Assignment 6 = Exam 2: Eigenfunction expansion and the wave equation Due Tuesday November 23, 2021

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December 11, 2021

Problem 1 *(eigenfunction expansion, Haberman 3.4.11) Consider the initial/boundary value problem for the forced 1-D heat equation*

$$\begin{cases} u_t = u_{xx} + 1 & \text{on } (0,2) \times (0,\infty) \\ u(0,y) = 0 = u(2,y), & y > 0 \\ u(x,0) = g(x), & 0 < x < 2 \end{cases}$$

where

$$g(x) = \begin{cases} -1, & 0 < x < 1 \\ 1, & 1 < x < 2. \end{cases}$$

(a) *Expand the constant forcing function*

$$1 = \sum_{j=1}^{\infty} c_j \sin(j\pi x/2)$$

in a Fourier sine series.

(b) *Expand the initial temperature g in a Fourier sine series*

$$g(x) = \sum_{j=1}^{\infty} g_j \sin(j\pi x/2).$$

(c) *Assume there is a solution of the form*

$$u = \sum_{j=1}^{\infty} B_j(t) \sin(j\pi x/2) \tag{1}$$

and write down the PDE in terms of this expression for u *and the expansion of part (a). Rearrange what you get so that it takes the form*

$$\sum_{j=1}^{\infty} L_j B_j \ \sin(j\pi x/2) = 0. \tag{2}$$

Make a similar vanishing series corresponding to the initial condition.

(d) *Solve a sequence of initial value problems to obtain the functions* $B_1, B_2, B_3, \ldots$ *(and solve the problem).*

(e) *Plot your solution with mathematical software and discuss the temperature evolution on the intervals* $[0, 1]$ *and* $[1, 2]$ *separately. Be sure to comment on the values* $u(1/2, t)$, $u(1, t)$ *and* $u(3/2, t)$.

Solution:

(a) I want to expand the constant function $f(x) = 1$ on the interval $(0, 2)$ using the eigenfunction basis $\{\sin(j\pi x/2)\}$. The coefficients c_j should satisfy

$$\frac{2}{j\pi}(1 - \cos(j\pi)) = \int_0^2 \sin(j\pi x/2)\, dx = c_j \int_0^2 \sin^2(j\pi x/2)\, dx = c_j.$$

That is,

$$c_j = \begin{cases} 4/(j\pi), & j \text{ odd} \\ 0, & j \text{ even.} \end{cases}$$

(b) Similarly,

$$\begin{aligned} g_j &= -\int_0^1 \sin(j\pi x/2)\, dx + \int_1^2 \sin(j\pi x/2)\, dx \\ &= \frac{2}{j\pi}[\cos(j\pi/2) - 1 + \cos(j\pi/2) - \cos(j\pi)] \\ &= \frac{2}{j\pi}[2\cos(j\pi/2) - 1 - (-1)^j]. \end{aligned}$$

The value of $\cos(j\pi/2)$ takes sequential values with a period of four starting with $j = 1$ given by $0, -1, 0, 1, 0, -1, 0, 1, 0, \ldots$. That is,

$$[2\cos(j\pi/2) - 1 - (-1)^j] = \begin{cases} 0, & j = 2k+1 \text{ odd} \\ -4, & j = 2(2k+1) \text{ even} \\ 0, & j = 4k \text{ even.} \end{cases}$$

so

$$g_j = \begin{cases} 0, & j = 2k+1 \text{ odd or } j = 4k \text{ even} \\ -\dfrac{4}{j\pi} = -\dfrac{4}{(2k+1)\pi}, & j = 2(2k+1) \text{ even.} \end{cases}$$

Note that in the cases $j = 2k+1$ and $j = 2(2k+1)$ the index $k = 0, 1, 2, 3, \ldots$, but in the case $j = 4k$ we use $k = 1, 2, 3, \ldots$.

(c) The PDE for the given series looks like

$$\sum_{j=1}^{\infty} B_j' \sin(j\pi x/2) = -\sum_{j=1}^{\infty} B_j \left(\frac{j\pi}{2}\right)^2 \sin(j\pi x/2) + \sum_{j=1}^{\infty} c_j \sin(j\pi x/2)$$

or

$$\sum_{j=1}^{\infty} \left[B_j' + \left(\frac{j\pi}{2}\right)^2 B_j - c_j\right] \sin(j\pi x/2) = 0.$$

The initial condition $u(x, 0) = g$ is

$$\sum_{j=1}^{\infty} [B_j(0) - g_j] \sin(j\pi x/2) = 0.$$

(d) The coefficients in the series for zero above must vanish. Therefore, we need to consider the initial value problems

$$\begin{cases} B_j' + \left(\frac{j\pi}{2}\right)^2 B_j - c_j = 0, & t > 0 \\ B_j(0) = g_j \end{cases}$$

for the ODE $B_j' + (j\pi)^2 B_j/4 - c_j = 0$ and $j = 1, 2, 3, \ldots$. This is a first order linear ODE, so we can use the integrating factor

$$h_j = e^{j^2\pi^2 t/4}$$

and write

$$[hB_j]' = c_j h.$$

Integrating from $t = 0$ to t we get

$$hB_j = B_j(0) + \frac{4c_j}{j^2\pi^2}\left(e^{j^2\pi^2 t/4} - 1\right) = g_j + \frac{4c_j}{j^2\pi^2}\left(e^{j^2\pi^2 t/4} - 1\right)$$

or

$$B_j(t) = \left(g_j - \frac{4c_j}{j^2\pi^2}\right) e^{-j^2\pi^2 t/4} + \frac{4c_j}{j^2\pi^2}. \tag{3}$$

When $j = 2k+1$ is odd, $g_j = 0$ and we have

$$B_j(t) = B_{2k+1}(t) = \frac{16}{(2k+1)^3\pi^3}\left(1 - e^{-(2k+1)^2\pi^2 t/4}\right).$$

In the cases where j is even we have $c_j = 0$, so (3) simplifies to

$$B_j(t) = g_j \; e^{-j^2\pi^2 t/4}.$$

When $j = 2(2k+1)$ is even, we have

$$B_j(t) = B_{2(2k+1)}(t) = -\frac{4}{(2k+1)\pi} \; e^{-(2k+1)^2\pi^2 t}.$$

And when $j = 4k$ is even, both g_j and c_j are zero so $B_j(t) = B_{4k}(t) = 0$.

We can break up the sum in the eigenfunction expansion (1) accordingly to get the solution as a series:

$$
\begin{aligned}
u &= \sum_{j=1}^{\infty} B_j(t)\sin\left(\frac{j\pi x}{2}\right) \\
&= \sum_{k=0}^{\infty} B_{2k+1}(t)\sin\left(\frac{(2k+1)\pi x}{2}\right) \\
&\qquad\qquad + \sum_{k=0}^{\infty} B_{2(2k+1)}(t)\sin((2k+1)\pi x) \\
&\qquad\qquad\qquad\qquad + \sum_{k=1}^{\infty} B_{4k}(t)\sin(2k\pi x) \\
&= \sum_{k=0}^{\infty} \frac{16}{(2k+1)^3\pi^3}\left(1-e^{-(2k+1)^2\pi^2 t/4}\right)\sin\left(\frac{(2k+1)\pi x}{2}\right) \\
&\qquad\qquad - \sum_{k=0}^{\infty} \frac{4}{(2k+1)\pi}\, e^{-(2k+1)^2\pi^2 t}\sin((2k+1)\pi x).
\end{aligned}
$$

(e) Notice first of all that at the end ($t \nearrow \infty$) we expect an equilibrium solution $u_*'' = -1$ with homogeneous Dirichlet boundary conditions $u_*(0) = 0 = u_*(2)$. This means a quadratic temperature distrubution

$$u_*(x) = \frac{1}{2}x(2-x)$$

corresponding to the forcing $f = 1$ with a constant input of thermal energy leading to positive temperatures along the rod with max temp of $u = 1/2$ in the middle.

The initial conditions are, of course, not consistent with the boundary conditions, so the values $u(0,t)$ and $u(2,t)$ are also, in principle, interesting. For the series, these values are simply zero, but at time $t = 0$ the limits of $u(x,t)$ as x tends to $x = 0$ from the right and as x tends to $x = 2$ from the left or minus and plus one respectively. These singularities resolve instantaneously for $t > 0$ due to the infinite propogation speed of the heat operator/equation, but in some sense we should expect a "faster" resolution at the left endpoint, i.e., more rapid increase of nearby values toward zero and a "slower" resolution at the right endpoint with nearby values decreasing more slowly due to the upward forcing. We'll see if we can capture this.

Let's see what we get. I found it essentially impossible to see the effect of the forcing in the resolving of the singularities, though there must be some effect there. I made a series of plots, and it is pretty clear I have the correct solution. In the figures below time increases from left to right. Animation is also very nice for illustrating this solution. One complication is that the number of terms must be restricted as the time increases because the exponentials get very large powers as t grows. Of course, when this happens, one is losing very little by leaving off higher order Fourier terms becuase those terms are very small, so with the proper adjustments the plots are all pretty good.

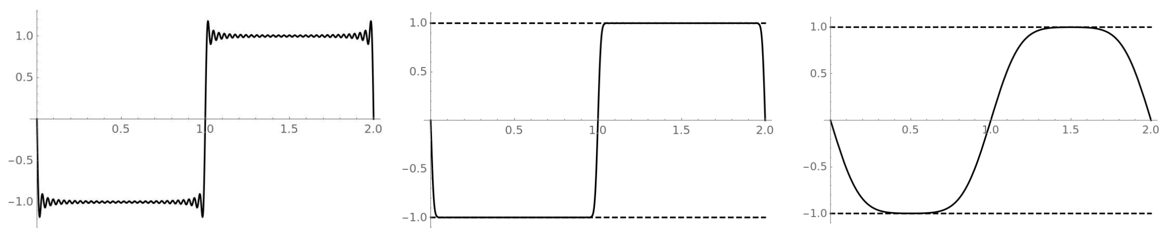

Figure 1: Plots with 62 nonzero terms ($k = 0, 1, 2, \ldots, 30$) of the series solution. Initial data $t = 0$ (left). Then $t = 0.0001$; one can see the essentially instantaneous resolution of the singularities/incompatibility of the initial data here. Finally, $t = 0.01$ (right). No indication of the (upward) forcing is apparent at these early times, at least from casual examination of these figures—and even somewhat more careful examination. This is a consequence of the fact that the forcing tends to have an effect on a larger time scale.

To summarize, on the spatial interval $[0, 1]$ the temperature starts at constant value $u \equiv -1$ and rises to the positive equilibrium solution $u_*(x) = t(1 - x/2)$. The fastest increase is at the endpoints where the boundary condition $u(0, t) = 0$ and the jump in initial value $u(1^+, 0) = 1$ play a role. The slowest rise is toward the middle of the interval with the temperature $u(1/2, t)$ increasing very slowly (essentially imperceptibly) as a function of t but eventually picking up speed and always increasing to the equilibrium temperature $u_*(1/2) = 3/8$. The temperature value at the endpoint $x = 1$ of the interval $[0, 1]$ is not well-defined for $t = 0$, but essentially instantaneously takes the value $u(1, t) \approx 0$ for $0 < t << 1$. Then $u(1, t)$ increases very slowly at first and eventually increases to the equilibrium temperature $u_*(1) = 1/2$.

The behavior of $u(x, t)$ for $x \in (1, 2]$ is quite different. The temperatures at

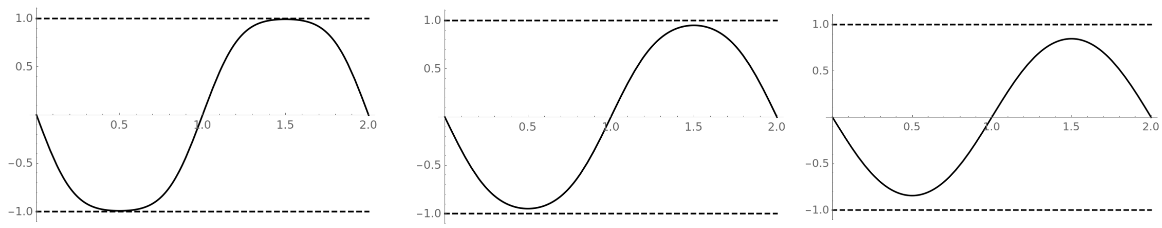

Figure 2: Plots with 22 nonzero terms ($k = 0, 1, 2, \ldots, 10$) of the series solution. Time snapshots at $t = 0.015$, $t = 0.025$, and $t = 0.04$. Still there is no very evident asymmetry due to the forcing.

