Assignment 5: Laplace's Equation (mean value property and maximum principle) Due Tuesday November 9, 2021

John McCuan

October 2, 2021

Problem 1 *(Laplace's equation in a strip, Haberman 2.5.15) Solve the boundary value problem for Laplace's equation*

$$\begin{cases} \Delta u = 0 & \text{on } (0, L) \times (0, \infty) \\ u(0, y) = 0 = u(L, y), & y > 0 \\ u_x(x, 0) = g(x), & 0 < x < L \end{cases}$$

Problem 2 *(mean value property) Consider*

$$f(r) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{p})} u$$

where $B_r(\mathbf{p}) = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x} - \mathbf{p}| < r\}$ *and* $u : \mathcal{U} \to \mathbb{R}$ *is a solution of Laplace's equation with* $\overline{B_r(\mathbf{p})} \subset \mathcal{U} \subset \mathbb{R}^2$.

(a) *Compute* $f'(r)$ *and show* $f'(r) = 0$. *Hint(s): Change variables so that you're integrating on the boundary of a fixed ball of radius* 1. *Differentiate under the integral sign, and use the divergence theorem.*

(b) *Use continuity to conclude*

$$u(\mathbf{p}) = \frac{1}{2\pi r} \int_{\partial B_r(\mathbf{p})} u.$$

(c) *(bonus) Show*

$$u(\mathbf{p}) = \frac{1}{\pi r^2} \int_{B_r(\mathbf{p})} u.$$

Problem 3 *(maximum principle, Haberman 2.5.13) Show that if $\mathcal{U}$ is an open, bounded, and* connected *domain (in $\mathbb{R}^n$) and $u : \mathcal{U} \to \mathbb{R}$ satisfies $\Delta u = 0$, then*

$$u(\mathbf{p}) > \min\{u(\mathbf{x}) : \mathbf{x} \in \overline{\mathcal{U}}\} \qquad \text{for all } \mathbf{p} \in \mathcal{U}$$

*unless u is constant. (***Connected** *means $\mathcal{U}$ cannot be written as the disjoint union of two nonempty open sets.)*

Problem 4 *(transport of mass, Haberman 2.5.17-18) If mass determined by a density $\rho = \rho(\mathbf{x}, t)$ is modeled by the transport of mass by a velocity $\mathbf{v}$, and the mass remains constant in space and time, then show*

$$\operatorname{div} \mathbf{v} = 0.$$

Problem 5 *(Fourier series, Haberman Chapter 3, sections 3.1-3.3)*

(a) *Find the Fourier sine series of the constant function $f(x) = 1$ on the interval of interest $(0, L)$.*

- **(i)** *Describe the convergence of the series expansion from part (a).*
- **(ii)** *Take $L = 1$, and if Gibb's phenomenon is present, verify it numerically.*

(b) *Find the Fourier cosine series of the constant function $f(x) = x$ on the interval of interest $(0, L)$.*

- **(i)** *Describe the convergence of the series expansion from part (b).*
- **(ii)** *Take $L = 1$, and if Gibb's phenomenon is present, verify it numerically.*

(c) *Find the full Fourier (sine and cosine) series of the constant function $f(x) = x^2$ on the interval of interest $(0, L)$.*

- **(i)** *Describe the convergence of the series expansion from part (c).*
- **(ii)** *Take $L = 2$, and if Gibb's phenomenon is present, verify it numerically.*

Problem 6 *(Fourier series, Haberman Chapter 3, sections 3.1-3.3) Find the Fourier sine series for the constant function* $f(x) = 1$ *on the interval (of interest)* $(0, \pi)$.

(a) *Evaluate your series at* $m\pi$ *for* $m \in \{0, \pm 1, \pm 2, \ldots\}$. *What is the relation between this evaluated value and the values of the constant function?*

(b) *Find the maximum value of the partial sum*

$$f(x) = \frac{4}{\pi} \sum_{j=0}^{N} \frac{1}{2j+1} \sin[(2j+1)x].$$

Hint(s): Differentiate, and try to solve $f'(x) = 0$. *Use induction to prove the trigonometric identity*

$$\sum_{j=0}^{N} \cos[(2j+1)x] = \frac{\sin[2(N+1)x]}{2 \sin x}.$$

Note that $2(N+1)x = (2N+3)x - x$. *Also,*

$$\frac{1}{2j+1} \sin[(2j+1)x] = \int_0^x \cos[(2j+1)\xi]\, d\xi.$$

The answer is

$$h(N) = \frac{4}{\pi} \int_0^{\pi/(2N+1)} \frac{\sin[2(N+1)\xi]}{2 \sin \xi}\, d\xi. \tag{1}$$

(c) *Approximate the limiting value*

$$\delta = \lim_{N \to \infty} h(N).$$

of the expression in (1) as N tends to infinity.

