Assignment 7: The Wave Equation
Due Tuesday November 21, 2023

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The wave equation in one space dimension has the form

$$u_{tt} = u_{xx}.$$

I'm going to give some problems to introduce the basics of the wave equation, then I'll come back to some problems involving the heat equation and Laplace's equation at the end.

Problem 1 (factoring the wave operator) The **wave operator** is also called the d'Alembertian, and it is denoted by

$$\Box u = u_{tt} - u_{xx}.$$

Like the Laplacian $\Delta : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ given by $\Delta u = u_{xx}$ and the heat operator $L : C^2(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty)$ by $Lu = u_t - u_{xx}$, the d'Alembertian is a second order partial differential operator.

Let $T : C^1(\mathbb{R} \times (0,\infty)) \to C^0(\mathbb{R} \times (0,\infty))$ be the first order transport operator given by

$$Tu = u_t - u_x.$$

(a) Find a first order linear operator $S : C^1(\mathbb{R}) \to C^0(\mathbb{R})$ for which

$$\Box u = S \circ T.$$

(b) Compute $T \circ S$.

(c) An operator $A : V \to W$ defined on a vector space V and taking values in a vector space W is **linear** if

$$A(cu) = cAu \qquad \text{for every } c \in \mathbb{R} \text{ and } u \in V$$

and

$$A(u + v) = Au + Av \qquad \text{for every } u, v \in V.$$

(i) Show the wave operator is linear on $C^2(\mathbb{R} \times (0, \infty))$.

(ii) Show the operator T is linear on $C^1(\mathbb{R} \times (0, \infty))$.

(iii) Show the Laplace operator Δ is linear on $C^2((a, b))$ where $a, b \in \mathbb{R}$ with $a < b$.

(iv) Show the heat operator L is linear on $C^2((a, b) \times (0, \infty))$ where $a, b \in \mathbb{R}$ with $a < b$.

Problem 2 (The wave equation on all of $\mathbb{R}$) Consider the initial value problem (IVP) for the wave equation:

$$\begin{cases} u_{tt} = u_{xx}, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \\ u_t(x, 0) = 0, & x \in \mathbb{R} \end{cases}$$

where u_0 is a given function with $u_0 \in C^2(\mathbb{R})$.

(a) If $u \in C^2(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty))$ satisfies the transport equation

$$Tu = w$$

for some $w \in C^1(\mathbb{R} \times (0, \infty)) \cap C^0(\mathbb{R} \times [0, \infty))$ and $Sw = 0$ where S and T are the factor operators from Problem 1, calculate $\Box u$.

(b) Find an appropriate initial condition for the problem

$$\begin{cases} Sw = 0, & \text{on } \mathbb{R} \times (0, \infty) \\ w(x, 0) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{1}$$

Problem 3 (first order linear equation) Consider the problem (1) with the initial condition you found in part **(b)** of Problem 2. Solve that problem by completing the following steps:

(a) Consider a parameterized path $\gamma : [0, \infty) \to \mathbb{R} \times [0, \infty)$ given by

$$\gamma(t) = (\xi(t), t)$$

for some real valued spatial function $x = \xi(t)$. Use the chain rule to compute

$$\frac{d}{dt} w \circ \gamma(t). \tag{2}$$

(b) Compare your result from (2) to the PDE from (1). Given any initial starting point $x_0 \in \mathbb{R}$ find an appropriate ODE for $\xi : [0, \infty) \to \mathbb{R}$ based on your comparison, and solve the ODE with the initial condition $\xi(0) = x_0$.

(c) With your solution for ξ from part **(b)** which should depend on x_0, consider for an arbitrary point $(x, t) \in \mathbb{R} \times (0, \infty)$ the equation

$$\gamma(t) = (x, t). \tag{3}$$

Choose x_0 so that (3) is satisfied.

(d) If you made the correct choice of ODE in part **(b)** you should now know the value of the quantity in (2), which should tell you

$$w(\xi(t), t) = w_0(x_0).$$

If you made the correct choice of w_0 in part **(b)** of Problem 2 you should now know the solution of the problem (1) in terms of u_0 from the IVP in Problem 2.

Problem 4 (another first order linear equation) Consider the inhomogeneous (forced) initial value problem for $u \in C^1(\mathbb{R} \times [0, \infty))$

$$\begin{cases} u_t - u_x = w, & \text{on } \mathbb{R} \times (0, \infty) \\ u(x, 0) = u_0(x), & x \in \mathbb{R} \end{cases} \tag{4}$$

where $u_0, w \in C^0(\mathbb{R})$ are given initial and spatial forcing functions.

(a) Consider a propagating curve $\gamma(t) = (\xi(t), t)$ with $\xi(0) = x_0 \in \mathbb{R}$. Calculate

$$\frac{d}{dt} u \circ \gamma(t) \tag{5}$$

and pose an appropriate ODE for ξ based on comparison with the operator $Tu = u_t - u_x$ in the PDE.

(b) Draw a picture of the curve parameterized by γ in $\mathbb{R} \times [0, \infty)$.

(c) How would you characterize the propagation of x_0 induced by γ? (Give speed and direction.)

(d) Derive an ODE for $u \circ \gamma$ based on your work above and computation of the derivative in (5).

(e) Couple your ODE from part **(d)** with an appropriate initial condition to find a formula for $u \circ \gamma(t)$ as a function of x_0. Be careful with the argument of w.

(f) Solve the equation $\gamma(t) = (x, t)$ for the starting point x_0.

(g) Substitute (x, t) in for $\gamma(t)$ along with the value you found for x_0 in part **(f)** into the formula you found in part **(e)** to solve the problem (4).

