Assignment 8 = Final Exam:
Classical Mathematical Methods in Engineering
Due Monday December 13, 2021

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Problem 1 *(d'Alembert's solution, Haberman 4.4.6-8) Consider the initial/boundary value problem*

$$\begin{cases} u_{tt} = \sigma^2 u_{xx} & \text{on } (0, L) \times (0, \infty) \\ u(0, t) = 0 = u(L, t), & t > 0 \\ u(x, 0) = g(x), & x \in (0, L) \\ u_t(x, 0) = h(x), & x \in (0, L) \end{cases}$$

for the one-dimensional wave equation where $g, h \in C^1[0, L]$ *are given continuously differentiable functions.*

(a) *Use separation of variables and superposition to find a Fourier sine series solution* $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *of this problem.*

(b) *Notice that your Fourier series solution* w *solves an initial value problem*

$$\begin{cases} w_{tt} = \sigma^2 w_{xx} & \text{on } \mathbb{R} \times (0, \infty) \\ w(x, 0) = u_0(x), & x \in \mathbb{R} \\ w_t(x, 0) = v_0(x), & x \in \mathbb{R} \end{cases}$$

where $u_0 = u_0(x)$ *and* $v_0 = v_0(x)$ *are some functions defined for* $x \in \mathbb{R}$.

(c) *Discuss the relation between* u_0 *and* g.

(d) *Discuss the relation between* v_0 *and* h.

(e) *Discuss the relation between the limits*

$$\lim_{x \searrow 0} w(x,0) \qquad \text{and} \qquad \lim_{t \searrow 0} w(0,t).$$

(f) *Discuss the relation between the limits*

$$\lim_{x \searrow 0} w_t(x,0) \qquad \text{and} \qquad \lim_{t \searrow 0} w_t(0,t).$$

(f) *Discuss the relation between the limits*

$$\lim_{x \searrow 0} w_x(x,0) \qquad \text{and} \qquad \lim_{t \searrow 0} w_x(0,t).$$

Problem 2 *(separation of variables and superposition, Haberman 2.5.1)*

(a) *Find a series solution of Laplace's equation on a rectangle* $R = [0,L] \times [0,M]$ *subject to the following boundary values:*

$$\begin{cases} u(x,0) = u_y(x,0), & 0 < x < L \\ u(x,M) = \chi_{[L/4,3L/4]}(x), & 0 < x < L \\ u(0,y) = 0 = u(L,y), & 0 < y < M \end{cases}$$

where $\chi_A : [0,L] \to \mathbb{R}$ *is a* **characteristic function** *defined for any subset* $A \subset [0,L]$ *by*

$$\chi_A(x) = \begin{cases} 0, & x \notin A \\ 1, & x \in A. \end{cases}$$

(b) *Choose specific numerical values for* L *and* M *and plot your solution.*

Problem 3 *(guitar strings; Haberman Chapter 4) In Problem 3 of Assignment 7 (or Chapter 4 of Haberman) you should find the wave equation model for transverse oscillations of a tightly stretched string involves the PDE*

$$\rho u_{tt} = T u_{xx}$$

where ρ *is the linear density of the string and* T *is the (constant) tension of the string. Most guitars have six strings with the two lowest strings (the sixth and fifth strings) producing the notes E and A at* 82.41 *Hz and* 110 *Hz respectively (when played as open strings). The units here are Herz (Hz) which is a measurement of frequency in cycles per second.*

I recently put new strings on my guitar, and I made the following measurements:

(i) *The mass of the low E string was* $m_6 = 5.3353$ *g and its total length was* 37 *inches.*

(i) *The mass of the low A string was* $m_5 = 2.8460$ *g and its total length was* 37.5 *inches.*