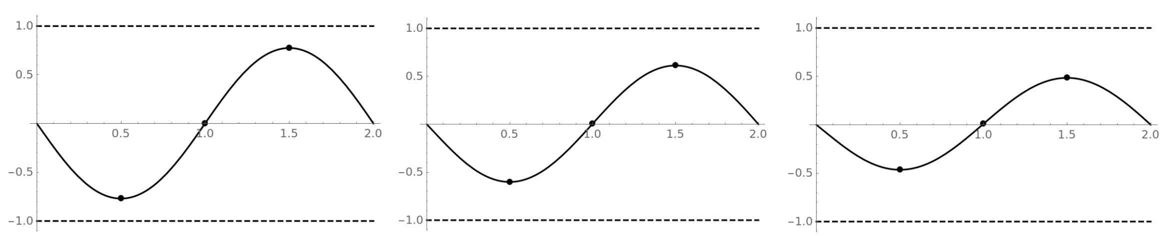

Figure 3: More plots with 22 nonzero terms ($k = 0, 1, 2, \ldots, 10$) of the series solution. Time snapshots at $t = 0.05$, $t = 0.075$, and $t = 0.1$. In these plots it is starting to become apparent that the middle point at $x = 1$ is starting to rise (due to the forcing). Also, in the last plot, it looks like the value $u(1/2, 0.1)$ satisfies $0 < -u(1/2, 0.1) < u(3/2, 0.1)$. We return to this time with different horizontal reference lines in the next snapshot.

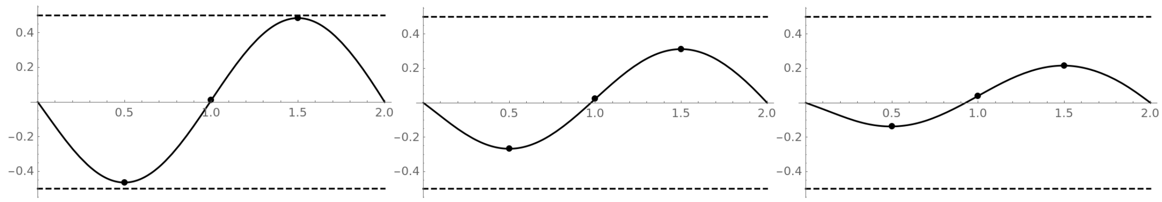

Figure 4: Plots with 22 nonzero terms ($k = 0, 1, 2, \ldots, 10$) of the series solution. Time snapshots at $t = 0.1$, $t = 0.15$. The reference lines ($u = \pm 0.5$) for the first plot at $t = -0.1$ make it clear that the forcing is causing asymmetry, and the midpoint clearly has a positive temperature in the middle snapshot (and is still rising while the temperature at $x = 3/2$ is still dropping. The snapshot on the right includes 12 terms ($k = 0, 1, 2, \ldots, 5$) of the series at time $t = 0.2$.

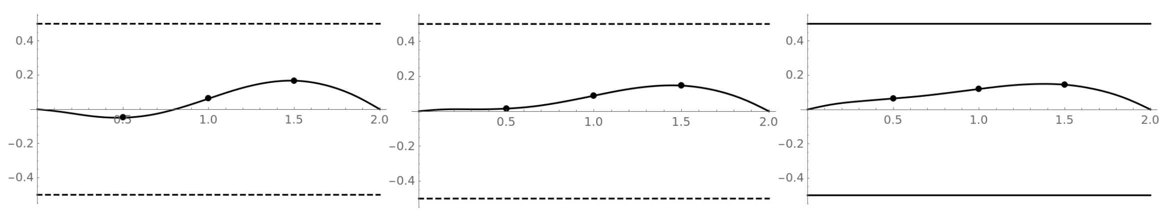

Figure 5: In this series, each with 12 nonzero terms ($k = 0, 1, 2, \ldots, 5$), the asymmetry becomes very evident. We have here $t = 0.25$, $t = 0.3$ and $t = 0.35$. The midpoint is still getting hotter and has increasing temperature throughout the evolution as does the temperature at $t = 1/2$. The latter is no surprise, but it may be noted that $u(1/2, 0.25) < 0 < u(1/2, 0.3)$. The temperature at $x = 3/2$ is probably still dropping in this series, but in the overall evolution it will eventually change direction and start to increase (due to the forcing).

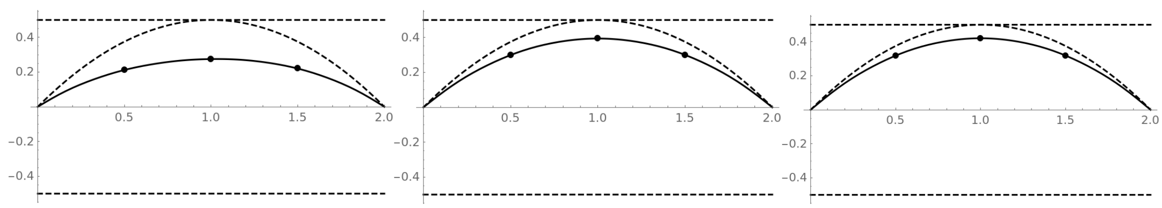

Figure 6: Plots with 10 nonzero terms ($k = 0, 1, 2, \ldots, 4$). These plots for $t = 0.6$, $t = 0.8$ and $t = 0.87$ represent convergence to the equilibrium solution with all temperatures increasing across the interval $[0, 2]$.

all these points start at $u(x, t) = 1$ and decrease. The decrease is rapid near the endpoints $x = 1$ and $x = 2$ and initially very slow in the middle of the interval. Eventually, all the points have noticeably decreasing temperatures, though all temperatures along this interval stay positive. The temperature at each point $x \in (1, 2)$ furthermore reaches a positive minimum in time—at which point in time the temperature begins to increase and the value subsequently increases to the equilibrium temperature $u_*(x) = x(1 - x/2)$. In particular, $u(3/2, t)$ decreases to a positive minimum and then increases to the equilibrium temperature $u_*(3/2) = 3/8$.

Getting a handle on the particular time $t_{\min}$ at which the minimum temperature occurs at $x = 3/2$ (or any other particular point $x \in (1, 2)$)—or the value of that minimum temperature—looks rather difficult. One would have to (or at least could) differentiate the series with respect to t to get $u_t(x, t_{\min})$ and set what you get equal to zero. This gives an equation for $t_{\min}$. Then one could attempt to evaluate the series solution at that particular time and location. Presumably, one could obtain a good numerical estimate for both the time $t_{\min}$ and the minimum temperature. According to the PDE the point $t_{\min} = t_{\min}(x)$ is also the time at which the condition $u_{xx}(x, t_{\min}) = -1$ holds in the spatial temperature profile at position x at time $t_{\min}$. (The PDE says $u_t = u_{xx} + 1$.) I don't see that this gives any particular advantage in terms of computation, but it does have a somewhat intersting consequence for the limiting temperature profile as $t \searrow 0$, namely, we expect that the limit

$$\lim_{x \searrow 1} t_{\min}(x) = 0,$$

and this means the value of the second homogeneous spatial derivative $u_{xx}(1, t)$

should limit to -1. Thus, there is a strong spatial asymmetry in the temperature profile at the level of the second (spatial) derivative all the way down to $t = 0$. This strong asymmetry is rather difficult to see in the plots. This is not altogether surprising because the spatial gradient $u_x(1,t)$ is blowing up to $+\infty$ as $t \searrow 0$.

Problem 2 *(eigenfunction expansion, Haberman 3.4.11) Consider*

$$v(x,t) = \left(1 - e^{-\pi^2 t/4}\right)\sin(\pi x/2)$$

and

$$w(x,t) = -e^{-\pi^2 t}\sin(\pi x)$$

on $(0,2) \times (0,\infty)$.

(a) *Find the initial/boundary value problem (for a forced heat equation) satisfied by* v.

(b) *Find the initial/boundary value problem satisfied by* w.

(c) *Find the initial/boundary value problem satisfied by* $v + w$.

(d) *Find a point* $x \in (0,2)$ *at which the temperature* $v(x,t)+w(x,t)$ *first decreases and then increases. Find a point* $x \in (0,2)$ *at which the temperature* $v(x,t)+w(x,t)$ *only increases. Can you increase the forcing to ensure the temperature at all points only increases? Hint(s): This means to consider* $av(x,t) + w(x,t)$ *for* $a > 1$. *If you're stuck you might want to look at part (e) first and then come back to this part.*

(e) *Animate* $v + w$. *Explain how* $v + w$ *relates to your discussion in part (e) of Problem 1.*

Solution:

(a) Differentiating

$$v(x,t) = \left(1 - e^{-\pi^2 t/4}\right)\sin(\pi x/2)$$

we find

$$v_t = \frac{\pi^2}{4}\, e^{-\pi^2 t/4}\sin(\pi x/2) \qquad \text{and} \qquad v_{xx} = -\frac{\pi^2}{4}\left(1 - e^{-\pi^2 t/4}\right)\sin(\pi x/2).$$

Therefore,

$$v_t = v_{xx} + \frac{\pi^2}{4}\ \sin(\pi x/2).$$

This forcing is rather similar to the forcing in Problem 1 in the sense that it is positive across (at least the interior) of the spatial interval. The boundary values are the same:

$$v(0,t) = 0 = v(2,t).$$

As for the initial value

$$v(x,0) \equiv 0.$$

So this is rather different, and we expect the temperature v to rise across the interval to the positive equilibrium temperature $v_* = v_*(x)$ determined by

$$v_*'' = -\frac{\pi^2}{4} \sin(\pi x/2).$$

That is,

$$v_*(x) = \sin(\pi x/2).$$

In summary, v satisfies

$$\begin{cases} v_t = v_{xx} + \frac{\pi^2}{4} \sin(\pi x/2) & \text{on } [0,2] \times [0,\infty) \\ v(0,t) = 0 = v(2,t), & t \geq 0 \\ v(x,0) \equiv 0, & 0 \leq x \leq 2. \end{cases}$$

(b) The same kind of computation applied to

$$w(x,t) = -e^{-\pi^2 t} \sin(\pi x)$$

gives

$$\begin{cases} w_t = w_{xx} & \text{on } [0,2] \times [0,\infty) \\ w(0,t) = 0 = w(2,t), & t \geq 0 \\ w(x,0) = -\sin(\pi x), & 0 \leq x \leq 2. \end{cases}$$

This one has no forcing (homogeneous PDE) but the initial value is somewhat similar to the initial value in Problem 1, in the sense that the temperature starts negative on the spatial interval $(0,1)$ and positive on the spatial interval $(1,2)$. In this case, we expect monotone decay (increasing with time for $x \in (0,1)$ and decreasing in time for $x \in (1,2)$) to the equilibrium solution $w_*(x) \equiv 0$.

(c) Let $u = v + w$. Then (just adding things up gives)

$$\begin{cases} u_t = u_{xx} + \frac{\pi^2}{4} \sin(\pi x/2) & \text{on } [0,2] \times [0,\infty) \\ u(0,t) = 0 = u(2,t), & t \geq 0 \\ u(x,0) = -\sin(\pi x), & 0 \leq x \leq 2. \end{cases}$$

So this initial/boundary problem is rather like a continuous/smooth version of the one in Problem 1.

(d) First decreasing and then increasing? I expect this on the second half of the interval. How about $x = 3/2$? We have

$$\begin{aligned} v(3/2,t) + w(3/2,t) &= \left(1 - e^{-\pi^2 t/4}\right) \frac{\sqrt{2}}{2} + e^{-\pi^2 t} \\ &= \frac{\sqrt{2}}{2} + e^{-\pi^2 t} - \frac{\sqrt{2}}{2}\, e^{-\pi^2 t/4}. \end{aligned}$$

Thus,

$$\frac{d}{dt}[v(3/2,t) + w(3/2,t)] = -\pi^2\, e^{-\pi^2 t} + \frac{\pi^2\sqrt{2}}{8}\, e^{-\pi^2 t/4}.$$

Setting this quantity equal to zero we conclude there is a uniqe zero $t = t_{\min}$ of the derivative:

$$8 = \sqrt{2}\, e^{\pi^2(1-1/4)t} \qquad \text{or} \qquad t_{\min} = \frac{4}{3\pi^2} \ln(4\sqrt{2}).$$

We can see that this is a minimum in a couple different ways. First of all, $v(3/2,0) + w(3/2,0) = 1$ and

$$\begin{aligned} v(3/2,t_{\min}) + w(3/2,t_{\min}) &= \frac{\sqrt{2}}{2} + e^{-\pi^2 t_{\min}} - \frac{\sqrt{2}}{2}\, 4\sqrt{2}\, e^{-\pi^2 t_{\min}} \\ &= \frac{\sqrt{2}}{2} - 3e^{-\pi^2 t_{\min}} \\ &= \frac{\sqrt{2}}{2} - 3e^{-(4/3)\ln(4\sqrt{2})} \\ &= \frac{\sqrt{2}}{2} - \frac{3}{(4\sqrt{2})^{4/3}} \\ &\doteq 0.409469 \\ &< 1. \end{aligned}$$

And it's clear that

$$\lim_{t \nearrow +\infty} [v(3/2,t) + w(3/2,t)] = \frac{\sqrt{2}}{2} \doteq 0.707107 > \frac{\sqrt{2}}{2} - 3e^{-\pi^2 t_{\min}}.$$

With only one unique critical point, it is clear from these observations that the value at the critical point is a minimum. In fact, only the last limiting observation is enough.