Problem 5 (Problem 4 above; forced transport)

(a) Take the specific choice(s)

$$u_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad w \equiv 0$$

in the formula you obtained in part **(g)** of Problem 4 and plot the graph of the solution u and then animate the profile of u with time as an animation parameter.

(b) If you think of the behavior of u_0 on its support as a "signal," how long does it take for information from the signal to be communicated at $x = -10$? At what time has the signal passed $x = -10$?

(c) Take the specific choice(s)

$$u_0(x) \equiv 0 \qquad \text{and} \qquad w(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

in the formula you obtained in part **(g)** of Problem 4 and plot the graph of the solution u and then animate the profile of u with time as an animation parameter.

(d) What information is propagated to $x = -10$ under this second evolution?

Problem 6 (weak solution)

(a) Take the specific choice(s)

$$u_0(x) = \begin{cases} 1 - |x|, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases} \qquad \text{and} \qquad w \equiv 0$$

in the formula you obtained in part **(g)** of Problem 4 and plot the graph of the solution u and then animate the profile of u with time as an animation parameter.

(b) Notice the "solution" $u = u(x,t)$ you obtain in this case is not differentiable in x or t. Formulate a notion of weak solutions of (4) which allows solutions in $u \in C^0(\mathbb{R} \times [0,\infty))$. Hint: Use the weak adjoint operator $\psi : C_c^\infty(\mathbb{R} \times [0,\infty)) \to \mathbb{R}$ by

$$W\psi = \int_{\mathbb{R}\times[0,\infty)} (\psi_x - \psi_t)u.$$

(c) Show the solution from part **(a)** satisfies your definition from part **(b)**.

Laplace's equation and the forced heat equation

Consider the following two-dimensional problems with rectangular spatial domain

$$R = (0,L) \times (0,M) = \{(x,y) \in \mathbb{R}^2 : 0 < x < L, 0 < y < M\}$$

where $L, M > 0$:

$$\begin{cases} u_t = \Delta u, & \text{on } R \times (0,\infty) \\ u\big|_{(x,y)\in\partial R} = g\big|_{(x,y)\in\partial R}, & t > 0 \\ u(x,y,0) = g(x,y), & (x,y) \in R \end{cases} \tag{6}$$

and

$$\begin{cases} \Delta w = 0, & \text{on } R \\ w\big|_{\partial R} = g\big|_{\partial R} \end{cases} \tag{7}$$

where $g : \mathbb{R}^2 \to \mathbb{R}$ by

$$g(x,y) = \frac{M^2}{4} - \left(y - \frac{M}{2}\right)^2.$$

Problem 7 (eigenfunction expansion) "Trade" the inhomogeneous initial and boundary conditions in (6) for a forced heat equation by setting $v = u - g$:

(a) Find a second initial/boundary value value problem for the forced heat equation satisfied by $v = u - g$.

(b) Solve the problem you found in part **(a)** using the **method of eigenfunction expansion**:

  (i) Apply separation of variables to the **associated homogeneous problem**, i.e., the problem with the same boundary and initial values as your problem form part **(a)** but without forcing.

  (ii) Use the spatial factors of your separated variables solution as a spatial eigen/Fourier basis

$$\{A_{ij}\}_{i,j=1}^{\infty}$$

  with $A_{ij} : R \to \mathbb{R}$ for $j = 1, 2, 3, \ldots$ and setting

$$v(x, y, t) = \sum_{i,j=1}^{\infty} \phi_{ij}(t) A_{ij}(x, y)$$

  solve the initial/boundary problem satisfied by v.

(c) Solve (6), i.e., write down the solution you have found.

Problem 8 (Laplace's equation) "Trade" the inhomogeneous initial and boundary conditions in (6) for a forced heat equation by setting $v = u - w$:

(a) Solve (7) as a sum of two superpositions.

(b) Find the initial/boundary value problem satisfied by $v = u - w$.

(c) Solve the problem you found in part **(b)** using separated variables and superposition.

(d) Solve (6), i.e., write down the solution you have found.

Problem 9 (Poincaré inequality) Show there is some constant $M > 0$ so that if $u \in C^0[a, b]$ and $u(a) = u(b) = 0$, then

$$\int_a^b [u(x)]^2\, dx \le M \int_a^b [u'(x)]^2\, dx. \tag{8}$$

Hints:

(a) Use the fundamental theorem of calculus to write

$$u(x) = \int_a^x u'(\xi)\,d\xi.$$

(b) Square the expression from part **(a)** and integrate.

(c) "Give up" the x dependence so you only need to estimate (the constant)

$$\int_a^b |u'(\xi)|\,d\xi.$$

Consider the integrand $|u'(\xi)|$ as a product $f(\xi)g(\xi)$ with $g(\xi) \equiv 1$ and apply the Cauchy-Schwarz inequality in $L^2(a,b)$.

(d) Last hint: $M = (b-a)^2$.

Follow up: Express the inequality (8) in terms of L^2 norms.

Problem 10 (heat dissipation) Use the Poincaré inequality to show a (classical) solution of the 1-D heat conduction problem

$$\begin{cases} u_t = u_{xx} & \text{on } (0,L)\times(0,\infty) \\ u(0,t) = 0 = u(\ell,t), & t > 0 \\ u(x,0) = u_0(x), & 0 < x < L, \end{cases} \tag{9}$$

satisfies

$$\frac{d}{dt}\int_{(0,L)} u^2 \le -C\int_{(0,L)} u^2 \tag{10}$$

for some positive constant C. This is sometimes called a Gronwall inequality.

Problem 11 (heat dissipation) Use the inequality (10) to show any classical solution $u \in C^2([0,L]\times[0,\infty))$ of (9) satisfies

$$\lim_{t\nearrow\infty}\int_{(0,L)} u^2 = 0$$

no matter what the initial temperature u_0 might be.