \theta) = \ln 2 + 4\cos 3\theta$*.*

Solution: The main thing is that on the exterior domain there is no requirement that solutions be bounded at $r = 0$. Separation of variables $u(x, y) = A(r)B(\theta)$ gives

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(AB)\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(AB) = 0.$$

See Haberman p. 304 (for example). That is,

$$-\frac{B_{\theta\theta}}{B} = \frac{r(rA_r)_r}{A} = \lambda.$$

This gives B with periodic boundary conditions is given by $\cos(j\theta)$ and/or $\sin(j\theta)$ with $\lambda_j = j^2$, and the radial ODE is

$$r^2A'' + rA' - j^2A = 0$$

which is an Euler equation unless $j = 0$.

When $j = 0$, we get $rA'' + A' = 0$ so that

$$\frac{A''}{A'} = \frac{d}{dr}\ln(A') = -\frac{1}{r} \qquad \text{or} \qquad \ln A' = A'(a) - \ln r + \ln a.$$

That is,

$$A' = \frac{c}{r} \qquad \text{and} \qquad A = A(a) + c\ln r - c\ln a$$

where $c = ae^{A'(a)}$. The θ equation $B'' = 0$ with periodic boundary conditions gives B is constant in this case, so the basis of solutions gives something interesting with regard to the constant boundary value $\ln 2$. To make this simple, we can take also $a = 2$. Then we get constant solutions for the $j = 0$ mode along with $\ln r$ as a solution, and two distinct solutions

$$u_1(x, y) = \ln 2 \qquad \text{and} \qquad u_2(x, y) = \ln\sqrt{x^2 + y^2}$$

for the exterior boundary value problem

$$\begin{cases} \Delta u = 0, & \text{on } \mathbb{R}^2 \backslash B_2(\mathbf{0}) \\ u(x,y) = \ln 2, & x^2 + y^2 = 4. \end{cases}$$

More generally, we get **non-uniqueness** for the exterior problem under consideration using

$$u_1(x,y) \equiv \ln 2 \qquad \text{and} \qquad u_2(x,y) = \ln 2 + c \ln \frac{\sqrt{x^2+y^2}}{a}.$$

This was, in some sense, the main point of this problem, especially with respect to Problem 10 below. Notice that we get the constant solution u_1 here by taking $c = 0$ in the solution u_2.

Taking $A = r^\alpha$, we get

$$\alpha(\alpha - 1) r^\alpha + \alpha r^\alpha - j^2 r^\alpha = 0 \qquad \text{or} \qquad \alpha = \pm j.$$

On the interior domain we would throw out $\alpha = -j$ to avoid a singularity, but on the exterior domain,

$$A(r) = c_1 r^{-j} + c_2 r^j.$$

Taking a look at the boundary values

$$w(a,\theta) = \ln 2 + 4\cos(3\theta)$$

in our original problem, we can see that we've already taken care of the possibility of the constant $\ln 2$, and we are particularly interested in the case $j = 3$. In that case, we have a solution

$$u_3(x,y) = \left(\frac{c_1}{r^3} + c_2 r^3 \right) \cos(3\theta),$$

and we are interested in any values of the constants c_1 and c_2 for which

$$\frac{c_1}{a^3} + c_2 a^3 = 4.$$

There is again a one parameter family of such solutions with $c_2 = 4/a^3 - \gamma/a^6$ and $c_1 = \gamma$ arbitrary.

Combining these observations, we obtain a two-parameter family of solutions of this problem: For any $c, \gamma \in \mathbb{R}$,

$$\begin{aligned} u(x,y) &= \ln 2 + c \ln \frac{\sqrt{x^2+y^2}}{a} + \left[\frac{\gamma}{r^3} + \frac{1}{a^3}\left(4 - \frac{\gamma}{a^3}\right) r^3 \right] \cos(3\theta) \\ &= \ln 2 + c \ln \frac{\sqrt{x^2+y^2}}{a} + \left[\frac{\gamma}{r^3} + \frac{1}{a^3}\left(4 - \frac{\gamma}{a^3}\right) r^3 \right] [\cos^3\theta - 3\cos\theta \sin^2\theta] \\ &= \ln 2 + c \ln \frac{\sqrt{x^2+y^2}}{a} + \left[\frac{\gamma}{(x^2+y^2)^3} + \frac{1}{a^3}\left(4 - \frac{\gamma}{a^3}\right) \right] [x^3 - 3xy^2]. \end{aligned}$$

Problem 10 *(Uniqueness of solutions for the Dirichlet problem, Haberman 2.5.12)*

(a) *Use the coordinate expression*

$$\operatorname{div} \mathbf{v} = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$$

for the divergence of a vector field $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ *defined on a region* $\mathcal{U} \subset \mathbb{R}^n$ *to derive the product formula*

$$\operatorname{div}(\phi \mathbf{v}) = D\phi \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v}$$

for the scaled field $\phi \mathbf{v}$ *where* $\phi : \mathcal{U} \to \mathbb{R}$ *is a scalar function.*

(b) *Obtain an identity for*

$$\int_{\mathcal{U}} w \Delta w.$$

Hint(s): Use part (a) and remember $\Delta w = \operatorname{div} Dw$.