A second way to see that a minimum occurs at $t_{\min}$ is, for example, to compute the second derivative

$$\frac{d^2}{dt^2}[v(3/2,t)+w(3/2,t)] = \pi^4\ e^{-\pi^2 t} - \frac{\pi^4\sqrt{2}}{32}\ e^{-\pi^2 t/4}.$$

and evaluate at $t = t_{\min}$. I'll leave it to you to convince yourself that you get a positive value. Mathematica says the value you should get is

$$\frac{d^2}{dt^2}[v(3/2,t)+w(3/2,t)]\Big|_{t=t_{\min}} = \frac{3\pi^4}{32\sqrt[3]{2}}.$$

If you need a third way to see this, you could just plot $v(3/2,t)+w(3/2,t)$ as a function of t using mathematical software.

For a point where the temperature only increases, we could try $x = 1/2$ or $x = 1$. For $x = 1/2$, we get

$$v(1/2,t)+w(1/2,t) = \left(1-e^{-\pi^2 t/4}\right)\frac{\sqrt{2}}{2} - e^{-\pi^2 t}.$$

Here

$$\frac{d}{dt}[v(1/2,t)+w(1/2,t)] = \pi^2\ e^{-\pi^2 t} + \frac{\pi^2\sqrt{2}}{8}\ e^{-\pi^2 t/4} > 0.$$

Similarly,

$$v(1,t)+w(1,t) = \left(1-e^{-\pi^2 t/4}\right),$$

and this clearly only increases.

As suggested in the hint, if we look at $u = \alpha v + w$, we get a solution of

$$\begin{cases} u_t = u_{xx} + \frac{\alpha\pi^2}{4}\ \sin(\pi x/2) & \text{on } [0,2]\times[0,\infty) \\ u(0,t) = 0 = u(2,t), & t \geq 0 \\ u(x,0) = -\sin(\pi x), & 0 \leq x \leq 2, \end{cases}$$

and the question is: If α is big enough, then is it true that u_t is always positive? This simply means you'd need to have the (positive) forcing dominate u_{xx}. I do not see why we couldn't do this. Notice that $u_{xx}(x,0)$ for the initial data is given by

$$u_{xx}(x,0) = w_{xx}(x,0) = \pi^2\sin(\pi x).$$

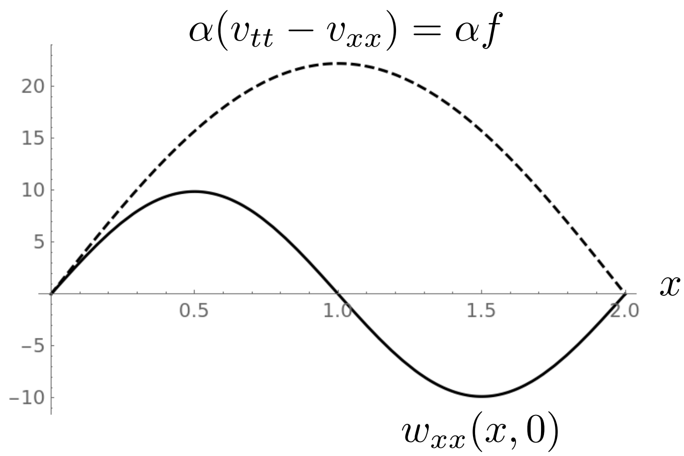

Figure 7: Comparison of the initial value of $u_{xx} = u_{xx}(x, 0)$ with the forcing $\alpha[v_t - v_{xx}] = \alpha f$. In this figure we have taken $\alpha = 9$, but any value greater than $\alpha = 8$ should work. Notice that the forcing in the PDE $u_t = u_{xx} + f$ depends only on x. As a consequence we can determine f from the equation $f(x) = u_t - u_{xx}$ even though u_t and u_{xx} may vary with time. (The difference $u_t - u_{xx}$ is always $f(x)$ independent of time.) Furthermore, since $w_t = w_{xx}$, we have also $f(x) = v_t - v_{xx}$. We are assuming the evolution does not lead to values of $u_{xx} = u_{xx}(x, t)$ which are more negative than those of $u_{xx} = u_{xx}(x, 0)$. This turns out to be the case, as you can see from animation with $\alpha = 9$. You can also figure this out directly, without any reasoning about α, just by doing animations of $\alpha v + w$ for increasing values of α; this was what was intended by the hint concerning the next part.

This has derivative of magnitude π^3 at the endpoints (and no larger anywhere else), so if we pick α so that the derivative of the forcing term, namely

$$\frac{\alpha\pi^3}{8} \cos(\pi x/2)$$

dominates at the endpoints, that is $\alpha > 8$, then we expect u_{xx} for positive time will only get smaller and the forcing will always dominate leading to the conclusion $u_t > 0$. This behavior is indicated in Figure 8, and I have included an animation in the accompanying Mathematica notebook.

(e) I've got an animation in my Mathematica notebook. I can try to use the same approach used in Figure 8 for part (d) (with varying lengths of dashes) to illustrate what happens. See Figure 9.

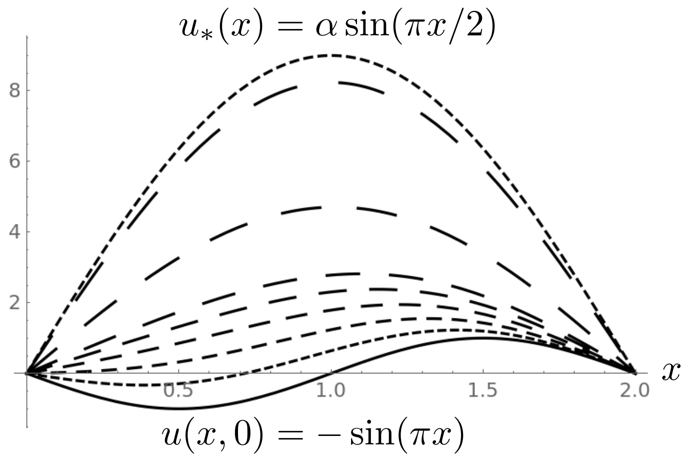

Figure 8: Here we have attempted to represent an animation of the temperature evolution associated with the increased forcing corresponding to $u = \alpha v + w$ with $\alpha = 9$. The initial temperature $u(x, 0) = -\sin(\pi x)$ is plotted with a solid line at the bottom. The passing of time is indicated with dashed plots of successively larger dash separation, and then the limiting equilibrium temperature u_* is indicated at the top with small dashing. As you can see, all temperatures increase during the evolution in agreement with our reasoning for choosing $\alpha > 8$ above.

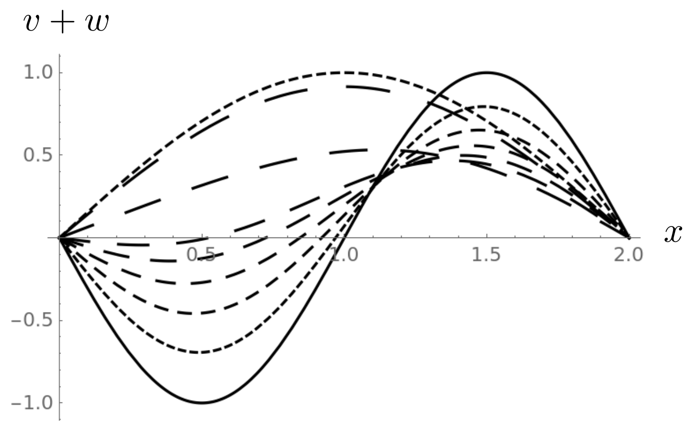

Figure 9: Here we have attempted to represent an animation of the temperature evolution associated with the increased forcing corresponding to $u = v+w$. The initial temperature $u(x, 0) = -\sin(\pi x)$ is plotted with a solid line. In this situation the initial temp is not "at the bottom" but overlaps subsequent temperature profiles. The passing of time is indicated with dashed plots of successively larger dash separation, and then the limiting equilibrium temperature u_* is indicated at the top with small dashing. The initially decreasing temperature on most of the interval $(1, 2)$ is evident in this figure.

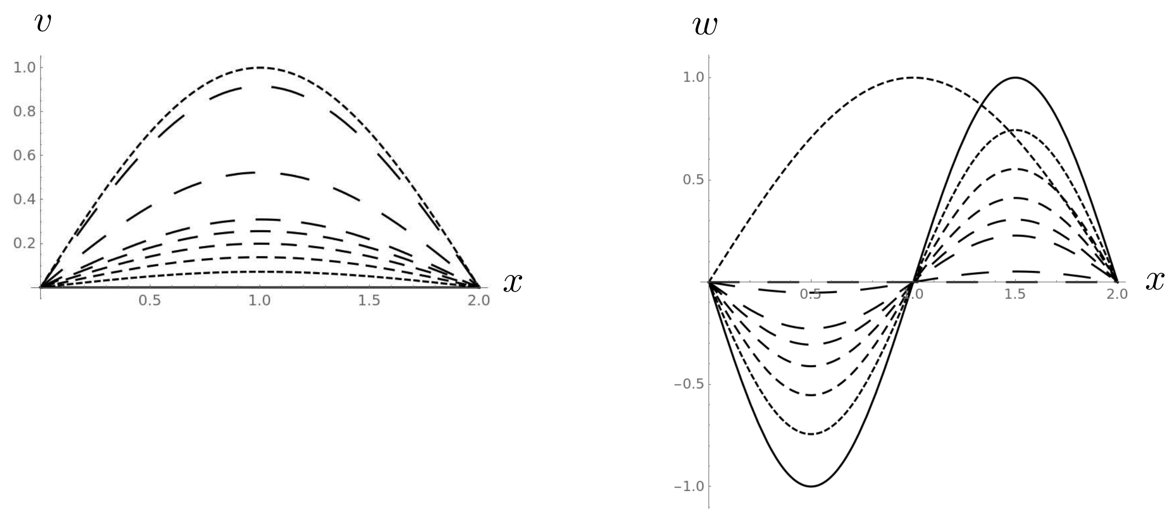

Figure 10: Separate evolutions of v (left) and w (right).

Notice in Figure 9 that while it appears all positions in the spatial interval $(0,1)$ and even $(0,1]$ experience monotone increasing temparature, it is not the case that all positions on the interval $(1,2)$ initially decrease. Nevertheless, all temperatures (at all points) appear to eventually be increasing to the equilibrium. The fact that some points in $(1,2)$, especially near the middle, experience only increasing temperatures is clearly to be expected from our discussion of part (d) above. To be precise, we know $u_{xx}(1,0) = 0$, so the positive forcing will clearly overwhelm the initial diffusion at nearby points and give $u_t = u_{xx} + f > 0$.

As for comparison to part (e) of Problem 1, we can easily view the evolution of $v + w$ as the superposition of v and w. That is to say, we have v initially at temp zero with $v(x,0) = 0$ and positive forcing leading to a monotone increasing temperature profile tending to the equilibrium. See Figure 10 (left). On the other hand (and on the right) we have w which begins with negative temperature on $(0,1)$ and positive temperature on $(1,2)$ and decays via simple diffusion to zero. We note that the temperature scale is the same in both "animations" and the time intervals are also harmonized so that the significance of the time dependence can be discerned. In particular, the finest dashed profile corresponding to $t = 0.03$, shows the temperature $w(3/2, 0.03)$ is much farther from $w(3/2, 0)$ than the temperature $v(3/2, 0.03)$ is from $v(3/2, 0)$. That is, the effect of diffusion at this point (for the superposition) is greater than that of the forcing (for small times). For larger times the diffusion diminishes and the forcing is dominant; compare the last two represented times with the largest dashes:

$$v(3/2, 1) - v(3/2, 0.3) > w(3/2, 0.3) - w(3/2, 1).$$

We saw this same kind of time dependence in Problem 1. For small times the initial diffusion of the square wave was dominant (leading to a decreasing temperature at $x = 3/2$ for example) and for later times the forcing became more significant (and caused a reversal and increasing temperature at $x = 3/2$ for example). One can, of course, "pull apart" the initial/boundary value problem in Problem 1 into a forced heat equation for a function v with constant forcing $f \equiv 1$ and a free diffusion for a function w with initial square wave temperature profile. Then the function u of Problem 1 is the superposition of these two.

Another way to relate Problem 2 and Problem 1 is that Problem 2 is essentially replacing the constant forcing $f \equiv 1$ with its first Fourier mode which is a multiple of $\sin(\pi x/2)$ and replacing the initial value also with its first Fourier mode which is a multiple of $-\sin(\pi x)$.

There are a couple further aspects of this kind of problem one might explore.

