Assignment 4 = Exam 1: Integration and The Heat Equation
Due Tuesday October 10, 2023

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Problem 1 (divergence) Let U be an open subset of $\mathbb{R}^2$ and assume $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field. Assume also that the coordinate functions v_1 and v_2 of $\mathbf{v} = (v_1, v_2)$ have continuous first partial derivatives on U. Take $\mathbf{p} = (p_1, p_2) \in U$ and consider for $\epsilon, \delta > 0$ a rectangular domain

$$R = (p_1 - \epsilon, p_1 + \epsilon) \times (p_2 - \delta, p_2 + \delta) = \{\mathbf{x} \in \mathbb{R}^2 : |x_1 - p_1| < \epsilon \text{ and } |x_2 - p_2| < \delta\}.$$

Finally, assume the closure

$$\overline{R} = [p_1 - \epsilon, p_1 + \epsilon] \times [p_2 - \delta, p_2 + \delta] = \{\mathbf{x} \in \mathbb{R}^2 : |x_1 - p_1| \le \epsilon \text{ and } |x_2 - p_2| \le \delta\}$$

satisfies $\overline{R} \subset U$.

(a) Express the boundary integral

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \sum_{j=1}^{4} I_j$$

where $\mathbf{n}$ is the outward unit normal field on ∂R as the sum of four elementary integrals of the form

$$I = \int_a^b f(t)\, dt$$

each corresponding to a single side of ∂R. Be careful to express the integrals I_j for $j = 1, 2, 3, 4$ precisely and in full detail so that the dependence of the arguments of v_1 and v_2 on the variable t and the lengths ϵ and δ is clearly indicated.

(b) Combine the integrals from part **(b)** above in pairs corresponding to opposite sides, and apply the mean value theorem to the resulting integrands. Hint: If the segment

$$\{(a, y) : y_1 \leq y \leq y_2\}$$

is a subset of U, then by the mean value theorem one can write

$$v_1(a, y_2) - v_1(a, y_1) = (y_2 - y_1)\frac{\partial v_1}{\partial y}(a, y_*)$$

for some y_* with $y_1 < y_* < y_2$.

(c) Use your expressions for part **(b)** to compute the following limits

(i)

$$\lim_{\epsilon \to 0} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(ii)

$$\lim_{\delta \to 0} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iii)

$$\lim_{\epsilon \to 0} \frac{1}{\text{length}(\partial R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iv)

$$\lim_{\delta \to 0} \frac{1}{\text{length}(\partial R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(d) The mean value theorem for integrals states that if f is continuous on the closed interval $[a, b]$, then there is some $x_{**} \in (a, b)$ for which

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(x_{**}).$$

Use this result along with your expression from part **(b)** above to write

$$\frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}$$

as a sum of two terms in which no integrals appear.

(e) Compute the limits

(i)

$$\lim_{\epsilon \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(ii)

$$\lim_{\delta \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iii)

$$\operatorname{div} \mathbf{v}(\mathbf{p}) = \lim_{\epsilon, \delta \to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

Problem 2 (2-D heat equation) Let U model a lamina on which the distribution of thermal energy evolves by conduction. Complete the following steps to derive the heat equation for the temperature $u : U \times [0, T) \to \mathbb{R}$:

(a) State the divergence theorem by filling in the blanks. If $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field having component functions $\mathbf{v} = (v_1, v_2)$ with continuous first partial derivatives and R is an open subset of $\mathbb{R}^2$ with closure

$$\overline{R} = R \cup \partial R \subset \underline{\qquad}$$

and well-defined continuous outward unit normal field

$$\mathbf{n} : \partial R \to \underline{\qquad},$$

then

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \underline{\qquad\qquad\qquad}.$$

(b) Letting $\theta : U \times [0, T) \to \mathbb{R}$ model the **thermal energy density** in the lamina, the physical dimensions of θ are given by

$$[\theta] = \underline{\qquad\qquad\qquad},$$

and the total thermal energy within the (sub)lamina corresponding to R is modeled by the integral expression

$$\underline{\qquad\qquad\qquad\qquad\qquad}.$$

(c) Letting $\vec{\phi} : U \times [0, T) \to \mathbb{R}^2$ model the **thermal flux** within U, the physical dimenions of $\vec{\phi}$ are given by

$$[\vec{\phi}] = \underline{\qquad\qquad\qquad},$$

and the integral expression

$$\int_{\partial R} \vec{\phi} \cdot \mathbf{n} \quad \text{models the rate } \underline{\qquad\qquad\qquad\qquad} \text{ exits } \underline{\qquad}.$$

(d) Assuming no independent thermal energy generation or depletion within the lamina, conservation of thermal energy is modeled by the integral equation

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which by differentiating under the integral sign and using the divergence theorem may be written

(1)

____

as the vanishing of a single integral expression.