(d) *The value* $(\delta - 1)/1$ *is called the Wilbraham-Gibbs constant. Explain the significance of this ratio as "overshoot."*

Problem 7 *(Calculus of variations, Troutman 1.1.3) Assume a boat maintains a relative velocity*

$$\mathbf{v} = (v_1, v_2) \qquad \textit{with constant magnitude } |\mathbf{v}| = v$$

while crossing a river of varying flow rate $\phi\ \mathbf{e}_2 = (0, \phi)$ *from a point*

$$(x(0), y(0)) = (0, 0) \tag{2}$$

to a point

$$(x(T), y(T)) = (L, Y). \tag{3}$$

This means the path of the boat is determined by the sum of these velocities according to the ODE

$$\left(\frac{dx}{dt}, \frac{dy}{dt}\right) = \mathbf{v} + \phi\ \mathbf{e}_2. \tag{4}$$

(a) *Show that the total time of transit for a path* $(x(t), y(t))$ *satisfying (2-4) is*

$$T = \int_0^L \frac{1}{v_1}\, dx.$$

Hint(s): $v_1 = dx/dt$. *(Assume this quantity is non-vanishing so that the time of travel can be expressed as a function of* x.*)*

(b) *If the path is expressed as the graph of a function* $y = u(x)$, *write down an appropriate admissible class and variational problem to determine the path giving the least time of travel across the river. Hint(s): Show that*

$$u' = \frac{\phi}{v_1} + \sqrt{\left(\frac{v}{v_1}\right)^2 - 1},$$

and solve this equation for $1/v_1$.

(c) *What can you say about the sign of* $v^2 - \phi^2$*?*

The following problems are related to Section 2.5.3 in Haberman and Haberman's exercises 2.5.17-27. You will (presumably) also need to consult my notes on Stokes' Flow around a Cylinder.

Problem 8 *(streamlines, Haberman 2.5.17-19)*

(a) *Show that the transport equation for the motion of mass under a velocity field* $\mathbf{v}$ *implies*

$$\operatorname{div} \mathbf{v} = 0 \tag{5}$$

when the mass density ρ *is constant.*

(b) *Show any velocity field given by* $\mathbf{v} = (\psi_y, -\psi_x)$ *where* $\psi : \mathcal{U} \to \mathbb{R}$ *with* ψ *a solution of Laplaces equation on some open set* $\mathcal{U} \subset \mathbb{R}^2$ *satisfies (5).*

(c) *Assuming the velocity field* $\mathbf{v} = (\psi_y, -\psi_x)$ *as in part (b), let* $\gamma : (a, b) \to L_h$ *be a parameterized curve with image in the level set*

$$L_h = \{(x, y) \in \mathcal{U} : \psi(x, y) = h\}.$$

Show that γ' *is parallel to* $\mathbf{v}$. *Hint(s): Note that* $\psi(\gamma(t)) = h$, *and differentiate.*

Problem 9 *(stream function, lift and drag, Haberman 2.5.21-24) Consider* $\psi : \mathbb{R}^2 \backslash B_1(\mathbf{0}) \to \mathbb{R}$ *by*

$$\psi(x, y) = -\alpha \ln \sqrt{x^2 + y^2} - y \left(1 - \frac{1}{x^2 + y^2}\right)$$

where $B_1(0) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$.

(a) *Show* ψ *satisfies the boundary value problem*

$$\begin{cases} \Delta\psi = 0, & x^2 + y^2 > 1 \\ \psi = 0, & x^2 + y^2 = 1. \end{cases}$$

(b) *Set* $\mathbf{v} = (\psi_y, -\psi_x)$ *and use Bernoulli's law*

$$P = P_0 - \rho \frac{|\mathbf{v}|^2}{2}$$

for the pressure P *to express the force vector exerted by the pressure on* $\partial\mathcal{U} = \partial B_1(\mathbf{0})$ *where* $\mathcal{U} = \mathbb{R}^2 \backslash \overline{B_1(\mathbf{0})}$ *as a flux integral over* $\partial\mathcal{U}$.

(c) *Calculate the horizontal force (drag) and the vertical force (lift) from your integral expression in part (b).*

Problem 10 *(streamlines, Haberman 2.5.25-26) Again consider*

$$\psi(x,y) = -\alpha \ln \sqrt{x^2+y^2} - y\left(1 - \frac{1}{x^2+y^2}\right)$$

and the associated velocity field $\mathbf{v} = (\psi_y, -\psi_x)$. *Assume* $\alpha > 0$.

(a) *Define* $\Psi(r,\theta) = \psi(r\cos\theta, r\sin\theta)$. *Determine the domain* $\mathcal{H}$ *of* Ψ *and plot numerically the level sets*

$$\mathcal{L}_h = \{(r,\theta) \in \Sigma : \Psi(r,\theta) = h\} \subset \Sigma$$

and

$$L_h = \{(x,y) \in \mathcal{U} : \psi(x,y) = h\} \subset \mathcal{U}$$

for $\alpha = h = 1/2$.

(b) *Show that for every* $h \in \mathbb{R}$ *and* $\alpha > 0$, *the level set* L_h *contains a curve* $\gamma : (a,b) \to \mathcal{U}$ *with*

$$\lim |\gamma(t)| = \infty. \tag{6}$$

In (6) the limit is taken as the parameter t *tends to some limit* T. *Setting* $\gamma = (\gamma_1, \gamma_2)$ *determine*

$$\lim_{t\to T} \gamma_2(t).$$

(c) *A* **stagnation point** *is a point* $(x,y) \in \overline{\mathcal{U}}$ *for which* $\mathbf{v}(x,y) = 0$. *For which values of* α *will there be a stagnation point on* $\partial\mathcal{U}$?

(d) *Consider*

$$\mathcal{L}_0 = \{(r,\theta) \in \Sigma : \Psi(r,\theta) = 0\} \subset \Sigma$$

and

$$L_0 = \{(x,y) \in \mathcal{U} : \psi(x,y) = h\} \subset \mathcal{U}.$$

Write down (carefully) a formula for each of these curves.