Assignment 4:
The Heat Equation and Laplace's Equation
(separation of variables)
Due Tuesday October 26, 2021

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Problem 1 *(1-D Heat Equation, Haberman 2.4.1) Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u_x(0,t) = 0 = u_x(L,t), & t > 0 \\ u(x,0) = 6 + 4\cos(3\pi x/L), & 0 < x < L \end{cases}$$

Problem 2 *(1-D heat equation, Haberman 2.4.1)*

(a) *Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,2)\times(0,\infty) \\ u_x(0,t) = 0 = u_x(2,t), & t > 0 \\ \\ u(x,0) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \end{cases}, & 0 < x < 2. \end{cases}$$

(b) *Use mathematical software to plot the graph of your solution.*

(c) *Use mathematical software to produce a time animation of your solution.*

Problem 3 *(1-D heat equation, Haberman 2.4.1) Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u_x(0,t) = 0 = u_x(L,t), & t>0 \\ \\ u(x,0) = \sin(\pi x/L), & 0<x<L. \end{cases}$$

Problem 4 *(1-D heat equation, Haberman 2.4.1)*

(a) *Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\pi)\times(0,\infty) \\ u_x(0,t) = 0 = u_x(\pi,t), & t>0 \\ \\ u(x,0) = \sin x, & 0<x<\pi. \end{cases}$$

(b) *Use mathematical software to plot the graph of your solution.*

(c) *Use mathematical software to produce a time animation of your solution.*

Problem 5 *(1-D heat conduction in a ring, Haberman 2.4.6)*

(a) *Solve the initial/boundary value problem for the 1-D heat equation*

$$\begin{cases} u_t = u_{xx} & \text{on } (0,2\pi)\times(0,\infty) \\ u(0,t) = u(2\pi,t), & t>0 \\ u_x(0,t) = u_x(2\pi,t), & t>0 \\ u(x,0) = \begin{cases} 0, & 0<x<\pi \\ 1, & \pi<x<2\pi \end{cases}, & 0<x<2\pi. \end{cases} \tag{1}$$

(b) *Use mathematical software to plot the graph of your solution.*

(c) *Use mathematical software to produce a time animation of your solution.*

(d) *Write down and solve the* **equilibrium problem** *associated with (1).*

(e) *Let $u = u(x,t)$ be the solution of (1) you found in part (a). Find*

$$\lim_{t\to\infty} u(x,t).$$

Problem 6 *(Laplace's equation in a rectangle, Haberman 2.5.1) Solve the boundary value problem for Laplace's equation*

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{on } (0, L) \times (0, M) \\ u_x(0, y) = 0 = u_x(L, y), & t > 0 \\ u(x, 0) = 0, & 0 < x < L. \\ u(x, M) = g(x), & 0 < x < L. \end{cases}$$

Problem 7 *(Laplace's equation in a rectangle, Haberman 2.5.2) Under what conditions does the following boundary value problem for Laplace's equation fail to have a solution?*

$$\begin{cases} u_{xx} + u_{yy} = 0 & \text{on } (0, L) \times (0, M) \\ u_x(0, y) = 0 = u_x(L, y), & t > 0 \\ u_y(x, 0) = 0, & 0 < x < L. \\ u_y(x, M) = g(x), & 0 < x < L. \end{cases}$$

Hint: Integrate the equation (PDE) over the rectangle and then use the divergence theorem.

Problem 8 *(Laplace's equation in a disk, Haberman 2.5.5) Solve Laplace's equation in a quarter disk*

$$\mathcal{U} = \{(r\cos\theta, r\sin\theta) : 0 < r < a, 0 < \theta < \pi/2\}$$

for a temperature $v = v(r, \theta)$ (in polar coordinates) subject to the boundary conditions

$$\begin{cases} v_\theta(r, 0) = 0, & 0 < r < a \\ v(r, \pi/2) = 0, & 0 < r < a \\ v(a, \theta) = g(\theta), & 0 < \theta < \pi/2. \end{cases}$$

Problem 9 *(Poisson integral formula, Haberman 2.5.4)*

(a) *Use the Fourier series/separated variables solution of*

$$\begin{cases} \Delta v = 0, & \text{on } \{(r\cos\theta, r\sin\theta) : 0 \le r < a, \theta \in \mathbb{R}\} \\ v(a, \theta) = g(\theta), & \theta \in \mathbb{R} \end{cases}$$

to show

$$v(r, \theta) = \frac{1}{\pi} \int_0^{2\pi} g(t) \left[-\frac{1}{2} + \sum_{j=0}^{\infty} \left(\frac{r}{a}\right)^j \cos[j(\theta - t)] \right] dt. \tag{2}$$

(b) *Substitute*

$$\cos[j(\theta - t)] = \mathrm{Re}\left[\cos[j(\theta - t)] + i\sin[j(\theta - t)]\right] = \mathrm{Re}\left[e^{ij(\theta - t)}\right]$$

into (2), where $i = \sqrt{-1}$, Re *denotes the* **real part of a complex number**, *and we* $\cos z + i \sin z = e^{iz}$ *is Euler's formula, to obtain the Poisson integral formula:*

$$v(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \, \frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta - t)} \, dt.$$

Hint(s): Taking the real part Re *can be treated as linear with respect to sums, series, and integrals; take* Re *outside the integral. Find a geometric series, and remember you know how to find the sum of such a series. Then use Euler's formula again and the fact that* $(a + bi)(a - bi) = a^2 + b^2$ *to write your integrand as* $f(t)(A + Bi)$*. Then take the real part back inside the integral.*

Problem 10 *(Uniqueness of solutions for the Dirichlet problem for Poisson's equation, Haberman 2.5.10) Prove* **any solution** u *of the problem*

$$\begin{cases} -\Delta u = f, \text{ on } \mathcal{U} \\ u\big|_{\partial u} = g, \end{cases}$$

where $f : \mathcal{U} \to \mathbb{R}$ *and* $g : \partial u \to \mathbb{R}$ *are given functions,* **is unique**. *Hint: Use the maximum priniple.*

Problem 11 *(Calculus of Variations, bonus)*

(a) *Derive the formula for the area of the surface of rotation determined by rotating the graph of a real valued function* $u : [a, b] \to \mathbb{R}$ *about the domain axis.*

(b) *Call the area calculated in part (a) the Bliss functional* $A : \mathcal{A} \to \mathbb{R}$ *where*

$$\mathcal{A} = \{u \in C^1[a, b] : u(a) = d, u(b) = d\}.$$

Calculate the first variation of A *and determine the first order necessary condition for minimizers.*

(c) *What can you say about the minimizers for this problem?*