Assignment 6 = Exam 2: Eigenfunction expansion and the wave equation Due Tuesday November 23, 2021

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Problem 1 *(eigenfunction expansion, Haberman 3.4.11) Consider the initial/boundary value problem for the forced 1-D heat equation*

$$\begin{cases} u_t = u_{xx} + 1 & \text{on } (0,2) \times (0,\infty) \\ u(0,y) = 0 = u(2,y), & y > 0 \\ u(x,0) = g(x), & 0 < x < 2 \end{cases}$$

where

$$g(x) = \begin{cases} -1, & 0 < x < 1 \\ 1, & 1 < x < 2. \end{cases}$$

(a) *Expand the constant forcing function*

$$1 = \sum_{j=1}^{\infty} c_j \sin(j\pi x/2)$$

in a Fourier sine series.

(b) *Expand the initial temperature g in a Fourier sine series*

$$g(x) = \sum_{j=1}^{\infty} g_j \sin(j\pi x/2).$$

(c) *Assume there is a solution of the form*

$$u = \sum_{j=1}^{\infty} B_j(t) \sin(j\pi x/2)$$

and write down the PDE in terms of this expression for u *and the expansion of part (a). Rearrange what you get so that it takes the form*

$$\sum_{j=1}^{\infty} L_j B_j \ \sin(j\pi x/2) = 0. \tag{1}$$

Make a similar vanishing series corresponding to the initial condition.

(d) *Solve a sequence of initial value problems to obtain the functions* $B_1, B_2, B_3, \ldots$ *(and solve the problem).*

(e) *Plot your solution with mathematical software and discuss the temperature evolution on the intervals* $[0,1]$ *and* $[1,2]$ *separately. Be sure to comment on the values* $u(1/2,t)$, $u(1,t)$ *and* $u(3/2,t)$.

Problem 2 *(eigenfunction expansion, Haberman 3.4.11) Consider*

$$v(x,t) = \left(1 - e^{-\pi^2 t/4}\right)\sin(\pi x/2)$$

and

$$w(x,t) = -e^{-\pi^2 t}\sin(\pi x)$$

on $(0,2)\times(0,\infty)$.

(a) *Find the initial/boundary value problem (for a forced heat equation) satisfied by* v.

(b) *Find the initial/boundary value problem satisfied by* w.

(c) *Find the initial/boundary value problem satisfied by* $v + w$.

(d) *Find a point* $x \in (0,2)$ *at which the temperature* $v(x,t)+w(x,t)$ *first decreases and then increases. Find a point* $x \in (0,2)$ *at which the temperature* $v(x,t)+w(x,t)$ *only increases. Can you increase the forcing to ensure the temperature at all points only increases? Hint(s): This means to consider* $av(x,t) + w(x,t)$ *for* $a > 1$. *If you're stuck you might want to look at part (e) first and then come back to this part.*

(e) *Animate* $v + w$. *Explain how* $v + w$ *relates to your discussion in part (e) of Problem 1.*

Problem 3 *(wave equation; Haberman 4.4.1) Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} = \kappa u_{xx} & \text{on } (0,1)\times(0,\infty) \\ u(0,t) = 0, & t > 0 \\ u_x(1,t) = 0, & t > 0 \\ u(x,0) = \sin(\pi x), & 0 < x < 1 \\ u_t(x,0) = 0, & 0 < x < 1. \end{cases}$$

for the wave equation.

(a) *Let* $w(x,t) = u(x,t) + x$ *and interpret the boundary conditions on* w *with respect to horizontal displacement of an elastic one-dimensional continuum. Do these conditions make sense?*

(b) *Find* $w(1,t)$.