(c) *Prove the boundary value problem*

$$\begin{cases} \Delta u = f & \text{on } \mathcal{U} \\ u\big|_{\partial \mathcal{U}} = g \end{cases} \tag{10}$$

for Poisson's equation has a unique solution. The boundary problem (10) with prescribed boundary values is called the **Dirichlet problem** *for Poisson's equation. Hint(s): Note that your identity in (b) holds for any function. Take* $w = u - v$ *where* u *and* v *are two solutions of (10).*

Solution:

(a)

$$\begin{aligned}
\operatorname{div}(\phi\mathbf{v}) &= \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\phi v_j) \\
&= \sum_{j=1}^{n} \left(\frac{\partial \phi}{\partial x_j}\, v_j + \phi \frac{\partial v_j}{\partial x_j} \right) \\
&= \sum_{j=1}^{n} \frac{\partial \phi}{\partial x_j}\, v_j + \sum_{j=1}^{n} \phi \frac{\partial v_j}{\partial x_j} \\
&= D\phi \cdot \mathbf{v} + \phi \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \\
&= D\phi \cdot \mathbf{v} + \phi \operatorname{div} \mathbf{v}.
\end{aligned}$$

(b)

$$\begin{aligned}
\int_{\mathcal{U}} w\Delta w &= \int_{\mathcal{U}} w \operatorname{div} Dw \\
&= \int_{\mathcal{U}} [\operatorname{div}(wDw) - Dw \cdot Dw \\
&= \int_{\mathcal{U}} \operatorname{div}(wDw) - \int_{\mathcal{U}} |Dw|^2 \\
&= \int_{\partial\mathcal{U}} \mathcal{U} w\, Dw \cdot \mathbf{n} - \int_{\mathcal{U}} |Dw|^2.
\end{aligned}$$

That's the identity:

$$\int_{\mathcal{U}} w\Delta w = \int_{\partial\mathcal{U}} \mathcal{U} w\, Dw \cdot \mathbf{n} - \int_{\mathcal{U}} |Dw|^2.$$

(c) Putting $w = u - v$ and applying the identity, we have

$$w\big|_{\partial\mathcal{U}} \equiv 0.$$

Thus, since $\Delta w = 0$,

$$\int_{\mathcal{U}} |Dw|^2 = 0.$$

This means $Dw \equiv \mathbf{0}$ from which it follows that on a connected open domain $\mathcal{U}$ w is constant. A nice technical proof of this assertion is as follows: Let $\mathbf{x}_0 \in \mathcal{U}$ so that $w_0 = w(\mathbf{x}_0)$ is a value taken by w. Consider

$$A = \{\mathbf{x} \in \mathcal{U} : w(\mathbf{x}) = w_0\}.$$

This set is closed because the complement

$$\{\mathbf{x} \in \mathcal{U} : w(\mathbf{x}) < w_0\} \cup \mathbf{x} \in \mathcal{U} : w(\mathbf{x}) > w_0\} \tag{11}$$

is a union of open sets. (Each of the sets in (11) is easily seen to be open by continuity.) On the other hand, given $\mathbf{x} \in A$, there is some open ball $B_r(\mathbf{x})$ with $B_r(\mathbf{x}) \subset A$. If $\mathbf{p} \in B_r(\mathbf{x})$, then the segment connecting $\mathbf{x}$ and $\mathbf{p}$ lies also in $B_r(\mathbf{x})$, and hence in $\mathcal{U}$. Therefore, we can apply the fundamental theorem of calculus and the chain rule as follows:

$$\begin{aligned} w(\mathbf{x}) - w_0 &= w(\mathbf{p}) - w(\mathbf{x}) \\ &= \int_0^1 \frac{d}{dt} w((1-t)\mathbf{x} + t\mathbf{p})\, dt \\ &= \int_0^1 Dw((1-t)\mathbf{x} + t\mathbf{p}) \cdot (\mathbf{p} - \mathbf{x})\, dt \\ &= 0. \end{aligned}$$

The vanishing of the last inequality follows since $Dw \equiv \mathbf{0}$. This means w takes the value w_0 on all of $B_r(\mathbf{x})$, which in turn means A is an open set.

Now, the definition of a set being **connected** is that it **cannot** be written as a union of disjoint, nonempty, open sets. But note that since A is both open and closed, the complement of A is also both open and closed. But

$$\mathcal{U} = A \cup A^c,$$

$\mathcal{U}$ is connected, and we know A is nonempty. This means A^c must be empty, i.e., there are no points $\mathbf{x}$ in $\mathcal{U}$ for which $w(\mathbf{x}) \neq w_0$.

Note: It can be shown that an open connected subset of $\mathbb{R}^n$ is also **path connected**. I haven't stated the definition of what it means to be path connected, but you can guess. An alternative proof that w is constant may be given using the fact that $\mathcal{U}$ is path connected. Also note that while the equivalence of connected and path connected holds for open sets in $\mathbb{R}^n$ it does not hold in general. There exist connected sets which are not path connected.

Finally, knowing w is constant, the constant value must be that attained on the boundary which is $w = 0$. This means, of course, that $u \equiv v$ as desired.

Note also that the same reasoning applies to the Neumann problem for which

$$Du \cdot \mathbf{n}\big|_{\partial\mathcal{U}} \equiv 0,$$

but the conclusion is not strict uniqueness. One obtains $Dw \equiv 0$ which implies w is constant as before. But the constant does not need to be zero. One concludes: Given two solutions u and v of the Neumann problem in which the normal derivative $Du \cdot \mathbf{n}$ is prescribed along the boundary is **unique up to an additive constant**, that is, given two solutions u and v, one has that there exists a constant c for which $v = u + c$.