Assignment 4 = Exam 1: Integration and The Heat Equation Solution for Problem 4 (under construction)

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Problem 1 (divergence) Let U be an open subset of $\mathbb{R}^2$ and assume $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field. Assume also that the coordinate functions v_1 and v_2 of $\mathbf{v} = (v_1, v_2)$ have continuous first partial derivatives on U. Take $\mathbf{p} = (p_1, p_2) \in U$ and consider for $\epsilon, \delta > 0$ a rectangular domain

$$R = (p_1 - \epsilon, p_1 + \epsilon) \times (p_2 - \delta, p_2 + \delta) = \{\mathbf{x} \in \mathbb{R}^2 : |x_1 - p_1| < \epsilon \text{ and } |x_2 - p_2| < \delta\}.$$

Finally, assume the closure

$$\overline{R} = [p_1 - \epsilon, p_1 + \epsilon] \times [p_2 - \delta, p_2 + \delta] = \{\mathbf{x} \in \mathbb{R}^2 : |x_1 - p_1| \le \epsilon \text{ and } |x_2 - p_2| \le \delta\}$$

satisfies $\overline{R} \subset U$.

(a) Express the boundary integral

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \sum_{j=1}^{4} I_j$$

where $\mathbf{n}$ is the outward unit normal field on ∂R as the sum of four elementary integrals of the form

$$I = \int_a^b f(t)\, dt$$

each corresponding to a single side of ∂R. Be careful to express the integrals I_j for $j = 1, 2, 3, 4$ precisely and in full detail so that the dependence of the arguments of v_1 and v_2 on the variable t and the lengths ϵ and δ is clearly indicated.

(b) Combine the integrals from part **(b)** above in pairs corresponding to opposite sides, and apply the mean value theorem to the resulting integrands. Hint: If the segment

$$\{(a, y) : y_1 \leq y \leq y_2\}$$

is a subset of U, then by the mean value theorem one can write

$$v_1(a, y_2) - v_1(a, y_1) = (y_2 - y_1)\frac{\partial v_1}{\partial y}(a, y_*)$$

for some y_* with $y_1 < y_* < y_2$.

(c) Use your expressions for part **(b)** to compute the following limits

(i)

$$\lim_{\epsilon \to 0} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(ii)

$$\lim_{\delta \to 0} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iii)

$$\lim_{\epsilon \to 0} \frac{1}{\text{length}(\partial R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(iv)

$$\lim_{\delta \to 0} \frac{1}{\text{length}(\partial R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}.$$

(d) The mean value theorem for integrals states that if f is continuous on the closed interval $[a, b]$, then there is some $x_{**} \in (a, b)$ for which

$$\frac{1}{b-a} \int_a^b f(x)\, dx = f(x_{**}).$$

Use this result along with your expression from part **(b)** above to write

$$\frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v} \cdot \mathbf{n}$$

as a sum of two terms in which no integrals appear.

(e) Compute the limits

(i)

$$\lim_{\epsilon\to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v}\cdot\mathbf{n}.$$

(ii)

$$\lim_{\delta\to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v}\cdot\mathbf{n}.$$

(iii)

$$\operatorname{div}\mathbf{v}(\mathbf{p}) = \lim_{\epsilon,\delta\to 0} \frac{1}{\text{area}(R)} \int_{\partial R} \mathbf{v}\cdot\mathbf{n}.$$

Problem 2 (2-D heat equation) Let U model a lamina on which the distribution of thermal energy evolves by conduction. Complete the following steps to derive the heat equation for the temperature $u : U \times [0, T) \to \mathbb{R}$:

(a) State the divergence theorem by filling in the blanks. If $\mathbf{v} : U \to \mathbb{R}^2$ is a vector field having component functions $\mathbf{v} = (v_1, v_2)$ with continuous first partial derivatives and R is an open subset of $\mathbb{R}^2$ with closure

$$\overline{R} = R \cup \partial R \subset \underline{\qquad}$$

and well-defined continuous outward unit normal field

$$\mathbf{n} : \partial R \to \underline{\qquad},$$

then

$$\int_{\partial R} \mathbf{v} \cdot \mathbf{n} = \underline{\qquad\qquad\qquad}.$$

(b) Letting $\theta : U \times [0, T) \to \mathbb{R}$ model the **thermal energy density** in the lamina, the physical dimensions of θ are given by

$$[\theta] = \underline{\qquad\qquad\qquad},$$

and the total thermal energy within the (sub)lamina corresponding to R is modeled by the integral expression

$$\underline{\qquad\qquad\qquad\qquad\qquad}.$$

(c) Letting $\vec{\phi} : U \times [0, T) \to \mathbb{R}^2$ model the **thermal flux** within U, the physical dimenions of $\vec{\phi}$ are given by

$$[\vec{\phi}] = \underline{\qquad\qquad\qquad},$$

and the integral expression

$$\int_{\partial R} \vec{\phi} \cdot \mathbf{n} \quad \text{models the rate } \underline{\qquad\qquad\qquad\qquad} \text{ exits } \underline{\qquad}.$$

(d) Assuming no independent thermal energy generation or depletion within the lamina, conservation of thermal energy is modeled by the integral equation

____

which by differentiating under the integral sign and using the divergence theorem may be written

(1)

____

as the vanishing of a single integral expression.

(e) Assuming

$$\frac{\partial \theta}{\partial t}$$

is continuous and $\vec{\phi}$ has component functions with continuous first spatial partial derivatives, we can use the

fundamental lemma of ____________________

to conclude

$$\frac{\partial \theta}{\partial t} + \operatorname{div}(\vec{\phi}) = 0 \qquad \text{on } U \times (0, T). \tag{2}$$

Equation (2) is a _____ order partial differential equation for _____ real valued functions.

(f) The **law of specific heat** asserts $\theta = c\rho u$ where $u : U \times [0, T) \to \mathbb{R}$ models the temperature and ρ is a mass density so that

$[\rho] =$ _______________ and $[c] =$ _______________.

(g) **Fourier's law of heat conduction** asserts

$$\vec{\phi} = K \underline{\qquad}$$

where K is called the conductivity and has physical units

$[K] =$ _______________.