1. We have seen that by increasing the positive forcing we can change the overall qualitative behavior of the solution and get, for example, strictly increasing temperatures at all interior points. In view of the comments concerning the time dependence above, this is an illustration of the simple assertion that incresing the positive forcing not only results in the temperature reaching a higher ultimate value but increased positive forcing also results in **faster heating**. Is there a qualitative change resulting from decreasing the positive forcing relative to the influence of the initial condition? That is, what happens if you consider $\alpha v + w$ for $\alpha > 0$ much smaller than 1?
2. Can you make the discussion of first Fourier modes just above precise and, furthermore, break Problem 1 up into two separate problems as suggested above and see the series solutions for those problems in the series solution for Problem 1?

Problem 3 *(wave equation; Haberman 4.4.1) Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} = \kappa u_{xx} & \text{on } (0,1)\times(0,\infty) \\ u(0,t) = 0, & t > 0 \\ u_x(1,t) = 0, & t > 0 \\ u(x,0) = \sin(\pi x), & 0 < x < 1 \\ u_t(x,0) = 0, & 0 < x < 1. \end{cases}$$

for the wave equation.

(a) *Let $w(x,t) = u(x,t) + x$ and interpret the boundary conditions on w with respect to horizontal displacement of an elastic one-dimensional continuum. Do these conditions make sense?*

(b) *Find $w(1,t)$.*

Solution:

(a) We have $w(0,t) = 0$. This means the left endpoint is fixed. That is simple enough. The condition at the right endpoint is

$$w_x(1,t) = 1.$$

This is a natural condition for a **free end**. Recall that the force within the deformed 1-D continuum is given by something like

$$F = \epsilon(w_x - 1)$$

where ϵ is the elasticity. In this problem we apparently have the relation $\kappa = \epsilon/\rho$ where ρ is a constant linear density. The point is that the condition $w_x(1,t)$ means that **no forces are acting at the end** $w(L)$, i.e., this is a free end. I think it makes good sense.

Note that this condition of a free end coincides with the "completely compressed" condition for a slinky, however, this is not quite the same thing. For a slinky, when you have $w_x = 1$ there can be no more "compression." With a standard oscillator you only need the admissibility condition $w_x > 0$, and the condition $w_x = 1$ really means no forces.

(b) This is a bit tricky (or unexpected) because the initial condition is not compatible with the free end condition. Mathematically, this means the initial condition

$u(x, 0) = \sin(\pi x)$, though a very nice function, is not a compatible Fourier mode for the problem, and we have to do a Fourier expansion of it in terms of the natural Fourier basis.

If you do the separation of variables, you find this basis to be

$$\left\{ \sin\left(\frac{1+2j}{2}\,\pi x\right) \right\}_{j=0}^{\infty}.$$

Thus, we write

$$u(x,t) = \sum_{j=0}^{\infty} [a_j \cos(\sqrt{\kappa}\,(1+2j)\pi t/2) + b_j \sin(\sqrt{\kappa}\,(1+2j)\pi t/2)] \sin\left(\frac{1+2j}{2}\,\pi x\right).$$

The zero initial velocity condition $u_t(x, 0) = 0$ tells us $b_j = 0$ for all j. Therefore, the superposition simplifies to

$$u(x,t) = \sum_{j=0}^{\infty} a_j \;\cos(\sqrt{\kappa}\,(1+2j)\pi t/2)\; \sin\left(\frac{1+2j}{2}\,\pi x\right).$$

the initial condition is

$$\sum_{j=0}^{\infty} a_j \;\sin\left(\frac{1+2j}{2}\,\pi x\right) = \sin(\pi x).$$

Assuming the usual orthogonality, we should get something like this:

$$a_j \int_0^1 \sin^2\left(\frac{1+2j}{2}\,\pi x\right)\,dx = \int_0^1 \sin(\pi x)\;\sin\left(\frac{1+2j}{2}\,\pi x\right)\,dx. \tag{4}$$

With any luck the integral on the left should still be $a_j/2$. The integral on the right looks a little unpleasant, but I think we can use a trig identity. Let's see...I know

$$\begin{aligned} \cos(A+B) &= \cos A \cos B - \sin A \sin B \quad \text{and} \\ \cos(A-B) &= \cos A \cos B + \sin A \sin B. \end{aligned}$$

I guess this means

$$2 \sin A \sin B = \cos(A-B) - \cos(A+B)$$

and

$$\sin(\pi x)\ \sin\left(\frac{1+2j}{2}\,\pi x\right) = \frac{1}{2}\left[\cos\left(\frac{2j-1}{2}\,\pi x\right) - \cos\left(\frac{2j+3}{2}\,\pi x\right)\right].$$

These I can integrate:

$$\begin{aligned}
\int_0^1 \cos\left(\frac{2j-1}{2}\,\pi x\right)\,dx &= \frac{2}{(2j-1)\pi}\sin\left(\frac{2j-1}{2}\,\pi x\right)\Big|_{x=0}^{1} \\
&= \frac{2}{(2j-1)\pi}\sin\left(\frac{2j-1}{2}\,\pi\right) \\
&= \frac{2(-1)^{j+1}}{(2j-1)\pi},
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 \cos\left(\frac{2j+3}{2}\,\pi x\right)\,dx &= \frac{2}{(2j+3)\pi}\sin\left(\frac{2j+3}{2}\,\pi x\right)\Big|_{x=0}^{1} \\
&= \frac{2}{(2j+3)\pi}\sin\left(\frac{2j+3}{2}\,\pi\right) \\
&= \frac{2(-1)^{j+1}}{(2j+3)\pi}.
\end{aligned}$$

Returning, more or less, to (4) I have

$$a_j = \frac{2(-1)^{j+1}}{(2j-1)\pi} - \frac{2(-1)^{j+1}}{(2j+3)\pi} = \frac{8(-1)^{j+1}}{(2j-1)(2j+3)\pi}.$$

At this point, I should turn to my Mathematica notebook and see if I've got the coefficients correct. They look correct (though I will admit I had a small error the first time I did the calculation). Lesson to learn: It's always good to check your Fourier coefficients by graphing partial sums with mathematical software.

So I have my solution:

$$u(x,t) = \sum_{j=0}^{\infty} \frac{8(-1)^{j+1}}{(2j-1)(2j+3)\pi}\ \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right)\ \sin\left(\frac{2j+1}{2}\,\pi x\right).$$

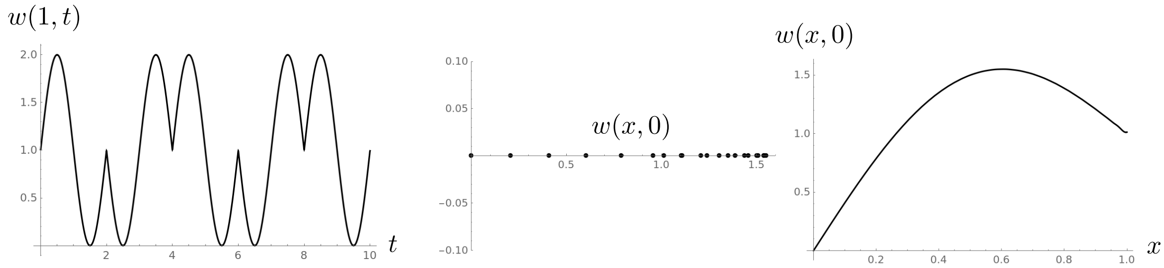

Figure 11: Plot of the endpoint position as a function of time (left). This one is interesting. Plot of the initial deformation (middle). Remembering that the endpoint is at $w = 1$ for the initial deformation, this looks like there is a problem, i.e., violation of the fundamental admissibility condition $w_x > 0$ for deformations. Plot of $w(x, 0)$ (right). This confirms that the value of w is decreasing, so we definitely have a problem.

Now I remember that $w = u + x$ and I'm supposed to find $w(1, t)$. That would be

$$
\begin{aligned}
w(1,t) &= 1 + \sum_{j=0}^{\infty} \frac{8(-1)^{j+1}}{(2j-1)(2j+3)\pi} \ \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right) \ \sin\left(\frac{2j+1}{2}\,\pi\right) \\
&= 1 + \sum_{j=0}^{\infty} \frac{8(-1)^{j+1}}{(2j-1)(2j+3)\pi} \ \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right) \ (-1)^j \\
&= 1 - \sum_{j=0}^{\infty} \frac{8}{(2j-1)(2j+3)\pi} \ \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right).
\end{aligned}
$$

Is this a reasonable value for the oscillation of the free end? Let's check by plotting. See Figure 11. The first thing to note is that the motion of the endpoint is not smooth—there are corners, but it's piecewise smooth, and it's conceivable that this is a reasonable representation of what the free end will actually do. We'd like to see it in an animation to make sure. When we go to do that, we notice there is a problem. The initial deformation is too big. That is to say, the initial deformation $w(x, 0)$ has points with $w(x, 0) > 1$ and yet $w(1, 0) = 1$. This means we must have $w_x < 0$ somewhere which is a violation of the fundamental admissibility condition for physical deformations. In order to fix this, we should make the initial deformation $u_0(x) = \sin(\pi x)$

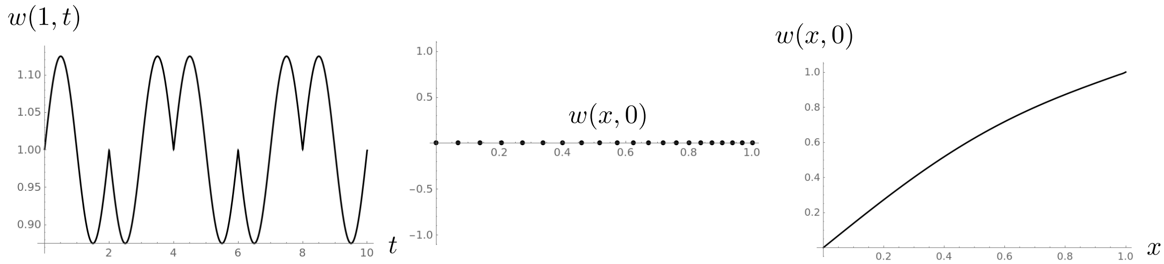

Figure 12: Plot of the endpoint position as a function of time (left). The interesting jerky motion of the endpoint is preserved. Plot of the initial deformation (middle). This looks better, and it looks like $w_x > 0$ at least initially. Plot of $w(x, 0)$ (right). This confirms the initial condition is okay.

smaller. At least that is one way to fix the problem and get an admissible initial deformation. Looking back over our solution above, I'll try replacing the initial/boundary value problem for u with

$$\begin{cases} u_{tt} = \kappa u_{xx} & \text{on } (0,1)\times(0,\infty) \\ u(0,t) = 0, & t > 0 \\ u_x(1,t) = 0, & t > 0 \\ u(x,0) = \sin(\pi x)/8, & 0 < x < 1 \\ u_t(x,0) = 0, & 0 < x < 1 \end{cases}$$

where I've divided the initial value by 8. (Notice this works out nicely with all the 8's in the coefficients of the Fourier expansion.) It also works. Figure 12 shows the relevant plots, and the (super cool) animation is in my Mathematica notebook. For new formulas we have (just removing factors of 8):

$$u(x,t) = \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j-1)(2j+3)\pi} \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right) \sin\left(\frac{2j+1}{2}\pi x\right).$$

$$w(x,t) = x + \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{(2j-1)(2j+3)\pi} \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right) \sin\left(\frac{2j+1}{2}\pi x\right).$$

$$w(1,t) = 1 - \sum_{j=0}^{\infty} \frac{1}{(2j-1)(2j+3)\pi} \cos\left(\frac{2j+1}{2}\sqrt{\kappa}\,\pi t\right).$$

Problem 4 *(damping, Haberman 4.4.3-5) Analyze the initial/boundary value problem*

$$\begin{cases} \rho u_{tt} = \epsilon u_{xx} - \beta u_t & (x,t) \in (0,L) \times (0,\infty) \\ u(x,0) = u_0(x), & x \in (0,L) \\ u_t(x,0) = v_0(x), & x \in (0,L) \\ u(0,t) = 0 = u(L,t), & t > 0 \end{cases}$$

where ρ, ϵ, and β are positive constants, and u_0 and v_0 are given functions. Here are some suggestions for your analysis:

(a) *Solve the problem in general using separation of variables and superposition.*

(b) *Solve the problem in general using eigenfunction expansion.*

Note: In parts (a) and (b) there should be multiple qualitative cases (underdamped, critically damped, and overdamped) depending on the magnitude of the damping coefficient β.