(e) Assuming

$$\frac{\partial \theta}{\partial t}$$

is continuous and $\vec{\phi}$ has component functions with continuous first spatial partial derivatives, we can use the

fundamental lemma of ____________________

to conclude

$$\frac{\partial \theta}{\partial t} + \operatorname{div}(\vec{\phi}) = 0 \qquad \text{on } U \times (0, T). \tag{2}$$

Equation (2) is a _____ order partial differential equation for _____ real valued functions.

(f) The **law of specific heat** asserts $\theta = c\rho u$ where $u : U \times [0, T) \to \mathbb{R}$ models the temperature and ρ is a mass density so that

$[\rho] =$ ________________ and $[c] =$ ________________ .

(g) **Fourier's law of heat conduction** asserts

$$\vec{\phi} = K \, \underline{\qquad}$$

where K is called the conductivity and has physical units

$[K] =$ ________________ .

(h) In view of Fourier's law and the law of specific heat, the integral equation (1) may be written in terms of the gradient

$$Du = \left(\underline{\hspace{3cm}}\ ,\ \underline{\hspace{3cm}} \right)$$

as

$$\underline{\hspace{7cm}},$$

and equation (2) may be written as the _____ order partial differential equation

$$\underline{\hspace{7cm}}$$

for _____.

Problems 3 and 4 below are about the initial/boundary value problem

$$\begin{cases} u_t = \Delta u, & (x, y, t) \in R \times (0, \infty) \\ u(x, y, 0) = u_0, & (x, y) \in R \\ u(x, y, t) = 0, & (x, y, t) \in \partial R \times (0, \infty) \end{cases}$$

for the 2-D heat equation where $R = (0, 4) \times (0, 2)$ is a rectangular spatial domain in $\mathbb{R}^2$.

Problem 3 (separated variables solutions)

(a) For this problem ignore the initial condition and find all solutions of the form

$$u(x, y, t) = a(x, y)b(t).$$

Hint: Set $a(x, y) = A(x)B(y)$ and find ODEs/Sturm-Liouville problems with appropriate boundary values for the functions A and B of one variable.

(b) One of your solutions u should have

$$a(x, y) = \sin\left(\frac{\pi x}{4}\right) \sin\left(\frac{\pi y}{2}\right).$$

Use mathematical software to plot (the graph of) $u(x, y, 0)$ for this solution.

(c) Use mathematical software to animate the time evolution of the graph

$$\mathcal{G}_t = \{(x, y, u(x, y, t)) : (x, y) \in R\}$$

where u is your solution from part **(b)**.

Problem 4 (superposition) Consider the initial temperature

$$u_0(x, y) = 2 - \max\{|x - 2|, 2|y - 1|\}$$

on $R = (0, 4) \times (0, 2)$.

(a) Plot (the graph of) u_0:

- **(i)** using mathematical software.
- **(ii)** by hand.

(b) Let

$$a_{k\ell}(x, y) = \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi y}{2}\right).$$

Compute the integrals

(i)

$$n_{k\ell} = \int_R [a_{k\ell}]^2.$$

(ii)

$$\nu_{k\ell} = \int_R a_{k\ell} u_0.$$

(c) Find a coefficient $c = c_{k\ell}$ for which

$$c n_{k\ell} = \nu_{k\ell}.$$

(d) Let

$$v_{k\ell}(x,y,t) = a_{k\ell}(x,y)b_{k\ell}(t)$$

be a separated variables solution you found in Problem 3 above. Use mathematical software to plot the following:

(i)

$$\sum_{k,\ell\leq 3} c_{k\ell}v_{k\ell}(x,y,0).$$

(ii)

$$\sum_{k,\ell\leq 5} c_{k\ell}v_{k\ell}(x,y,0).$$

(iii)

$$\sum_{k,\ell\leq 7} c_{k\ell}v_{k\ell}(x,y,0).$$

(iv)

$$\sum_{k,\ell\leq 9} c_{k\ell}v_{k\ell}(x,y,0).$$

(d) Use mathematical software to animate the evolution of the following graphs:

(i)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 3} c_{k\ell}v_{k\ell}(x,y,t) \right) : (x,y) \in R \right\}.$$

(ii)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 5} c_{k\ell}v_{k\ell}(x,y,t) \right) : (x,y) \in R \right\}.$$

(iii)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 7} c_{k\ell}v_{k\ell}(x,y,t) \right) : (x,y) \in R \right\}.$$

(iv)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 9} c_{k\ell}v_{k\ell}(x,y,t) \right) : (x,y) \in R \right\}.$$