(h) In view of Fourier's law and the law of specific heat, the integral equation (1) may be written in terms of the gradient

$$Du = \left(\underline{\hspace{3cm}} \, , \, \underline{\hspace{3cm}} \right)$$

as

$$\underline{\hspace{7cm}} \, ,$$

and equation (2) may be written as the _____ order partial differential equation

$$\underline{\hspace{7cm}}$$

for _____.

Problems 3 and 4 below are about the initial/boundary value problem

$$\begin{cases} u_t = \Delta u, & (x,y,t) \in R \times (0,\infty) \\ u(x,y,0) = u_0, & (x,y) \in R \\ u(x,y,t) = 0, & (x,y,t) \in \partial R \times (0,\infty) \end{cases}$$

for the 2-D heat equation where $R = (0,4) \times (0,2)$ is a rectangular spatial domain in $\mathbb{R}^2$.

Problem 3 (separated variables solutions)

(a) For this problem ignore the initial condition and find all solutions of the form

$$u(x,y,t) = a(x,y)b(t).$$

Hint: Set $a(x,y) = A(x)B(y)$ and find ODEs/Sturm-Liouville problems with appropriate boundary values for the functions A and B of one variable.

(b) One of your solutions u should have

$$a(x,y) = \sin\left(\frac{\pi x}{4}\right)\sin\left(\frac{\pi y}{2}\right).$$

Use mathematical software to plot (the graph of) $u(x,y,0)$ for this solution.

(c) Use mathematical software to animate the time evolution of the graph

$$\mathcal{G}_t = \{(x,y,u(x,y,t)) : (x,y) \in R\}$$

where u is your solution from part **(b)**.

Problem 4 (superposition) Consider the initial temperature

$$u_0(x,y) = 2 - \max\{|x-2|, 2|y-1|\}$$

on $R = (0,4) \times (0,2)$.

(a) Plot (the graph of) u_0:

- **(i)** using mathematical software.
- **(ii)** by hand.

(b) Let

$$a_{k\ell}(x,y) = \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi y}{2}\right).$$

Compute the integrals

(i)

$$n_{k\ell} = \int_R [a_{k\ell}]^2.$$

(ii)

$$\nu_{k\ell} = \int_R a_{k\ell} u_0.$$

(c) Find a coefficient $c = c_{k\ell}$ for which

$$c n_{k\ell} = \nu_{k\ell}.$$

(d) Let

$$v_{k\ell}(x, y, t) = a_{k\ell}(x, y)b_{k\ell}(t)$$

be a separated variables solution you found in Problem 3 above. Use mathematical software to plot the following:

(i)

$$\sum_{k,\ell\leq 3} c_{k\ell}v_{k\ell}(x, y, 0).$$

(ii)

$$\sum_{k,\ell\leq 5} c_{k\ell}v_{k\ell}(x, y, 0).$$

(iii)

$$\sum_{k,\ell\leq 7} c_{k\ell}v_{k\ell}(x, y, 0).$$

(iv)

$$\sum_{k,\ell\leq 9} c_{k\ell}v_{k\ell}(x, y, 0).$$

(d) Use mathematical software to animate the evolution of the following graphs:

(i)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 3} c_{k\ell}v_{k\ell}(x, y, t) \right) : (x, y) \in R \right\}.$$

(ii)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 5} c_{k\ell}v_{k\ell}(x, y, t) \right) : (x, y) \in R \right\}.$$

(iii)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 7} c_{k\ell}v_{k\ell}(x, y, t) \right) : (x, y) \in R \right\}.$$

(iv)

$$\mathcal{G}_t = \left\{ \left(x, y, \sum_{k,\ell\leq 9} c_{k\ell}v_{k\ell}(x, y, t) \right) : (x, y) \in R \right\}.$$

Problem 4 solution: Notice the initial temperature is given by

$$\begin{aligned} u_0(x, y) &= 2 - \max\{|x - 2|, 2|y - 1|\} \\ &= \min\{2 - |x - 2|, 2(1 - |y - 1|)\}. \end{aligned}$$

(a) Plotting (the graph of) u_0:

(i) This is relatively easy using Mathematica. The code

```
uzero[x_, y_] = Min[2 - Abs[x - 2], 2 (1 - Abs[y - 1])]
Plot3D[uzero[x, y], {x, 0, 4}, {y, 0, 2},
    ViewPoint -> 2 {1.3, 0.4, 0.5}, BoxRatios -> {4, 2, 2},
    PlotStyle -> {GrayLevel[0.8], Opacity[0.2]}]
```

produces the output in Figure 1.

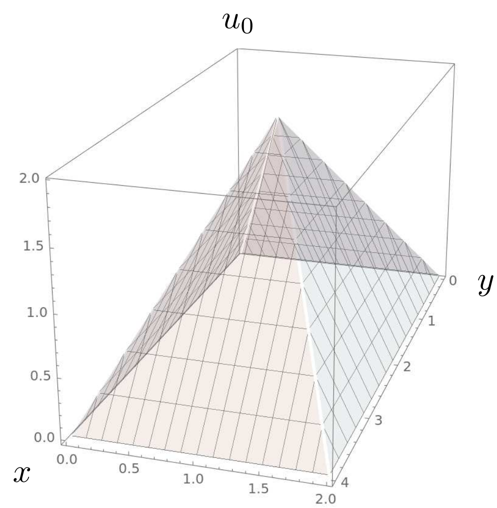

Figure 1: The way mathematical software plots the graph of u_0.