(c) *Choose some specific values of the constants (including L) and initial position and velocity to see some simple separated variable solutions illustrating each qualitative case. Animations of the standard (Haberman) "string" model could be good.*

(d) *For at least one choice of "more interesting" initial conditions that require a superposition write down and illustrate the solution.*

Solution: We let $u = A(x)B(t)$ so that the PDE becomes

$$AB'' = A''B - \beta AB'.$$

Dividing by AB we get a separation

$$\frac{B'' + \beta B'}{B} = \frac{A''}{A} = -\mu^2.$$

In view of the boundary conditions $A(0) = 0 = A(L)$, we get $\mu = j\pi/L$,

$$A_j(x) = \sin\left(\frac{j\pi}{L}x\right)$$

and

$$B_j'' + \beta B_j' + \frac{j^2\pi^2}{L^2}B_j = 0.$$

This is a standard linear oscillator ODE which we, i.e., you, should know everything about. In particular, the positive constant β is the unit mass (or normalized) damping constant and the positive constant $j^2\pi^2/L^2$ represents the restoring force (divided by the mass). The qualitative properties of the solutions are determined by the relative sizes of these constants in the sense that if

$$\beta^2 < 4\frac{j^2\pi^2}{L^2}, \tag{5}$$

then the oscillator modeled by B_j is **underdamped** and

$$B_j(t) = e^{-\beta t/2}\left[a_j \cos(\omega t) + b_j \sin(\omega t)\right] \qquad \text{where} \qquad \omega = \frac{1}{2}\sqrt{4\frac{j^2\pi^2}{L^2} - \beta^2}; \tag{6}$$

if

$$\beta^2 = 4\frac{j^2\pi^2}{L^2}, \qquad \text{i.e.,} \qquad \beta = \frac{2j\pi}{L}, \tag{7}$$

then the oscillator modeled by B_j is **critically damped** and

$$B_j(t) = e^{-\beta t/2}\left[a_j + b_j t\right], \tag{8}$$

and if

$$\beta^2 > 4\frac{j^2\pi^2}{L^2}, \tag{9}$$

then the oscillator modeled by B_j is **overdamped** and

$$B_j(t) = a_j e^{-r_1 t} + b_j e^{-r_2 t} \tag{10}$$

where

$$-r_m = -\frac{\beta}{2} + (-1)^m\sqrt{\frac{\beta^2}{4} - \frac{j^2\pi^2}{L^2}}, \ m = 1, 2. \tag{11}$$

You may remember also that the characteristic property of critically damped and overdamped harmonic oscillators is that there can be **at most one pass through the equilibrium position**, no matter what the initial conditions, while the characteristic property of underdamped oscillators is that all nonzero initial conditions lead to infinitely many passes through the equilibrim position.

Of course we should realize in this case that B_j does not really model a simple 1-D harmonic oscillator but rather a mode of a spatially extended 1-D continuum.

This means things are a little more complicated. But the crucial thing to note is that the eigenvalues (according to Sturm-Liouville theory)

$$\frac{j^2\pi^2}{L^2},\ j = 1, 2, 3, \ldots$$

comprise an **increasing sequence** tending to $+\infty$. The relevant consequence here is that **all high enough modes will be underdamped** and fall into the category of (5-6). With a single continuum damping constant β like we have here, this observation has an interesting physical consequence (or at least suggests it):

> *Lower frequency modes are more likely to be damped and are easier to damp; high frequency modes are less likely to be damped and are successively more difficult to damp.*

Taking these comments into consideration, there are essentially three cases for the series solution

$$u(x,t) = \sum_{j=1}^{\infty} A_j(x)B_j(t) = \sum_{j=1}^{\infty} B_j(t)\sin\left(\frac{j\pi}{L}\,x\right).$$

CASE 1 ($\beta L < 2\pi$) This would be the special case considered by Haberman in his Exercise 4.4.3. Here every mode is underdamped, and it makes sense to say the continuum is also underdamped; we would expect temporal oscillation and infinitely many passes through the equilibrium position (at least in some sense). The form of the solution becomes

$$u(x,t) = e^{-\beta t/2}\sum_{j=1}^{\infty}\left[a_j\cos(\omega t) + b_j\sin(\omega t)\right]\sin\left(\frac{j\pi}{L}\,x\right).$$

Notice the damping rate does not depend on the index j of the mode, but the (temporal) frequency of oscillation does. The temporal frequency ω, however, is also dependent on the damping and will not be the same as the spatial frequency of the oscillation. Of course, the temporal frequency of the oscillation will also depend in general on the wave speed σ which we have taken to be $\sigma = 1$ in this problem. (Note that Haberman uses $\sigma = T/\rho$ in his problem where T is the tension and ρ is the linear density for modeling transverse oscillations.

Setting $t = 0$ we can use the initial position as usual:

$$u(x,0) = \sum_{j=1}^{\infty} a_j\sin\left(\frac{j\pi}{L}\,x\right) = u_0(x).$$

This means

$$a_j = \frac{2}{L} \int_0^L u_0(x) \sin\left(\frac{j\pi}{L} x\right) dx.$$

Differentiating with respect to t we have

$$\begin{aligned} u_t(x,t) = &-\frac{\beta}{2} e^{-\beta t/2} \sum_{j=1}^{\infty} [a_j \cos(\omega t) + b_j \sin(\omega t)] \sin\left(\frac{j\pi}{L} x\right) \\ &+ \omega e^{-\beta t/2} \sum_{j=1}^{\infty} [b_j \cos(\omega t) - a_j \sin(\omega t)] \sin\left(\frac{j\pi}{L} x\right). \end{aligned}$$

Therefore, the initial velocity condition becomes

$$u_t(x,0) = \sum_{j=1}^{\infty} \left[\omega b_j - \frac{\beta}{2} a_j\right] \sin\left(\frac{j\pi}{L} x\right) = v_0(x).$$

This means

$$\begin{aligned} b_j &= \frac{1}{\omega} \left[\frac{\beta}{2} a_j + \frac{2}{L} \int_0^L v_0(x) \sin\left(\frac{j\pi}{L} x\right) dx\right] \\ &= \frac{2}{\omega L} \left[\frac{\beta}{2} \int_0^L u_0(x) \sin\left(\frac{j\pi}{L} x\right) dx + \int_0^L v_0(x) \sin\left(\frac{j\pi}{L} x\right) dx\right] \\ &= \frac{2}{\omega L} \int_0^L \left(\frac{\beta}{2} u_0(x) + v_0(x)\right) \sin\left(\frac{j\pi}{L} x\right) dx. \end{aligned}$$

It is understood in this formula that $\omega = \omega_j$ is given in (6), and at this point we can consider the problem solved in this case.

CASE 2 ($\beta L > 2\pi$ and $\beta L/(2\pi)$ is not an integer) The special case with

$$1 < \frac{\beta L}{2\pi} < 2$$

is considered by Haberman in his Exercise 4.4.5.

In this case, there will be at least one overdamped mode corresponding to $j = 1$ and perhaps finitely many others so that for some integer $N \geq 2$

$$N - 1 < \frac{\beta L}{2\pi} < N,$$

and we can write the solution as

$$u(x,t) = \sum_{j=1}^{N-1} [a_j e^{-r_1 t} + b_j e^{-r_2 t}] \sin\left(\frac{j\pi}{L}\, x\right)$$
$$+ e^{-\beta t/2} \sum_{j=N}^{\infty} [a_j \cos(\omega t) + b_j \sin(\omega t)] \sin\left(\frac{j\pi}{L}\, x\right).$$

The first $N-1$ modes are overdamped and no mode is critically damped. The determination of the coefficients proceeds as follows:

Setting $t = 0$ we can use the initial position as usual:

$$u(x,0) = \sum_{j=1}^{N-1} [a_j + b_j] \sin\left(\frac{j\pi}{L}\, x\right) + \sum_{j=N}^{\infty} a_j \sin\left(\frac{j\pi}{L}\, x\right) = u_0(x).$$

This means

$$a_j = \frac{2}{L} \int_0^L u_0(x)\, \sin\left(\frac{j\pi}{L}\, x\right)\, dx$$

for $j = N, N+1, N+2, \ldots$ as before, but

$$a_j + b_j = \frac{2}{L} \int_0^L u_0(x)\, \sin\left(\frac{j\pi}{L}\, x\right)\, dx \qquad \text{for} \qquad j = 1, 2, \ldots, N-1.$$

Differentiating with respect to t we have

$$u_t(x,t) = -\sum_{j=1}^{N-1} [r_1 a_j e^{-r_1 t} + r_2 b_j e^{-r_2 t}] \sin\left(\frac{j\pi}{L}\, x\right)$$
$$- \frac{\beta}{2} e^{-\beta t/2} \sum_{j=N}^{\infty} [a_j \cos(\omega t) + b_j \sin(\omega t)] \sin\left(\frac{j\pi}{L}\, x\right)$$
$$+ \omega e^{-\beta t/2} \sum_{j=N}^{\infty} [b_j \cos(\omega t) - a_j \sin(\omega t)] \sin\left(\frac{j\pi}{L}\, x\right).$$

Therefore, the initial velocity condition becomes

$$u_t(x,0) = -\sum_{j=1}^{N-1} [r_1 a_j + r_2 b_j] \sin\left(\frac{j\pi}{L}\, x\right)$$
$$+ \sum_{j=N}^{\infty} \left[\omega b_j - \frac{\beta}{2} a_j\right] \sin\left(\frac{j\pi}{L}\, x\right) = v_0(x).$$

This means

$$b_j = \frac{2}{\omega L}\int_0^L \left(\frac{\beta}{2}u_0(x) + v_0(x)\right) \sin\left(\frac{j\pi}{L}\,x\right)\,dx$$

for $j = N, N+1, N+2, \ldots$ but

$$r_1 a_j + r_2 b_j = -\frac{2}{\omega L}\int_0^L v_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx \qquad \text{for} \qquad j = 1, 2, \ldots, N-1.$$

For the overdamped modes, $j = 1, 2, \ldots, N-1$ we have then

$$\begin{cases} a_j + b_j = \dfrac{2}{L}\int_0^L u_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx \\[2ex] r_1 a_j + r_2 b_j = -\dfrac{2}{L}\int_0^L v_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx. \end{cases}$$

Since $r_2 - r_1 \neq 0$ (in fact, $r_2 - r_1 < 0$) we can solve this system to obtain for $j = 1, 2, \ldots, N-1$

$$\begin{aligned} a_j &= \frac{2}{L(r_2 - r_1)}\left[r_2\int_0^L u_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx + \int_0^L v_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx\right] \\ &= \frac{2}{L(r_2 - r_1)}\int_0^L [r_2 u_0(x) + v_0(x)]\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx \end{aligned}$$

and

$$\begin{aligned} b_j &= \frac{2}{L(r_1 - r_2)}\left[\int_0^L v_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx + r_1\int_0^L u_0(x)\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx\right] \\ &= \frac{2}{L(r_1 - r_2)}\int_0^L [v_0(x) + r_1 u_0(x)]\,\sin\left(\frac{j\pi}{L}\,x\right)\,dx. \end{aligned}$$

Thus, the problem is solved in the second case.

CASE 3 The final case is when there is some integer $N \geq 2$ for which

$$\frac{\beta L}{2\pi} = N - 1.$$

The mode $j = N-1$ is critically damped. If $N = 2$, then there is no overdamped mode and

$$u(x,t) = e^{-\beta t/2}[a_1 + b_1 t]\sin\left(\frac{\pi}{L}\,x\right) + e^{-\beta t/2}\sum_{j=2}^{\infty}[a_j\cos(\omega t) + b_j\sin(\omega t)]\sin\left(\frac{j\pi}{L}\,x\right).$$

If $N > 2$, then all three modes are present:

$$u(x,t) = \sum_{j=1}^{N-2} [a_j e^{-r_1 t} + b_j e^{-r_2 t}] \sin\left(\frac{j\pi}{L}\, x\right)$$
$$+ e^{-\beta t/2} [a_{N-1} + b_{N-1} t] \sin\left(\frac{(N-1)\pi}{L}\, x\right)$$
$$+ e^{-\beta t/2} \sum_{j=N}^{\infty} [a_j \cos(\omega t) + b_j \sin(\omega t)] \sin\left(\frac{j\pi}{L}\, x\right).$$

The coefficients for the overdamped (if present) and underdamped modes are obtained as above:

$$a_j = \frac{2}{L(r_2 - r_1)} \int_0^L [r_2 u_0(x) + v_0(x)] \, \sin\left(\frac{j\pi}{L}\, x\right) \, dx$$
$$b_j = \frac{2}{L(r_1 - r_2)} \int_0^L [v_0(x) + r_1 u_0(x)] \, \sin\left(\frac{j\pi}{L}\, x\right) \, dx$$

for $j = 1, 2, \ldots, N-2$ and

$$a_j = \frac{2}{L} \int_0^L u_0(x) \, \sin\left(\frac{j\pi}{L}\, x\right) \, dx \tag{12}$$
$$b_j = \frac{2}{\omega L} \int_0^L \left(\frac{\beta}{2} u_0(x) + v_0(x)\right) \, \sin\left(\frac{j\pi}{L}\, x\right) \, dx$$

for $j = N, N+1, N+2, \ldots$.