Problem 4 *(damping, Haberman 4.4.3-5) Analyze the initial/boundary value problem*

$$\begin{cases} \rho u_{tt} = \epsilon u_{xx} - \beta u_t & (x,t) \in (0,L) \times (0,\infty) \\ u(x,0) = u_0(x), & x \in (0,L) \\ u_t(x,0) = v_0(x), & x \in (0,L) \\ u(0,t) = 0 = u(L,t), & t > 0 \end{cases}$$

where ρ, ϵ, and β are positive constants, and u_0 and v_0 are given functions. Here are some suggestions for your analysis:

(a) *Solve the problem in general using separation of variables and superposition.*

(b) *Solve the problem in general using eigenfunction expansion.*

Note: In parts (a) and (b) there should be multiple qualitative cases (underdamped, critically damped, and overdamped) depending on the magnitude of the damping coefficient β.

(c) *Choose some specific values of the constants (including L) and initial position and velocity to see some simple separated variable solutions illustrating each qualitative case. Animations of the standard (Haberman) "string" model could be good.*

(d) *For at least one choice of "more interesting" initial conditions that require a superposition write down and illustrate the solution.*

Problem 5 *(sagging equilibrium, Haberman 4.2.1) Consider a deformation $w_* : [0,L] \to [0,L]$ of a one-dimensional elastic continuum with constant equilibrium density $\rho > 0$ and constant elasticity ϵ with $w_*(0) = 0$, $w_*(L) = L$, and $w_*' > 0$. Let $y : [0,L] \to [-L,0]$ by $y(x) = -w_*(x)$ give a vertical representation of the deformation. Assume w_* is an equilibrium for the forced wave equation*

$$\rho w_{tt} = \epsilon w_{xx} + \rho g$$

where $g > 0$ is a gravitational constant.

(a) *Find w_* and determine conditions under which w_* is admissible. Hint: Non-admissibility may arise is $w_*(x) \notin [0,L]$ for some $x \in (0,L)$ or if $w_*'(x) < 0$ for some x. You may wish to consider the relation of these two conditions and the borderline condition in which $w_*'(x) = 0$ for some x.*

(b) *Use mathematical software to illustrate the hanging (and sagging) configuration given by y (for some specific values of the constants).*

(c) *Let $u_* : [0, L] \to \mathbb{R}$ by $u_*(x) = -y(x) - x$. Find the boundary value problem satisfied by u_* and plot the graph of u_* (for some specific values of the constants).*

I received a request to make my assignments shorter, starting with Assignment 6 (this assignment). I will make this assignment shorter in the following sense:

You have my official permission to consider the five problems above to be the entirety of Assignment 6 = Exam 2. I think the problems below are very interesting, and you can learn many potentially useful things if you do them. I will, however, make an effort to exclude the things you might learn from being required for future assignments in this course. I don't make any guarantees concerning the success of that effort.

Problem 6 *(Hamilton's action principle for the motion of a point mass) Show that motions $\mathbf{x} : [0, T] \to \mathbb{R}^n$ satisfying Newton's second law*

$$M\ddot{\mathbf{x}} = \mathbf{f}$$

are stationary points for the action functional $\mathcal{A} : X \to \mathbb{R}$ by

$$\mathcal{A}[\mathbf{x}] = \int_0^T \left[\Phi(\mathbf{x}, t) - \frac{1}{2} M \, |\mathbf{v}|^2 \right] dt$$

where X is the admissible class

$$X = \{\mathbf{x} \in C^2([0, T] \to \mathbb{R}^n) : \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{p}\}.$$

Hint: I'm leaving it to you to figure out the relation between the potential Φ and the force $\mathbf{f}$.