(ii) This is more involved to do by hand, but one can understand the function better by doing it. First consider a fixed value of x with $0 < x < 2$. The corresponding function of y given by $u_0(x, y)$ is given by

$$u_0(x, y) = \min\{x, 2(1 - |y - 1|)\}.$$

Plots of the constant function value x for a value $x = 0.8$ and the function values $2(1-|y-1|)$ are shown on the left in Figure 2. One can see that as y

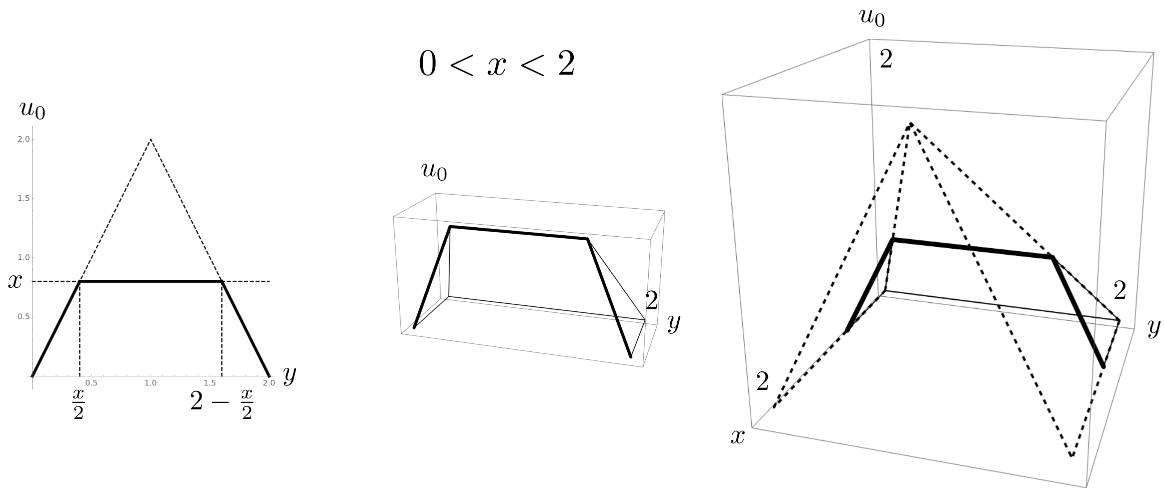

Figure 2: The way a human might plot the graph of u_0. Cross-section of the graph of u_0 corresponding to x fixed with $0 < x < 2$ (left). In this case, $x = 0.8$.

increases, the minimum is obtained by the latter values and transitions to the constant value x at $y_1 = x/2$. The minimum value transitions back to $2(1 - |y - 1|)$ when $y_2 = 2 - x/2$ as indicated in the figure. The remaining drawings indicate how this cross-section fits into the graph for $0 < x < 0.8$ and (on the right) for $0 < x < 2$. Here one can see the "back" part of the pyramid indicated in Figure 1.

Similar considerations apply for x fixed with $2 < x < 4$ except the constant value is then given by $2 - |x - 2| = 4 - x$ and the transitions occur at $y_1 = 2 - x/2 < x/2$ and at $y_2 = x/2$. The result gives the "front" of the pyramid as indicated in Figure 3 where we obtain a sketch essentially equivalent to the one shown in Figure 1.

(b) Setting

$$a_{k\ell}(x, y) = \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell \pi y}{2}\right),$$

we have

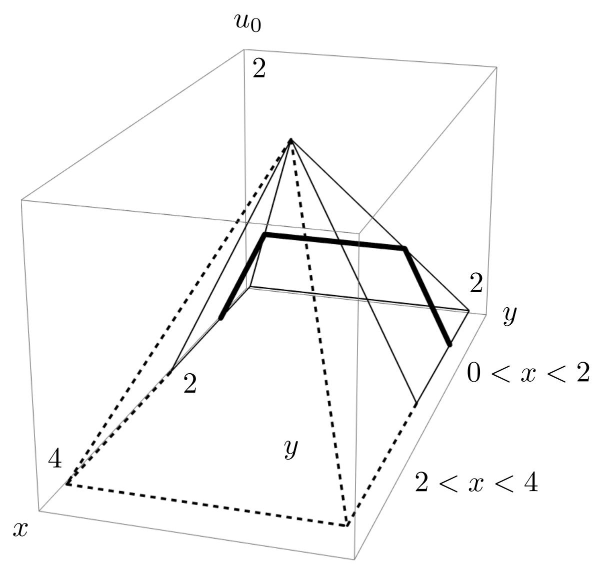

Figure 3: The way a human might plot the graph of u_0.

(i)

$$
\begin{aligned}
n_{k\ell} &= \int_R [a_{k\ell}]^2 \\
&= \int_0^4 \left(\int_0^2 \sin^2\left(\frac{k\pi x}{4}\right) \sin^2\left(\frac{\ell\pi y}{2}\right) \, dy \right) \, dx \\
&= \int_0^4 \int_0^2 \sin^2\left(\frac{k\pi x}{4}\right) \left(\int_0^2 \sin^2\left(\frac{\ell\pi y}{2}\right) \, dy \right) \, dx \\
&= \left(\int_0^2 \sin^2\left(\frac{\ell\pi y}{2}\right) \, dy \right) \left(\int_0^4 \int_0^2 \sin^2\left(\frac{k\pi x}{4}\right) \, dx \right) \\
&= \frac{1}{2} \left(\int_0^2 1 \, dy \right) \frac{1}{2} \left(\int_0^4 1 \, dx \right) \\
&= 2,
\end{aligned}
$$

and

(ii)

$$\begin{aligned}
\nu_{k\ell} &= \int_R a_{k\ell} u_0 \\
&= \int_0^4 \left(\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\, dy \right) dx \\
&= \int_0^2 \left(\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\, dy \right) dx \\
&\quad + \int_2^4 \left(\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\, dy \right) dx \\
&= I_1 + I_2. \qquad (3)
\end{aligned}$$

Here, we have broken the integration into two pieces, namely integration over the "back" region

$$I_1 = \int_0^2 \left(\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\, dy \right) dx$$

and integration over the "front" region

$$I_2 = \int_2^4 \left(\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\, dy \right) dx.$$

Roughly speaking, in view of the "symmetry" involved in the problem, one should expect the overall integration over the region R to eventually reduce to some kind of integration over only one quarter of the region R, so we will work toward expressing the entire integral in terms of an integral over $Q = (0,2) \times (0,1)$. A first step might be[1] change variables using $\xi = 4 - x$ in I_2. I will take the point of view, however, that in order to see the more immediate and full effect of this change of variables, it is better