We can deal with the coefficients for the critically damped mode independently. Let us set

$$U = u_{N+1} = e^{-\beta t/2} [a_{N-1} + b_{N-1} t] \sin\left(\frac{(N-1)\pi}{L}\, x\right).$$

Then

$$U(x,0) = a_{N-1} \sin\left(\frac{(N-1)\pi}{L}\, x\right)$$

and we conclude that formula (12) holds also for $j = N-1$. Differentiating with respect to t, we find

$$U_t(x,t) = e^{-\beta t/2} \left[\left(1 - \frac{\beta}{2}\, t\right) b_{N-1} - \frac{\beta}{2} a_{N-1}\right] \sin\left(\frac{(N-1)\pi}{L}\, x\right)$$

and

$$U_t(x,0) = \left[b_{N-1} - \frac{\beta}{2} a_{N-1}\right] \sin\left(\frac{(N-1)\pi}{L}\, x\right).$$

It follows that

$$\begin{aligned} b_{N-1} &= \frac{\beta}{2} a_{N-1} + \frac{2}{L}\int_0^L v_0(x)\, \sin\left(\frac{(N-1)\pi}{L}\, x\right)\, dx \\ &= \frac{2}{L}\int_0^L \left[\frac{\beta}{2}\, u_0(x) + v_0(x)\right]\, \sin\left(\frac{(N-1)\pi}{L}\, x\right)\, dx \end{aligned}$$

Again, we have found the coefficients for every mode.

The only difference using eigenfunction expansion is that we begin by assuming a solution of the form

$$u(x,t) = \sum_{j=1}^{\infty} B_j(t) \sin\left(\frac{j\pi}{L}\, x\right)$$

and plug this into the PDE. From this we get

$$\rho \sum_{j=1}^{\infty} B_j''(t) \sin\left(\frac{j\pi}{L}\, x\right) = -\epsilon \sum_{j=1}^{\infty} \frac{j^2\pi^2}{L^2}\, B_j(t) \sin\left(\frac{j\pi}{L}\, x\right) - \beta \sum_{j=1}^{\infty} B_j'(t) \sin\left(\frac{j\pi}{L}\, x\right).$$

Combining the series we get

$$\sum_{j=1}^{\infty} \left\{\rho B_j'' + \beta B_j' + \frac{\epsilon j^2 \pi^2}{L^2}\, B_j\right\} \sin\left(\frac{j\pi}{L}\, x\right) = 0.$$

Thus, we have a Fourier expansion of the (spatial) constant zero function, so all the coefficients should vanish. This gives us the ODEs

$$\rho B_j'' + \beta B_j' + \frac{\epsilon j^2 \pi^2}{L^2}\, B_j = 0$$

for $j = 1, 2, 3, \ldots$ which should be precisely the same ODEs considered above. Thus, the analysis proceeds as above from here.

At this point we have essentially complete parts (a) and (b) of the hint. We have been able to apply the qualitative designations of underdamped, critically damped, and overdamped to specific modes (or separated variable solutions) of the continuum modeled by the wave equation, but it is still not entirely clear what this means for the 1-D continuum as an oscillator. In order to explore the relation between the damping coefficient β and the qualitative behavior of the continuum, we turn first to the behavior of specific separated variables solutions. If we take the fundamental (or lowest frequency) mode we obtain for $\beta L < 2\pi$ (CASE 1)

$$u_1(x,t) = e^{-\beta t/2}[a_1 \cos(\omega t) + b_1 \sin(\omega t)] \sin\left(\frac{\pi x}{L}\right)$$

where

$$\omega = \omega_1 = \sqrt{\frac{\pi^2}{L^2} - \frac{\beta^2}{4}}.$$

We can further simplify the oscillation by considering full initial displacement and zero initial velocity:

$$u_1(x,t) = ae^{-\beta t/2}\ \cos(\omega t)\ \sin\left(\frac{\pi x}{L}\right). \tag{13}$$

Thus, the wave form $\sin(\pi x/L)$ decays exponentially but still oscillates passing through the equilibrium position $u \equiv 0$ (though not **at equilibrium** because the velocity is nonzero) infinitely many times. This is completely analogous to the ODE oscillator giving simple harmonic motion; this is very much as expected. See the animation with $L = \beta = 1$. It's not so easy to represent these oscillations using static figures, so I'm just going to refer to the Mathematica notebook. It is relatively easy to visualize the evolution represented by (13).

From this case of a single underdamped fundamental mode, there are two obvious directions in which to proceed. We can add further (underdamped modes) and/or we can increase the damping coefficient β. Let us do the latter first. For $\beta L = 2\pi$, we have

$$u_1(x,t) = e^{-\pi t/L}[a_1 + b_1 t]\ \sin\left(\frac{\pi x}{L}\right).$$

We see from our computation of the coefficients in CASE 3 when $N = 2$ and $j = N-1$ that even if the initial velocity $v_0 \equiv 0$ we do not get zero for b_1. More precisely, for pure displacement to the fundamental wave form $a \sin(\pi x/L)$ and zero initial velocity

$$a_1 = \frac{2a}{L}\int_0^L \sin^2\left(\frac{\pi x}{L}\right)\,dx = a$$

and

$$b_1 = \frac{\beta}{2}\,a_1 = \frac{\pi a}{L}.$$

Therefore, in this case the simplified expression for u_1 is

$$u_1(x,t) = ae^{-\pi t/L}\left[1 + \frac{\pi t}{L}\right]\ \sin\left(\frac{\pi x}{L}\right).$$

We see, as expected, that this solution tends monotonically to the equilibrium position (without ever reaching it) just like a 1-D harmonic oscillator. See the animation.

We can arrange to have one pass through the equilibrium position by choosing the initial position to be the equilibrium position $u \equiv 0$ along with some nonzero initial velocity v_0. Or we can also, in principle, choose u_0 close to $u \equiv 0$ and v_0 large, but this is rather more difficult—maybe a challenge problem. Let us consider the simpler possibility. We cannot choose any initial velocity v_0 arbitrarily or else we will introduce other modes. In this case, we know all other modes are underdamped, so if we take $u_0 \equiv 0$, and we want all the coefficients for the other modes to vanish we need simply

$$\int_0^L v_0(x) \sin\left(\frac{j\pi}{L}x\right) = 0 \qquad \text{for} \qquad j = 2, 3, 4, \ldots$$

This comes from considering the underdamped coefficients in CASE 3. The natural choice is

$$v_0(x) = b \sin\left(\frac{j\pi}{L}x\right).$$

With this choice $a_1 = 0$ and

$$b_1 = \frac{2b}{L}\int_0^L \sin^2\left(\frac{\pi x}{L}\right)\, dx = b.$$

The animation gives what we would expect. Running slightly backwards in time we can see a solution of the challenge problem.

Before we leave the critically damped fundamental mode, I'll leave you with one more obvious question: We've noted that starting in the position of full extension with zero velocity does **not** correspond to $b_1 = 0$. What initial condition(s) for the critically damped fundamental mode do correspond to $a_1 = a \neq 0$ and $b_1 = 0$?

The only thing that should really change when the fundamental mode is overdamped with $\beta L > 2\pi$ is the rate of decay, and that is more or less clear from the formula:

$$u_1(x,t) = \left[a_1 e^{-r_1 t} + b_1 e^{-r_2 t}\right] \sin\left(\frac{\pi x}{L}\right).$$

Here we have from (11)

$$r_1 = \frac{\beta}{2} + \sqrt{\frac{\beta^2}{4} - \frac{\pi^2}{L^2}} > r_2 = \frac{\beta}{2} - \sqrt{\frac{\beta^2}{4} - \frac{j^2\pi^2}{L^2}}.$$

The discussion above makes it essentially clear that a similar behavior occurs for each isolated mode of a particular spatial frequency $j\pi/L$, $j = 2, 3, 4, \ldots$. In this case, when $\beta L < j\pi$, there will be oscillations (at a higher frequency) of the continuum with the appropriate wave form; see the animation for $j = 2$. We've used

$$u_2(x,t) = e^{-\beta t/2}[a_1 \cos(\omega_1 t) + b_1 \sin(\omega_1 t)] \sin(2\pi x/L)$$

with $\beta = 1/5$, $a_1 = 1$, $b_1 = 0$, and $L = 1$ and

$$\omega_1 = \sqrt{\frac{4\pi^2}{L^2} - \beta^2} = \frac{1}{5}\sqrt{100\pi^2 - 1}.$$

For the critically and overdamped cases with such an isolated mode, one will also get relatively rapid decay (with possibly one pass through equilibrium) to the equilibrium with essentially no oscillation. See the animation for $j = 3$ where we have used

$$u_3(x,t) = e^{-3\pi t}(a_1 + b_1 t) \sin(3\pi x)$$

with $b_1 = 0$ to gain possibly some insight to the question about the critically damped case above.

The next thing to do is to start mixing modes, i.e., including more than one wave form. We can start with the first mode underdamped and then the second mode will also be underdamped. We will use the expression for u_2 above with $\beta = 1/5$ along with u_1 from (13) with $\beta = 1/5$ and consider

$$u(x,t) = u_1(x,t) + \mu u_2(x,t).$$

We can change the multiple μ of the second mode as well as the time shift determined by the coefficients a_2 and b_2 of the oscillation terms in u_2. Our first animation is of a small amplitude perturbation of u_1 with the second mode also starting from zero velocity and full extension. Various animations are given. It is not entirely clear what qualitative commentary can be made about these somewhat complicated superpositions.

It is clear that if the damping increases and various modes (spatial wave forms) are present so that both overdampled/critically damped modes are present along with underdamped modes, then the lower frequency modes will damp out first and the higher (underamped) modes will continue to oscillate. Let's move on to the final part of the hint.

The final part of the hint, part (d), is to try something that requires a full series solution with all the modes. I chose a square wave $u_0(x) = 1$ with $v_0 = 0$. Of course, this is not compatible with the boundary conditions, but in some sense that just makes things more interesting. You can see the result in my Mathematica notebook. When $\beta L < 2\pi$ we have what we would expect with all modes underdamped, namely oscillation. It becomes a bit difficult to distinguish the oscillations of the higher modes.

When β is increased so that $\beta L = 2\pi$ and the first mode becomes critically damped, the behavior is quite striking. We can see the square wave execute critically damped decay with the higher modes oscillating upon it. See the Mathematica notebook. So at least in this case, the critical damping more or less dominates the evolution of the continuum.

Problem 5 *(sagging equilibrium, Haberman 4.2.1) Consider a deformation* $w_* : [0, L] \to [0, L]$ *of a one-dimensional elastic continuum with constant equilibrium density* $\rho > 0$ *and constant elasticity* ϵ *with* $w_*(0) = 0$, $w_*(L) = L$, *and* $w_*' > 0$. *Let* $y : [0, L] \to [-L, 0]$ *by* $y(x) = -w_*(x)$ *give a vertical representation of the deformation. Assume* w_* *is an equilibrium for the forced wave equation*

$$\rho w_{tt} = \epsilon w_{xx} + \rho g$$

where $g > 0$ *is a gravitational constant.*

(a) *Find* w_* *and determine conditions under which* w_* *is admissible. Hint: Non-admissibility may arise is* $w_*(x) \notin [0, L]$ *for some* $x \in (0, L)$ *or if* $w_*'(x) < 0$ *for some* x. *You may wish to consider the relation of these two conditions and the borderline condition in which* $w_*'(x) = 0$ *for some* x.

(b) *Use mathematical software to illustrate the hanging (and sagging) configuration given by* y *(for some specific values of the constants).*

(c) *Let* $u_* : [0, L] \to \mathbb{R}$ *by* $u_*(x) = -y(x) - x$. *Find the boundary value problem satisfied by* u_* *and plot the graph of* u_* *(for some specific values of the constants).*

Note: There was a typo in the original posting of this problem. Specifically, the values of u_* *were given as* $u_*(x) = -y(x) + x$. *This is incorrect. The motivation behind this part of the problem is that* w_* *should be obtained by adding* x *to a solution of the same equation with homogeneous boundary values. That is,* $w_* = u_* + x$. *This means* $-y = u_* + x$ *or* $u_* = -y - x$. *Sorry about that.*

Solution:

(a) The equilibrium equation associated with the PDE above is

$$w_*'' = -\frac{\rho g}{\epsilon} \tag{14}$$

This has general solution

$$w_* = -\frac{\rho g}{2\epsilon}\, x^2 + ax + b,$$

and we conclude from the boundary conditions $w_*(0) = 0$ and $w_*(L) = L$ that $b = 0$ and $a = 1 + \rho g L/(2\epsilon)$. Therefore we have

$$w_*(x) = -\frac{\rho g}{2\epsilon}\, x^2 + \left(1 + \frac{\rho g}{2\epsilon}\, L\right) x. \tag{15}$$

Note: For a free-hanging elastic continuum like the slinky we can use a balance of forces to obtain the equilibrium equation (14), but this is not so easy with the "lower" endpoint fixed at $x = L$. My notes on the wave equation contain a detailed variational derivation of the equilibrium equation for w_* under these conditions. This is equation (13) near the top of page 10 in the notes. The derivations of the wave equation in those notes can also be adapted to give the PDE above.