Problem 7 *(equilibrium under tension) Derive a model for the elastic deformation/motion with respect to time for a function* $w : [0, L_0] \times [0, T) \to [0, L_0]$ *in*

$$\mathcal{W} = \{w \in C^2([0, L_0] \times [0, T)) : w(0, t) = 0, w(L_0, t) = L_0, w_x(0, t) > 0 \text{ for } t \geq 0\}$$

under the following assumptions: The evolving one-dimensional continuum is modeled on an **initial equilibrium interval** $[0, L]$ *with* $L < L_0$ *using an initial extension* $w_0 : [0, L] \to [0, L_0]$ *by* $w_0(x) = L_0 x/L$ *and initial tension given by*

$$F = -\epsilon(w_0' - 1).$$

You may assume constant density ρ *and elasticity* ϵ*. You may use any of the three approaches presented in my notes on the wave equation (or some other approach if you like), that is, Newton's second law according to continuum assumption A, the momentum force relation of continuum assumption B, or Hamilton's principle of stationary action.*

Problem 8 *(slinky/modeling) Note that the equilibrium of Problem 5 above requires that compression from the equilibrium* ($w' < 0$) *is possible, and this is not possible for a slinky. Using the result of Problem 7 above, model the equilibrium position of an elongated slinky suspended vertically and sagging due to constant downward gravitational acceleration* g *within an interval* $[0, L_0]$ *of length* $L < L_0$*. Hint: There should be three distinct cases depending on whether or not* L_0 *exceeds the length of the slinky with a free hanging end.*

Problem 9 *(center of mass) Consider the modeling of the motion of a one-dimensional elastic continuum by a function* $w : [0, L] \times [0, T) \to \mathbb{R}$ *in*

$$\mathcal{W} = \{w \in C^2([0, L] \times [0, T)) : w_x > 0\}$$

where the elasticity $\epsilon = \epsilon(x)$ *is spatially dependent and in the presence of a potential field* $\Phi : \mathbb{R} \to \mathbb{R}$ *according to which the potential energy associated with the field* Φ *for a given configuration* w *is given by*

$$E_\Phi = E_\Phi(t) = \int_0^L \Phi(w)\, dx.$$

(a) *Using Hamilton's principle of stationary action (see my notes on the wave equation), derive a forced wave equation for the evolution of* w*. Hint: Your answer should be (something like)*

$$\rho w_{tt} = [\epsilon(w_x - 1)]_x + f$$

where $f(w, x, t) = -\Phi_z(w, x, t)$.

(b) *Let $[a,b] \subset (0,L)$ be an equilibrium subinterval and let*

$$p_{\text{cm}} = \frac{\displaystyle\int_a^b \rho w(x,t)\,dx}{\displaystyle\int_a^b \rho\,dx}$$

be the center of mass of the deformed interval $[w(a,t), w(b,t)]$. Show that

$$M\ddot{p}_{\text{cm}} = \left(\int_a^b \rho\,dx\right)\frac{d^2}{dt^2}p_{\text{cm}}$$

is the sum of the forces at the endpoints of $[w(a,t), w(b,t)]$. Hint(s): Differentiate under the integral sign and then use the PDE. The forces you should get are of two kinds: tension forces from the deformation and field forces from the external forcing.

Problem 10 *(Conservation of energy; Haberman 4.4.9-13) Consider the potential energy*

$$E(t) = \frac{\epsilon}{2}\int_0^L (w_x - 1)^2\,dx,$$

the kinetic energy

$$K(t) = \frac{1}{2}\int_0^L \rho w_t^2\,dx,$$

and the **total energy** *$\mathcal{E}(t) = E(t) + K(t)$ associated with a one-dimensional elastic motion $w : [0,L] \times [0,T) \to [0,L]$ satisfying*

$$\begin{cases} \rho w_{tt} = \epsilon w_{xx}, & \text{on } (0,L)\times(0,T) \\ w(0,t) = 0 = w(L,t), & t > 0. \end{cases} \tag{2}$$

(a) *Compute the derivative $\dot{\mathcal{E}}(t)$ of the energy with respect to time to obtain the general formula*

$$\dot{\mathcal{E}}(t) = [w_x(L,t) - 1]w_t(L,t) - [w_x(0,t) - 1]w_t(0,t).$$

(b) *Conclude that the total energy is conserved for solutions of (2).*

(c) *What other (natural) boundary conditions result in conservation of energy?*