[1] It can be an instructive exercise at this point to take this "first step" and obtain

$$\begin{aligned}
I_2 &= \int_0^2 \sin\left(k\pi - \frac{k\pi\xi}{4}\right) \left(\int_0^2 \sin\left(\frac{\ell\pi y}{2}\right) u_0(4-\xi,y)\, dy \right) d\xi \\
&= -(-1)^k \int_0^2 \sin\left(\frac{k\pi x}{4}\right) \left(\int_0^2 \sin\left(\frac{\ell\pi y}{2}\right) u_0(4-x,y)\, dy \right) dx.
\end{aligned}$$

to introduce the explicit expression for u_0 on intervals/regions where this expression simplifies. Thus, I turn attention to I_1 and write

$$\begin{aligned}
I_1 = &\int_0^2 \left(\int_0^{x/2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\,dy \right) dx \\
&\quad + \int_0^2 \left(\int_{x/2}^{2-x/2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\,dy \right) dx \\
&\qquad + \int_0^2 \left(\int_{2-x/2}^{2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) u_0(x,y)\,dy \right) dx \\
= &\int_0^2 \left(\int_0^{x/2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) 2y\,dy \right) dx \\
&\quad + \int_0^2 \left(\int_{x/2}^{2-x/2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) x\,dy \right) dx \\
&\qquad + \int_0^2 \left(\int_{2-x/2}^{2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) 2(2-y)\,dy \right) dx.
\end{aligned}$$

Notice that we have used the division points $y_1 = x/2 < 2 - x/2$ and $y_2 = 2 - x/2$ for $0 < y < 2$ associated with Figure 2. Specifically, we obtain

$$J_1 = \int_0^2 \left(\int_0^{x/2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) 2y\,dy \right) dx$$

because $u_0(x,y) = \min\{x, 2(1-|y-1|)\} = 2(1-|y-1|) = 2y$ for $0 < y < x/2 < 1$,

$$J_2 = \int_0^2 \left(\int_{x/2}^{2-x/2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) x\,dy \right) dx$$

where $u_0(x,y) = \min\{x, 2(1-|y-1|)\} = x$ when $x/2 < y < 2 - x/2$, and

$$J_3 = \int_0^2 \left(\int_{2-x/2}^{2} \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi y}{2}\right) 2(2-y)\,dy \right) dx$$

with $u_0(x,y) = \min\{x, 2(1-|y-1|)\} = 2(1-|y-1|) = 2(2-y)$ for $1 < 2 - x/2 < y < 2$. We record for future reference the decomposition

$$I_1 = J_1 + J_2 + J_3. \tag{4}$$

Note that

$$J_1 = 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \left(\int_0^{x/2} y \sin\left(\frac{\ell\pi y}{2}\right) dy\right) dx \tag{5}$$

$$= 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right) K_1\, dx,$$

$$J_2 = \int_0^2 x \sin\left(\frac{k\pi x}{4}\right) \left(\int_{x/2}^{2-x/2} \sin\left(\frac{\ell\pi y}{2}\right) dy\right) dx \tag{6}$$

$$= \int_0^2 x \sin\left(\frac{k\pi x}{4}\right) K_2\, dx, \text{ and}$$

$$J_3 = 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right) \left(\int_{2-x/2}^2 (2-y) \sin\left(\frac{\ell\pi y}{2}\right) dy\right) dx \tag{7}$$

$$= 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right) K_3\, dx$$

where

$$K_1 = \int_0^{x/2} y \sin\left(\frac{\ell\pi y}{2}\right) dy,$$

$$K_2 = \int_{x/2}^{2-x/2} \sin\left(\frac{\ell\pi y}{2}\right) dy, \text{ and}$$

$$K_3 = \int_{2-x/2}^2 (2-y) \sin\left(\frac{\ell\pi y}{2}\right) dy.$$

Taking each of these integrals in turn, we find

$$K_1 = -\frac{2y}{\ell\pi} \cos\left(\frac{\ell\pi y}{2}\right)\Big|_{y=0}^{x/2} + \frac{2}{\ell\pi}\int_0^{x/2} \cos\left(\frac{\ell\pi y}{2}\right) dy$$

$$= -\frac{x}{\ell\pi} \cos\left(\frac{\ell\pi x}{4}\right) + \frac{4}{\ell^2\pi^2} \sin\left(\frac{\ell\pi y}{2}\right)\Big|_{y=0}^{x/2}$$

$$= -\frac{x}{\ell\pi} \cos\left(\frac{\ell\pi x}{4}\right) + \frac{4}{\ell^2\pi^2} \sin\left(\frac{\ell\pi x}{4}\right). \tag{8}$$

For K_2 we write

$$K_2 = \int_{x/2}^1 \sin\left(\frac{\ell\pi y}{2}\right) dy + \int_1^{2-x/2} \sin\left(\frac{\ell\pi y}{2}\right) dy$$

and change variables using $\eta = 2 - y$ in the second integral so that

$$\begin{aligned} K_2 &= \int_{x/2}^{1} \sin\left(\frac{\ell\pi y}{2}\right) dy + \int_{x/2}^{1} \sin\left(\ell\pi - \frac{\ell\pi\eta}{2}\right) d\eta \\ &= \int_{x/2}^{1} \sin\left(\frac{\ell\pi y}{2}\right) dy - \cos(\ell\pi) \int_{x/2}^{1} \sin\left(\frac{\ell\pi\eta}{2}\right) d\eta \\ &= [1 - (-1)^{\ell}] \int_{x/2}^{1} \sin\left(\frac{\ell\pi y}{2}\right) dy \end{aligned} \tag{9}$$

since $\sin(\ell\pi) = 0$ and $\cos(\ell\pi) = (-1)^{\ell}$. Continuing by setting

$$K_{1/2} = \int_{x/2}^{1} \sin\left(\frac{\ell\pi y}{2}\right) dy,$$

we find

$$\begin{aligned} K_{1/2} &= -\frac{2}{\ell\pi} \cos\left(\frac{\ell\pi y}{2}\right)\Big|_{y=x/2}^{1} \\ &= -\frac{2}{\ell\pi}\left[\cos\left(\ell\,\frac{\pi}{2}\right) - \cos\left(\frac{\ell\pi x}{4}\right)\right] \\ &= \frac{2}{\ell\pi}\left[\cos\left(\frac{\ell\pi x}{4}\right) - \cos\left(\ell\,\frac{\pi}{2}\right)\right]. \end{aligned} \tag{10}$$