For admissibility we need $w_*' > 0$, that is,

$$-\frac{\rho g}{\epsilon}\, x + 1 + \frac{\rho g}{2\epsilon}\, L > 0 \qquad \text{for} \qquad 0 \le x \le L.$$

Notice that the lowest value of this expression occurs when $x = L$, we we need (and it is sufficient to have)

$$-\frac{\rho g}{\epsilon}\, L + 1 + \frac{\rho g}{2\epsilon}\, L = 1 - \frac{\rho g}{2\epsilon}\, L > 0,$$

that is,

$$\frac{\rho g}{2\epsilon}\, L < 1.$$

This condition along with the boundary conditions $w_*(0) = 0$ and $w_*(L) = L$ ensure that $w_*(x)$ satisfies $0 < w_*(x) < L$ for $0 < x < L$, and any such equilibrium should be physically admissible at least in the absence of other conditions (like the condition $w_*' \ge 1$ for the slinky). The borderline case in which

$$\frac{\rho g}{2\epsilon}\, L = 1$$

leads to a configuration in which $w_*'(L) = 0$. Of course, this would not be admissible for a slinky, but even for a more general elastic continuum this corresponds to a deformation with unbounded material density as the material density is given (as discussed on page 5 of my notes on the wave equation) by

$$\mu(x) = \frac{\rho}{w_*'(x)}.$$

Generally, this sagging equilibrium has maximum density at the "lower" $w_*(L) = L$, but presumably the "bunching" that takes place at this end may be assumed bounded.

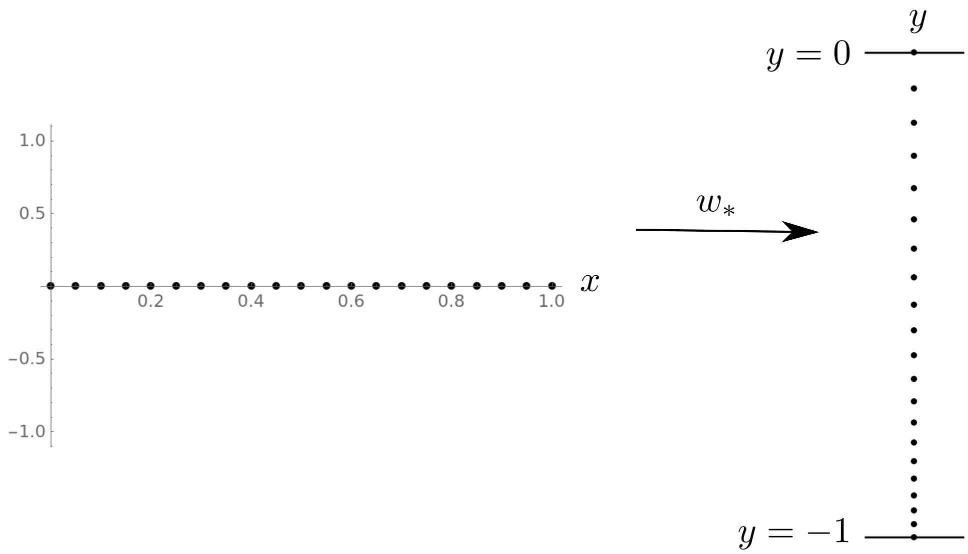

Figure 13: Sagging equilibrium (Part (b))

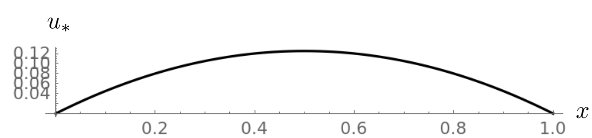

Figure 14: The sagging function u_*. (Part (c))

(b) $\rho g L = \epsilon$ should be okay.

(c) If $u_* = -y - x = w_* - x$, then

$$\begin{cases} u_*'' = -\rho g/\epsilon, & 0 < x < L \\ u_*(0) = 0 = u_*(L). \end{cases}$$

We have from the formula in (15) above

$$u_*(x) = w_*(x) - x = -\frac{\rho g}{2\epsilon}\, x^2 + \frac{\rho g}{2\epsilon}\, L\, x.$$

With $\rho g L = \epsilon$ this becomes $u_*(x) = x(1-x)/2$. This is trivial to plot.

I received a request to make my assignments shorter, starting with Assignment 6 (this assignment). I will make this assignment shorter in the following sense:

You have my official permission to consider the five problems above to be the entirety of Assignment 6 = Exam 2. I think the problems below are very interesting, and you can learn many potentially useful things if you do them. I will, however, make an effort to exclude the things you might learn from being required for future assignments in this course. I don't make any guarantees concerning the success of that effort.

Problem 6 *(Hamilton's action principle for the motion of a point mass) Show that motions* $\mathbf{x} : [0, T] \to \mathbb{R}^n$ *satisfying Newton's second law*

$$M\ddot{\mathbf{x}} = \mathbf{f}$$

are stationary points for the action functional $\mathcal{A} : X \to \mathbb{R}$ *by*

$$\mathcal{A}[\mathbf{x}] = \int_0^T \left[\Phi(\mathbf{x}, t) - \frac{1}{2}\, M\, |\mathbf{v}|^2\right] \, dt$$

where X *is the admissible class*

$$X = \{\mathbf{x} \in C^2([0, T] \to \mathbb{R}^n) : \mathbf{x}(0) = \mathbf{x}_0, \mathbf{x}(T) = \mathbf{p}\}.$$

Hint: I'm leaving it to you to figure out the relation between the potential Φ *and the force* $\mathbf{f}$*.*

Solution: The first variation of the action here is calculated as follows:

$$\frac{d}{d\epsilon} \int_0^T \left[\Phi(\mathbf{x} + \epsilon\xi, t) - \frac{1}{2}\, M\, |\dot{\mathbf{x}} + \epsilon\dot{\xi}|^2\right] \, dt = \int_0^T \left[D\Phi(\mathbf{x} + \epsilon\xi, t) \cdot \xi - M\, (\dot{\mathbf{x}} + \epsilon\dot{\xi} \cdot \dot{\xi}\right] \, dt.$$

Therefore,

$$\delta\mathcal{A}_{\mathbf{x}}[\xi] = \int_0^T \left[D\Phi(\mathbf{x}, t) \cdot \xi - M\, \dot{\mathbf{x}} \cdot \dot{\xi}\right] \, dt.$$

For compactly supported variations ξ we can integrate by parts to write

$$\int_0^T \dot{\mathbf{x}} \cdot \dot{\xi} \, dt = -\int_0^T \ddot{\mathbf{x}} \cdot \xi \, dt.$$

Making this substitution we have for compactly supported variations

$$\delta \mathcal{A}_{\mathbf{x}}[\xi] = \int_0^T [D\Phi(\mathbf{x}, t) + M\,\dot{\mathbf{x}}] \cdot \dot{\xi}\, dt.$$

If this vanishes for all compactly supported ξ, then we obtain the vector equation

$$M\ddot{\mathbf{x}} = -D\Phi(\mathbf{x}, t).$$

Note that we can, for example, take ξ to be nonzero in only one component and thus obtain the vector equation componentwise (in each component one at a time). In any case, a potential function Φ for a field of force $\mathbf{f}$ is a real valued function for which

$$-D\Phi(\mathbf{x}, t) = \mathbf{f}(\mathbf{x}, t).$$

Thus, we obtain Newton's second law:

$$M\ddot{\mathbf{x}} = \mathbf{f}.$$

Problem 7 *(equilibrium under tension) Derive a model for the elastic deformation/motion with respect to time for a function* $w : [0, L_0] \times [0, T) \to [0, L_0]$ *in*

$$\mathcal{W} = \{w \in C^2([0, L_0] \times [0, T)) : w(0, t) = 0, w(L_0, t) = L_0, w_x(0, t) > 0 \text{ for } t \geq 0\}$$

under the following assumptions: The evolving one-dimensional continuum is modeled on an **initial equilibrium interval** $[0, L]$ *with* $L < L_0$ *using an initial extension* $w_0 : [0, L] \to [0, L_0]$ *by* $w_0(x) = L_0 x/L$ *and initial tension given by*

$$F = -\epsilon(w_0' - 1).$$

You may assume constant density ρ *and elasticity* ϵ. *You may use any of the three approaches presented in my notes on the wave equation (or some other approach if you like), that is, Newton's second law according to continuum assumption A, the momentum force relation of continuum assumption B, or Hamilton's principle of stationary action.*

Solution: I will use Hamilton's principle starting on page 17 of my notes on the wave equation. The key observation is that the action for the motion $w : [0, L_0] \times [0, T] \to [0, L_0]$ may be calculated with respect to the initial interval $[0, L]$ using the composition:

$$\mathcal{A}[w] = \frac{1}{2} \int_0^T \int_0^L \left\{ \epsilon[(w \circ w_0)_x - 1]^2 - \rho[(w \circ w_0)_t]^2 \right\} \, dx \, dt.$$

Then we can change variables $\xi = w_0(x)$ with $d\xi = w_0' \, dx = (L_0/L) \, dx$ to write

$$\mathcal{A}[w] = \frac{L}{2L_0} \int_0^T \int_0^{L_0} \left\{ \epsilon[w_x - 1]^2 - \rho[w_t]^2 \right\} \, dx \, dt.$$

Then the derivation goes as before with

$$\begin{aligned} \delta\mathcal{A}_w[\phi] &= \frac{L}{L_0} \int_0^T \int_0^{L_0} \{\epsilon[w_x - 1]\phi_x - \rho w_t \phi_t\} \, dx \, dt \\ &= \frac{L}{L_0} \int_0^T \int_0^{L_0} \{-\epsilon w_{xx} + \rho w_{tt}\} \, \phi \, dx \, dt \end{aligned}$$

for compactly supported variations $\phi : (0, L_0) \times (0, T) \to \mathbb{R}$. Thus, deformations $w : [0, L_0] \times [0, T] \to [0, L_0]$ with stationary action satisfy

$$\rho w_{tt} = \epsilon w_{xx}$$

by the fundamental lemma of the calculus of variations.

Problem 8 *(slinky/modeling) Note that the equilibrium of Problem 5 above requires that compression from the equilibrium ($w' < 0$) is possible, and this is not possible for a slinky. Using the result of Problem 7 above, model the equilibrium position of an elongated slinky suspended vertically and sagging due to constant downward gravitational acceleration g within an interval $[0, L_0]$ of length $L < L_0$. Hint: There should be three distinct cases depending on whether or not L_0 exceeds the length of the slinky with a free hanging end.*

Solution: The natural (free hanging) length would be that satisfying

$$w_* = -\frac{\rho g}{2\epsilon}\, x^2 + ax + b,$$

from Problem 5 with $w_*(0) = 0$ and $w'(L) = 0$ corresponding to no tension at the free end. This would imply

$$w_*(x) = -\frac{\rho g}{2\epsilon}\, x^2 + \left(1 + \frac{\rho g}{\epsilon}\, L\right) x. \tag{16}$$

If the end at L_0 just happens to be fixed at

$$w_*(L) = L + \frac{\rho g}{2\epsilon}\, L^2 \tag{17}$$

given by this particular deformation, then this will be the solution. This gives us a kind of middle case.

The shorter length case is when

$$L < L_0 < L + \frac{\rho g}{2\epsilon}\, L^2.$$

In this case, we should expect some portion of the slinky to "pile up" with $w_*'(x) \equiv 1$ on some interval $L_1 \leq w \leq L_0$. The length L_1 will be the free hanging length of a shorter portion $[0, L - (L_0 - L_1)]$ of the original slinky. Substituting $L - (L_0 - L_1) = L + L_1 - L_0$ in for L in (17) and setting the result equal to L_1, we find

$$L + L_1 - L_0 + \frac{\rho g}{2\epsilon}\,(L + L_1 - L_0)^2 = L_1.$$

This means we must have

$$L_1 = L_0 - L + \sqrt{\frac{2\epsilon(L_0 - L)}{\rho g}}. \tag{18}$$

Thus, for sagging deformations with fixed endpoint at $L_0 > L$ satisfying

$$L < L_0 < L + \frac{\rho g}{2\epsilon} L^2. \tag{19}$$

we have

$$w_*(x) = \begin{cases} -\dfrac{\rho g}{2\epsilon} x^2 + \left(1 + \dfrac{\rho g}{\epsilon} [L - (L_0 - L_1)]\right) x, & 0 \le x \le L - (L_0 - L_1) \\ L_0 - L + x, & L - (L_0 - L_1) \le x \le L \end{cases}$$

where the "pile up" length L_1 is given by (18). Notice that for this to make sense we need $0 < L_0 - L_1 < L$. For the left inequality, note that by (19)

$$L_0 - L < \frac{\rho g}{2\epsilon} L^2.$$

Both sides in this inequality are positive so that this implies

$$L > \sqrt{\frac{2\epsilon}{\rho g}(L_0 - L)}.$$

In view of (18) this means $L_1 < L_0$. Also in view of (18) the inequality $L_0 - L_1 < L$ follows immediately because

$$L_0 - L_1 = L - \sqrt{\frac{2\epsilon}{\rho g}(L_0 - L)} < L.$$

The discussion would also work if we have equality $L_0 = L$, but this is a rather uninteresting case since then the slinky is not extended at all and $w_*(x) = x$.