Since $\cos(\ell\pi/2) = 0$ when ℓ is odd and $1 - (-1)^{\ell} = 0$ when ℓ is even, we conclude

$$K_{1/2} = \frac{2}{\ell\pi} \cos\left(\frac{\ell\pi x}{4}\right) \qquad \text{when } \ell = 2j+1 \text{ is odd,} \tag{11}$$

and

$$K_2 = [1 - (-1)^{\ell}] K_{1/2} = \begin{cases} 0, & \ell \text{ even} \\ \dfrac{4}{\ell\pi} \cos\left(\dfrac{\ell\pi x}{4}\right), & \ell = 2j+1 \text{ odd.} \end{cases} \tag{12}$$

The change of variables $\eta = 2 - y$ applied to K_3 gives

$$\begin{aligned} K_3 &= \int_0^{x/2} \eta \sin\left(\ell\pi - \frac{\ell\pi\eta}{2}\right) d\eta \\ &= -(-1)^{\ell} \int_0^{x/2} \eta \sin\left(\frac{\ell\pi\eta}{2}\right) d\eta \\ &= -(-1)^{\ell} K_1. \end{aligned} \tag{13}$$

Substituting the expression for K_2 in (9) into J_2 we find

$$J_2 = [1-(-1)^\ell] \int_0^2 x \sin\left(\frac{k\pi x}{4}\right) \left(\int_{x/2}^1 \sin\left(\frac{\ell\pi y}{2}\right) dy\right) dx$$
$$= [1-(-1)^\ell] \int_0^2 x \sin\left(\frac{k\pi x}{4}\right) K_{1/2}\, dx \qquad (14)$$

so that taking account of the calculation (10-12)

$$J_2 = \begin{cases} 0, & \ell \text{ even} \\ \dfrac{4}{\ell\pi} \displaystyle\int_0^2 x \sin\left(\frac{k\pi x}{4}\right) \cos\left(\frac{\ell\pi x}{4}\right) dx, & \ell = 2j+1 \text{ odd.} \end{cases} \qquad (15)$$

Similarly, substituting the expression for K_3 from (13) into J_3 we have

$$J_3 = -2(-1)^\ell \int_0^2 \sin\left(\frac{k\pi x}{4}\right) K_1\, dx = -(-1)^\ell J_1$$

and

$$J_1 + J_3 = 2[1-(-1)^\ell] \int_0^2 \sin\left(\frac{k\pi x}{4}\right) K_1\, dx \qquad (16)$$

so that

$$J_1 + J_3 = \begin{cases} 0, & \ell \text{ even} \\ 4 \displaystyle\int_0^2 \sin\left(\frac{k\pi x}{4}\right) K_1\, dx, & \ell = 2j+1 \text{ odd.} \end{cases} \qquad (17)$$

In view of the calculation (8) giving an expression for K_1 which we apply when $\ell = 2j+1$ is odd, we see that in that case

$$J_1 + J_3 = \frac{16}{\ell^2\pi^2} \int_0^2 \sin\left(\frac{k\pi x}{4}\right) \sin\left(\frac{\ell\pi x}{4}\right) dx$$
$$- \frac{4}{\ell\pi} \int_0^2 x \sin\left(\frac{k\pi x}{4}\right) \cos\left(\frac{\ell\pi x}{4}\right) dx. \qquad (18)$$

We could press forward to use the trigonometric identities

$$\cos\left(\frac{(k-\ell)\pi x}{4}\right) = \cos\left(\frac{k\pi x}{4}\right)\cos\left(\frac{\ell\pi x}{4}\right) + \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right) \tag{19}$$

$$\cos\left(\frac{(k+\ell)\pi x}{4}\right) = \cos\left(\frac{k\pi x}{4}\right)\cos\left(\frac{\ell\pi x}{4}\right) - \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right) \tag{20}$$

$$\sin\left(\frac{(k-\ell)\pi x}{4}\right) = \sin\left(\frac{k\pi x}{4}\right)\cos\left(\frac{\ell\pi x}{4}\right) - \cos\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right) \tag{21}$$

$$\sin\left(\frac{(k+\ell)\pi x}{4}\right) = \sin\left(\frac{k\pi x}{4}\right)\cos\left(\frac{\ell\pi x}{4}\right) + \cos\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right) \tag{22}$$

to evaluate the integrals

$$L_1 = \int_0^2 \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right)\,dx \tag{23}$$

and

$$L_2 = \int_0^2 x\,\sin\left(\frac{k\pi x}{4}\right)\cos\left(\frac{\ell\pi x}{4}\right)\,dx \tag{24}$$

appearing in (15) and (18). We postpone these calculations for the moment and turn instead to consideration of

$$I_2 = \int_2^4 \left(\int_0^2 \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi y}{2}\right)\,u_0(x,y)\,dy\right)\,dx$$

corresponding to integration over the "front" region $(2,4)\times(0,2)$. We do note first however, that given the form of K_1 in (16) and the form of $K_{1/2}$ in (14) we can write

$$\begin{aligned} J_1 + J_2 + J_3 = [1-(-1)^\ell]\Bigg\{ & 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right)\left(\int_0^{x/2} y\sin\left(\frac{\ell\pi y}{2}\right)\,dy\right)\,dx \\ & + \int_0^2 x\,\sin\left(\frac{k\pi x}{4}\right)\left(\int_{x/2}^1 \sin\left(\frac{\ell\pi y}{2}\right)\,dy\right)\,dx\Bigg\} \\ = [1-(-1)^\ell]\Bigg\{ & 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right)K_1\,dx \\ & + \int_0^2 x\,\sin\left(\frac{k\pi x}{4}\right)K_{1/2}\,dx\Bigg\}, \end{aligned} \tag{25}$$

and we have indeed reduced $J_1 + J_2 + J_3$ to an expression in terms of integrals over subregions of the quarter domain $Q = (0,2) \times (0,1)$.

By the change of variables $\xi = 4 - x$,

$$\begin{aligned} I_2 &= -(-1)^k \int_0^2 \sin\left(\frac{k\pi\xi}{4}\right) \left(\int_0^2 \sin\left(\frac{\ell\pi y}{2}\right) u_0(4-\xi, y)\, dy \right) d\xi \\ &= -(-1)^k \int_0^2 \sin\left(\frac{k\pi\xi}{4}\right) K_4\, d\xi \end{aligned} \tag{26}$$

where

$$K_4 = \int_0^2 \sin\left(\frac{\ell\pi y}{2}\right) u_0(4-\xi, y)\, dy. \tag{27}$$

For $2 < x < 4$, the division points are different so that

$$u_0(x,y) = \min\{4-x, 2(1-|y-1|)\} = \begin{cases} 2y, & 0 < y < 2 - x/2 \\ 4-x, & 2 - x/2 < y < x/2 \\ 2(2-y), & x/2 < y < 2. \end{cases}$$

Taking account of our change of variables $x = 4 - \xi$ with $0 < \xi < 2$ this transforms to

$$u_0(4-\xi, y) = \begin{cases} 2y, & 0 < y < \xi/2 \\ \xi, & \xi/2 < y < 2 - \xi/2 \\ 2(2-y), & 2 - \xi/2 < y < 2. \end{cases}$$

Therefore,

$$\begin{aligned} K_4 = 2 \int_0^{\xi/2} y \sin\left(\frac{\ell\pi y}{2}\right) dy & \\ + \xi \int_{\xi/2}^{2-\xi/2} \sin\left(\frac{\ell\pi y}{2}\right) dy & \\ + 2 \int_{2-\xi/2}^{2} (2-y) \sin\left(\frac{\ell\pi y}{2}\right) dy. & \end{aligned}$$

It follows that by simply changing the name of the variable of integration from

ξ to x in (26) we see from the expressions (5-7)

$$
\begin{aligned}
I_2 = -(-1)^k \Bigg\{ 2\int_0^2 \sin\left(\frac{k\pi x}{4}\right)\left(\int_0^{x/2} y\,\sin\left(\frac{\ell\pi y}{2}\right)\,dy\right)\,dx \\
+\int_0^2 x\,\sin\left(\frac{k\pi x}{4}\right)\left(\int_{x/2}^{2-x/2} \sin\left(\frac{\ell\pi y}{2}\right)\,dy\right)\,dx \\
+2\int_0^2 \sin\left(\frac{k\pi x}{4}\right)\left(\int_{2-x/2}^{2} (2-y)\sin\left(\frac{\ell\pi y}{2}\right)\,dy\right)\,dx\Bigg\} \\
= -(-1)^k(J_1+J_2+J_3).
\end{aligned}
$$

Combining this observation with (4), (3), and (25)

$$
\nu_{k\ell} = [1-(-1)^k](J_1+J_2+J_3) \tag{28}
$$

$$
= [1-(-1)^k][1-(-1)^\ell]\left\{2\int_0^2 \sin\left(\frac{k\pi x}{4}\right)K_1\,dx + \int_0^2 x\,\sin\left(\frac{k\pi x}{4}\right)K_{1/2}\,dx\right\}. \tag{29}
$$

Evidently then if k is even or ℓ is even we have $\nu_{k\ell}=0$. We proceed under the assumption

$$
k = 2i+1 \text{ and } \ell = 2j+1 \text{ are both odd.}
$$

In this case, $\nu_{k\ell} = 2(J_1+J_2+J_3)$ and this quantity is also given by

$$
\nu_{k\ell} = 4\left\{2\int_0^2 \sin\left(\frac{k\pi x}{4}\right)K_1\,dx + \int_0^2 x\,\sin\left(\frac{k\pi x}{4}\right)K_{1/2}\,dx\right\}.
$$

Using the expression for J_2 from (15) and the expression for J_1+J_3 from (18) we see there is a cancellation and

$$
\nu_{k\ell} = \frac{32}{\ell^2\pi^2}\int_0^2 \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right)\,dx. \tag{30}
$$

Consequently, we need only calculate one integral,[2] namely

$$
L_1 = \int_0^2 \sin\left(\frac{k\pi x}{4}\right)\sin\left(\frac{\ell\pi x}{4}\right)\,dx,
$$

[2]That is, one integral from among (23) and (24).

using the trigonometric identities (19-22) and, in fact, we only need (19) and (20). We proceed in cases: Remember we are assuming $k = 2i+1$ and $\ell = 2j+1$ are both odd. If $k \neq \ell$, then

$$\begin{aligned} L_1 &= \int_0^2 \frac{1}{2}\left[\cos\left(\frac{(k-\ell)\pi x}{4}\right) - \cos\left(\frac{(k+\ell)\pi x}{4}\right)\right] dx \\ &= \frac{2}{(k-\ell)\pi}\sin\left(\frac{(k-\ell)\pi x}{4}\right)\Big|_{x=0}^{2} - \frac{2}{(k+\ell)\pi}\sin\left(\frac{(k+\ell)\pi x}{4}\right)\Big|_{x=0}^{2} \\ &= \frac{2}{\pi}\left[\frac{1}{(k-\ell)\pi}\sin\left((k-\ell)\frac{\pi}{2}\right) - \frac{1}{(k+\ell)\pi}\sin\left((k+\ell)\frac{\pi}{2}\right)\right]. \end{aligned}$$

Substituting $k - \ell = 2(i-j)$ and $k + \ell = 2(i+j+1)$ we find

$$\begin{aligned} \sin\left((k-\ell)\frac{\pi}{2}\right) &= \sin\left((i+j)\pi\right) = 0 \qquad \text{and} \\ \sin\left((k+\ell)\frac{\pi}{2}\right) &= \sin\left((i+j+1)\pi\right) = 0. \end{aligned}$$