The final case is when the slinky is extended beyond its natural free hanging length with

$$L_0 > L + \frac{\rho g}{2\epsilon} L^2. \tag{20}$$

In this situation we return to the form from Problem 5

$$w_* = -\frac{\rho g}{2\epsilon} x^2 + ax + b,$$

but require $w_*(0) = 0$ and $w_*(L) = L_0$. This gives

$$w_*(x) = -\frac{\rho g}{2\epsilon} x^2 + \frac{1}{L}\left(L_0 + \frac{\rho g}{2\epsilon} L^2\right) x. \tag{21}$$

In this case, the derivative is given by

$$w_*'(x) = \frac{1}{L}\left(L_0 + \frac{\rho g}{2\epsilon}\,L^2\right) - \frac{\rho g}{\epsilon}\,x$$

which takes its smallest value at $x = L$ with

$$w_*'(L) = \frac{1}{L}\left(L_0 + \frac{\rho g}{2\epsilon}\,L^2\right) - \frac{\rho g}{\epsilon}\,L = \frac{L_0}{L} - \frac{\rho g}{2\epsilon}\,L.$$

In this case we can write the condition (20) as

$$L_0 - \frac{\rho g}{2\epsilon}L^2 > L$$

so that dividing by L we have

$$w_*'(x) \geq w_*'(L) = \frac{L_0}{L} - \frac{\rho g}{2\epsilon}\,L > 1.$$

Thus, the stretched extension (21) is admissible and models the sagging due to gravity in this case.

Problem 9 *(center of mass) Consider the modeling of the motion of a one-dimensional elastic continuum by a function $w : [0, L] \times [0, T) \to \mathbb{R}$ in*

$$\mathcal{W} = \{w \in C^2([0, L] \times [0, T)) : w_x > 0\}$$

where the elasticity $\epsilon = \epsilon(x)$ is spatially dependent and in the presence of a potential field $\Phi : \mathbb{R} \to \mathbb{R}$ according to which the potential energy associated with the field Φ for a given configuration w is given by

$$E_\Phi = E_\Phi(t) = \int_0^L \Phi(w)\,dx.$$

(a) *Using Hamilton's principle of stationary action (see my notes on the wave equation), derive a forced wave equation for the evolution of w. Hint: Your answer should be (something like)*

$$\rho w_{tt} = [\epsilon(w_x - 1)]_x + f$$

where $f(w, x, t) = -\Phi_z(w, x, t)$.

(b) *Let $[a, b] \subset (0, L)$ be an equilibrium subinterval and let*

$$p_{\rm cm} = \frac{\displaystyle\int_a^b \rho w(x,t)\,dx}{\displaystyle\int_a^b \rho\,dx}$$

be the center of mass of the deformed interval $[w(a,t), w(b,t)]$. Show that

$$M\ddot{p}_{\rm cm} = \left(\int_a^b \rho\,dx\right)\frac{d^2}{dt^2}p_{\rm cm}$$

is the sum of the forces at the endpoints of $[w(a,t), w(b,t)]$. Hint(s): Differentiate under the integral sign and then use the PDE. The forces you should get are of two kinds: tension forces from the deformation and field forces from the external forcing.

Solution:

(a) We can use essentially the same action used on my notes on page 17 of my notes and in Problem 7 above except that we do not need to worry about an initial homogeneous deformation, so the composition can be avoided, and (most importantly) we need to complement the potential portion of the action with a time integral of the potential energy due to the field:

$$\mathcal{A}[w] = \frac{1}{2}\int_0^T\int_0^L \left\{\epsilon[w_x - 1]^2 + \Phi(w) - \rho[w_t]^2\right\}\,dx\,dt.$$

Remembering that elasticity depends on x, the first variation is given by

$$\begin{aligned}\delta\mathcal{A}_w[\phi] &= \int_0^T\int_0^L \{\epsilon(w_x - 1)\phi_x + \Phi'(w)\phi - \rho w_t\phi_t\}\,dx\,dt\\ &= \int_0^T\int_0^L \{-[\epsilon(w_x - 1)]_x + \Phi'(w) + \rho w_{tt}\}\,\phi\,dx\,dt\end{aligned}$$

for compactly supported variations $\phi : (0, L) \times (0, T) \to \mathbb{R}$. Thus, deformations $w : [0, L] \times [0, T] \to \mathbb{R}$ with stationary action satisfy

$$\rho w_{tt} = [\epsilon(w_x - 1)]_x - \Phi'(w)$$

by the fundamental lemma of the calculus of variations. Naturally, in the 1-D spatial domain the gradient of the potential function is $D\Phi = \Phi' = -f$, so our PDE is

$$\rho w_{tt} = [\epsilon(w_x - 1)]_x + f.$$

A couple notes: I've done the integration by parts both here and in Problem 7 in a little bit of a cavalier[1] manner just integrating by parts in each variable separately. This is okay in the sense that I get the correct answer. If I wanted to do this correctly, I should recognize the double integral containing the terms with ϕ_x and ϕ_t factors as a dot product of a certain vector field with the **full space and time gradient** $D\phi = (\phi_x, \phi_t)$. Then I would use the product rule and the divergence theorem as usual for the higher dimensional version of integration by parts. I think I did all this pretty carefully in my notes on the wave equation and certainly in some other notes related to the calculus of variations for functions of several variables, so you can go back and read about it there if you like.

One thing to note about this whole business is that usually when we write a gradient $D\phi$ in the context of evolution equations (like the heat equation and the wave equation or even Laplace's equation where there is no time dependence) we mean only the **spatial gradient**, the vector containing the partial derivatives with respect to the spatial variables. In this case, the spatial gradient would be $D\phi = \phi'$, but the full space and time gradient used above is a different thing.

Finally, I didn't mention it, but the derivation here is okay if the density ρ depends on x. This is the case because the t derivative in the integration by parts does not effect $\rho = \rho(x)$. Now, if we had $\rho = \rho(x, t)$ with time dependent density, we would get something different, but that would model a very different physical situation, e.g., it's somewhat difficult to imagine a one-dimensional gas with an elastic energy.

And as a final final note there is a small typo at the end of the statement of part (a) above. It should read *where* $f(x) = -\Phi'(w(x))$. We're only considering a fixed spatially dependent potential function $\Phi : \mathbb{R} \to \mathbb{R}$ with $\Phi = \Phi(w)$. The potential for the equilibrium reference position with $x \in [0, L]$ and the potential on the line for the motion are the same, so it doesn't make sense to have Φ to depend on both w and x positions. On the other hand, time dependence would be okay with $\Phi = \Phi(x, t)$, but then we should have $\Phi : \mathbb{R} \times [0, T) \to \mathbb{R}$

[1]cavalier (adj.) having or showing no concern for something that is important or serious

which is different from what is stated in the first part of the description of the problem. You can of course think of Φ as the constant gravitational potential, i.e., $\Phi \equiv -\rho g x$.

(b) Differentiating under the integral sign we have

$$\ddot{p}_{\rm cm} = \frac{1}{M}\int_a^b \rho w_{tt}\, dx = \frac{1}{M}\int_a^b \{[(\epsilon(w_x - 1)]_x + f\}\, dx$$

where

$$M = \int_a^b \rho\, dx.$$

Notice that here I really am considering $\rho = \rho(x)$. You get $M = \rho(b-a)$ for the mass of the deformed segment if ρ is constant. The fundamental theorem of calculus may be applied to the first term:

$$M\ddot{p}_{\rm cm} = \epsilon(b)[w_x(b,t) - 1] - \epsilon(a)[w_x(a,t) - 1] + \int_a^b f(w(x))\, dx.$$

Thus, we have Newton's second law applied to the center of mass. The tension forces at the ends

$$\epsilon(b)[w_x(b,t) - 1] - \epsilon(a)[w_x(a,t) - 1] = \tau(b) - \tau(a)$$

are given first and then the force from the field

$$\int_a^b f(w(x))\, dx.$$

I neglected to mention the field force term in the statement of the problem because I had in mind the special case when $f = 0$, i.e., there is no potential. But if the potential field is there, this is a natural term to see in Newton's second law. We can interpret this scalar quantity as a single force acting at the center of mass $p_{\rm cm}$. This can be made a bit more precise (and convincing) if we take $\Phi = -\rho g x$ with ρ constant. Then we can write

$$\int_a^b f(w(x))\, dx = \int_a^b \rho g\, dx = Mg.$$

Note that we have the positive sign here because we are using the horizontal model in which gravity points to the right. Probably the cleanest way to think about this, however, is the one unintentially suggested in the statement of the problem with $\Phi \equiv 0$. Then you see the regular wave equation has, as a consequence, that the center of mass for each deformed subinterval moves according to Newton's second law subject to the tension forces at the ends. Thus, at least this is a consistent assumption to use in deriving the wave equation; see the discussion of **continuum assumption A** in my notes on the wave equation. In that particular discussion I was also assuming constant elasticity ϵ, but the discussion can easilly be generalized to spatially dependent elasticity.

Problem 10 *(Conservation of energy; Haberman 4.4.9-13) Consider the potential energy*

$$E(t) = \frac{\epsilon}{2} \int_0^L (w_x - 1)^2 \, dx,$$

the kinetic energy

$$K(t) = \frac{1}{2} \int_0^L \rho w_t^2 \, dx,$$

and the **total energy** $\mathcal{E}(t) = E(t) + K(t)$ *associated with a one-dimensional elastic motion* $w : [0, L] \times [0, T) \to [0, L]$ *satisfying*

$$\begin{cases} \rho w_{tt} = \epsilon w_{xx}, & \text{on } (0, L) \times (0, T) \\ w(0, t) = 0 = w(L, t), & t > 0. \end{cases} \tag{22}$$

(a) *Compute the derivative $\dot{\mathcal{E}}(t)$ of the energy with respect to time to obtain the general formula*

$$\dot{\mathcal{E}}(t) = \epsilon[w_x(L, t) - 1]w_t(L, t) - \epsilon[w_x(0, t) - 1]w_t(0, t).$$

(b) *Conclude that the total energy is conserved for solutions of (22).*

(c) *What other (natural) boundary conditions result in conservation of energy?*

Solution:

(a)

$$\begin{aligned}
\dot{\mathcal{E}}(t) &= \frac{d}{dt}\left\{\frac{\epsilon}{2}\int_0^L (w_x-1)^2\,dx + \frac{1}{2}\int_0^L \rho w_t^2\,dx\right\} \\
&= \int_0^L \left[\epsilon(w_x-1)w_{xt} + \rho w_t w_{tt}\right]\,dx \\
&= \int_0^L \left[\epsilon(w_x-1)\frac{\partial}{\partial x}w_t + \epsilon w_t w_{xx}\right]\,dx \\
&= \epsilon\int_0^L \left[(w_x-1)\frac{\partial}{\partial x}w_t + w_{xx}w_t\right]\,dx \\
&= \epsilon\int_0^L \frac{\partial}{\partial x}\left[(w_x-1)w_t\right]\,dx \\
&= \epsilon[w_x(L,t)-1]w_t(L,t) - \epsilon[w_x(0,t)-1]w_t(0,t).
\end{aligned}$$

(b) If $w(0,t)=0$ for all t, we can differentiate this relation and conclude

$$w_t(0,t)=0.$$

Similarly, we have for (22) that $w_t(L,t)=0$. Thus, $\dot{\mathcal{E}}(t)$ vanishes as claimed. Note: One cannot conclude from these boundary conditions that $w_x(L,t)=0$ or $w_x(0,t)=0$ or $w_x(0,t)=1$ or anything like that because there is no x dependence in the boundary condition $w(0,t)=0$ to differentiate.

(c) Among the alternative boundary conditions leading to conservation of energy are those in which there prevails a free end condition $w_x - 1 = 0$:

$$w(0,t)=0,\ w_x(L,t)=1,\ t>0 \qquad \text{(left end fixed, right end free)}$$

or

$$w_x(0,t)=1,\ w(L,t)=0,\ t>0 \qquad \text{(left end free, right end fixed)}$$

or

$$w_x(0,t)=0,\ w_x(L,t)=1,\ t>0 \qquad \text{(both ends free).}$$

Note there could be lots of other possibilities in which the two ends have positions and tensions changing with time so as to cancel and make the quantity $\dot{\mathcal{E}}(t)$ vanish.