It follows that $\nu_{k\ell} = 0$ when $k \neq \ell$. The final case is when $k = 2i+1 = \ell$, and

$$L_1 = \int_0^2 \sin^2\left(\frac{k\pi x}{4}\right) dx = \frac{1}{2}\int_0^2 1\, dx = 1.$$

Thus,

$$\nu_{k\ell} = \begin{cases} 0, & \text{if } k \text{ is even or } \ell \text{ is even or } k \neq \ell \\ \dfrac{32}{k^2\pi^2} = \dfrac{32}{(2i+1)^2\pi^2}, & \text{if } k = 2i+1 = \ell \text{ is odd.} \end{cases} \tag{31}$$

It is possible to use a mathematical software package like Mathematica make this entire calculation, but due to the various cases a little care may be required. With `uzero` defined as in part **(a)(i)** above, the command

```
Integrate[ Sin[k Pi x/4] Sin[ell Pi y/2] uzero[x, y],
           {x, 0, 4}, {y, 0, 2}]
```

(after some time) returns a complicated expression (which no one wants to deal with directly) but

```
Simplify[Integrate[ Sin[k Pi x/4] Sin[ell Pi y/2] uzero[x, y],
            {x, 0, 4}, {y, 0, 2}]]
```

(after some time) returns something like

$$-\left(\frac{64\ \text{Sin}\left[\frac{k\pi}{2}\right]\left(k\ (-1 + \text{Cos}[ell\ \pi])\ \text{Cos}\left[\frac{k\pi}{2}\right] + ell\ \text{Sin}[ell\ \pi]\ \text{Sin}\left[\frac{k\pi}{2}\right]\right)}{(ell\ (ell - k)\ k\ (ell + k)\ \pi^3)}\right)$$

which in more traditional notation looks like

$$-\frac{64\sin(k\pi/2)[k(-1+\cos(\ell\pi))\cos(k\pi/2)+\ell\sin(\ell\pi)\sin(k\pi/2)]}{\ell(\ell-k)k(\ell+k)\pi^3}. \tag{32}$$

In any case, one can see here that the factor $\ell - k$ in the denominator is going to be a problem when $\ell = k$. Essentially, you're going to get an indeterminate form in that case, but Mathematica very well may not know what to do with the expression and give a division by zero error. The way I usually deal with this, on the one hand, is to make a separate computation for the case $k = \ell$ using

```
Simplify[Integrate[ Sin[k Pi x/4] Sin[k Pi y/2] uzero[x, y],
            {x, 0, 4}, {y, 0, 2}]]
```

which returns

$$-\frac{32\ \text{Sin}\left[\frac{k\pi}{2}\right]^2 (-k\pi + \text{Sin}[k\pi])}{k^3\pi^3}. \tag{33}$$

On the other hand, looking at (32) one can see the $\sin(k\pi/2)$ factor is going to give $\nu_{k\ell} = 0$ when k is even. Also, the factors $(-1+\cos(\ell\pi))$ and $\sin(\ell\pi)$ are going to make $\nu_{k\ell} = 0$ when ℓ is even. This reduces consideration to the case $k = 2i + 1$ and $\ell = 2j + 1$ odd. In fact, the factor $\sin(\ell\pi)$ is always zero, so we can simplify (32) in the case $k \neq \ell$ to

$$-\frac{64k\sin(k\pi/2)(-1+\cos(\ell\pi))\cos(k\pi/2)}{\ell(\ell-k)k(\ell+k)\pi^3} = 128k\ \frac{\sin(k\pi/2)\cos(k\pi/2)}{\ell(\ell-k)k(\ell+k)\pi^3}.$$

This can also be written as

$$64k\ \frac{\sin(k\pi)}{\ell(\ell-k)k(\ell+k)\pi^3} = 0$$

or put another way, one of the two factors $\sin(k\pi/2)$ and $\cos(k\pi/2)$ will always be zero. This tells us the only possible situation in which $\nu_{k\ell}$ is nonzero is when $k = \ell$ is odd, and we should return to (33). The factor $\sin(k\pi)$ appearing in (33) is always zero as well, so we get the simplification

$$\nu_{kk} = \frac{32}{k^2\pi^2} \sin^2\left(\frac{k\pi}{2}\right).$$

which for k odd gives the same answer recorded in (31). It is also possible to carry out this computation/calculation in a somewhat more automated manner using "deferred evaluation" to avoid division by zero errors. In mathematica an expression defined using "`:=`" is not evaluated until specific values are given. Specifically, one can make an assignment like

```
nu[k_,ell_] := Integrate[ Sin[k Pi x/4] Sin[ell Pi y/2] uzero[x, y],
              {x, 0, 4}, {y, 0, 2}]
```

and then only evaluate when specific numerical values of k and ℓ are used. The call

```
nu[1,1]
```

for example then produces the desired answer $32/\pi^2$.

(c) Finding the coefficients $c = c_{k\ell}$ for which

$$cn_{k\ell} = \nu_{k\ell}$$

is now quite easy:

$$c_{k\ell} = \begin{cases} 0, & \text{if } k \text{ is even or } \ell \text{ is even or } k \neq \ell \\ \dfrac{16}{k^2\pi^2} = \dfrac{16}{(2i+1)^2\pi^2}, & \text{if } k = 2i+1 = \ell \text{ is odd.} \end{cases}$$

Specifically, we can consider only the coefficients

$$C_i = \frac{16}{(2i+1)^2\pi^2} \qquad \text{for } i = 0, 1, 2, \ldots$$

corresponding to the separated variables solution(s)

$$v_{k,k}(x,y,t) = e^{-\frac{5(2i+1)^2\pi^2}{16}t} \sin\left(\frac{(2i+1)\pi x}{4}\right) \sin\left(\frac{(2i+1)\pi y}{2}\right).$$

In particular, we should have

$$u_0(x,y) = \sum_{i=0}^{\infty} C_i \sin\left(\frac{(2i+1)\pi x}{4}\right) \sin\left(\frac{(2i+1)\pi y}{2}\right).$$

The next part should give the opportunity to verify we have the coefficients correct.

(d) (i)

$$\sum_{k,\ell\le 3} c_{k\ell} v_{k\ell}(x,y,0) = \sum_{i=0}^{1} C_i \sin\left(\frac{(2i+1)\pi x}{4}\right) \sin\left(\frac{(2i+1)\pi y}{2}\right).$$

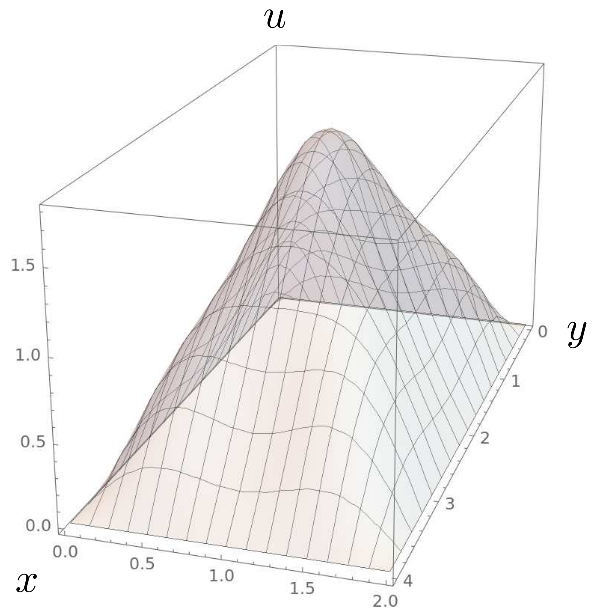

Figure 4: First two nonzero terms in the two variable Fourier approximation of u_0.

For this plot and the plots below I used code along the following lines:

```
coeffs = Table[nu[2 i+1, 2 i+1],{i,0,4}]
a[t_,i_] = E^(-5(2 i+1)^2 Pi^2 t/16)
b[x_,y_,i_] = Sin[(2 i+1) Pi x/4] Sin[(2 i+1) Pi y/2]
appox[x_, y_, k_] := Sum[coeffs[[i + 1]] a[x, y, i], i, 0, k]
first[x_,y_] = approx[x,y,1]
Plot3D[second[x, y], {x, 0, 4}, {y, 0, 2},
       PlotStyle -> {GrayLevel[0.8], Opacity[0.2]},
       ViewPoint -> 2 {1.3, 0.4, 0.5}, BoxRatios -> {4, 2, 2}]
```

(ii)

$$\sum_{k,\ell\leq 5} c_{k\ell} v_{k\ell}(x,y,0) = \sum_{i=0}^{2} C_i \sin\left(\frac{(2i+1)\pi x}{4}\right) \sin\left(\frac{(2i+1)\pi y}{2}\right).$$

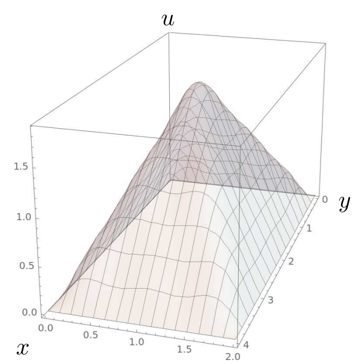

Figure 5: First three nonzero terms in the two variable Fourier approximation of u_0.

(iii)

$$\sum_{k,\ell\leq 7} c_{k\ell} v_{k\ell}(x,y,0) = \sum_{i=0}^{3} C_i \sin\left(\frac{(2i+1)\pi x}{4}\right) \sin\left(\frac{(2i+1)\pi y}{2}\right).$$

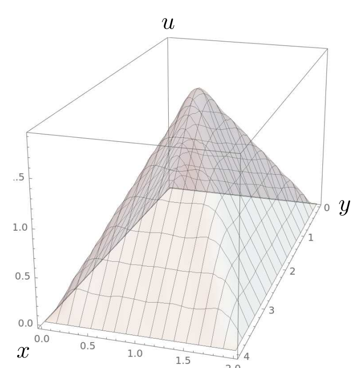

Figure 6: First four nonzero terms in the two variable Fourier approximation of u_0.

(iv)

$$\sum_{k,\ell\leq 9} c_{k\ell}v_{k\ell}(x,y,0) = \sum_{i=0}^{4} C_i \sin\left(\frac{(2i+1)\pi x}{4}\right)\sin\left(\frac{(2i+1)\pi y}{2}\right).$$

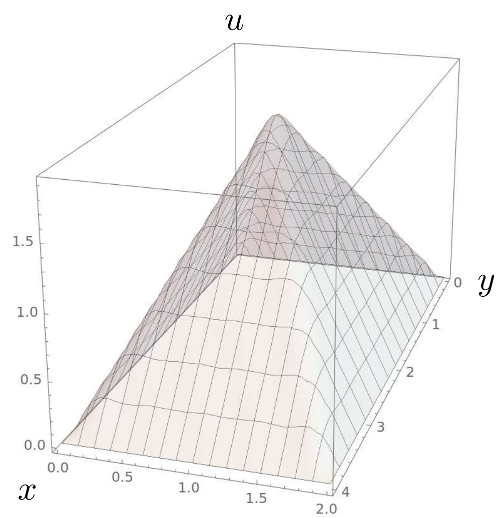

Figure 7: First five nonzero terms in the two variable Fourier approximation of u_0.

(d) I can't show the corresponding animations here, but here is the relevant code:

```
anim[x_, y_, t_, k_] := Sum[coeffs[[i + 1]] a[x, y, i] b[t, i],
            {i, 0, k}]
firstanim[x_, y_, t_] = anim[x, y, t, 1]
Animate[Plot3D[fanim[x, y, t], {x, 0, 4}, {y, 0, 2},
      PlotStyle -> {GrayLevel[0.8], Opacity[0.2]},
      ViewPoint -> 2 {1.3, 0.4, 0.5},
      PlotRange -> {{0, 4}, {0, 2}, {0, 2}},
      BoxRatios -> {4, 2, 2}], {t, 0, 1}]
```

The animations do not display as much resolution as one would like, as is sometimes the case with Mathematica, but the general behavior looks as expected.