To do this problem you need to know a little about the construction of a guitar. When the strings are installed on the guitar they stretch from the **nut** *which is a bar crossing the neck of the guitar near the tuning pegs to the* **saddle** *which is a bar on the opposite side of the guitar from the neck. The distance between the nut and the saddle on my guitar is* $26 + 1/16$ *inches (for both strings). The remainder of the length of each string is used to secure the string to a pin just past the saddle and to wind around the tuning pegs which are used to tension the string.*

(a) *Determine the tension on the two lowest strings of my guitar when they are tuned correctly. Hint: When played as an open string the frequency of oscillation determining the note you hear is the* **fundamental frequency** *or lowest frequency you find as a separated variables solution. This is also sometimes called the first Fourier mode associated with the particular combination of density, tension, and length.*

There are many other bars crossing the neck of a guitar; these are called **frets**. *By pushing down on a string between a fret and the nut one makes the length of the portion of the string that is vibrating shorter and, consequently, increases the fundamental frequency which is heard when that string is played. The frets are numbered starting with the one closest to the nut.*

(b) *If I play the sixth string (the open low E string) with my finger pressed down on it between the fourth and fifth fret, then it plays an A at (approximately)* 110 *Hz. What is the distance between the fifth fret and the saddle? Give your answer to the nearest* $1/16$ *of an inch.*

(c) *The actual measurement from the fifth fret to the saddle on my guitar is* $19 + 9/16$ *inches. Your answer from part (b) is probably shorter then this actual measurement. Why is that the case?*

(d) *If you know the likely cause of the mismatch between your calculation and the actual measurement in part (c), calculate the magnitude of the quantity involved.*

Problem 4 *(2D wave equation, Haberman 7.7.5)*

Let $u = u(x, y, t)$ satisfy the initial/boundary value problem

$$\begin{cases} u_{tt} = \sigma^2 \Delta u, & \text{on } \mathcal{U} \times (0, \infty) \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}), & \mathbf{x} \in \mathcal{U} \\ u_t(\mathbf{x}, 0) = v_0(\mathbf{x}), & \mathbf{x} \in \mathcal{U} \\ u\Big|_{\partial\mathcal{U}} \equiv 0, & t > 0 \end{cases}$$

for the wave equation on the plane domain

$$\mathcal{U} = \{(x, y) \in \mathbb{R}^2 : a^2 < x^2 + y^2 < b^2, \ x, y > 0\}$$

which is the portion of an annulus with inner radius $a > 0$ and outer radius $b > 0$ in the first quadrant. Find the fundamental frequencies associated with this problem and animate some of the vibrations (with time as the animation parameter and for specific choices of the radii and wave speed).

Problem 5 *(hanging slinky) Consider a deformation $w : [0, L] \to [0, w(L)]$ of a one-dimensional elastic continuum with constant equilibrium density $\rho > 0$ and elasticity ϵ with $w(0) = 0$ and $w' > 1$. We will also use the gravitational constant $g > 0$, e.g., $g = 9.8$ m/s^2.*

(a) *Assuming the elasticity ϵ is constant, compute the first variation of the potential energy*

$$E[w] = \int_0^L \left[\frac{\epsilon}{2}(w' - 1)^2 - \rho g w \right] dx$$

to obtain the ODE

$$\epsilon w'' = -\rho g \tag{1}$$

for the equilibrium extension.

(b) *Still assuming constant elasticity, find the solution of (1) satisfying the conditions $w'(L) = 1$ and $w(L) = 52.5$ inches with the physical data*

$$\rho = 3.82671 \ kg/m, \qquad \text{and} \qquad L = 0.05474 \ m.$$

Hint: Use the end measurement $w(L)$ to find the elasticity constant.

(c) *Compare the model function you found in part (b) to the measured data*

$$\begin{aligned} w(0.006676) &= 0.31433 \\ w(0.013351) &= 0.58896 \\ w(0.020027) &= 0.81915 \\ w(0.026703) &= 1.00965 \\ w(0.033378) &= 1.15253 \\ w(0.040054) &= 1.25730 \\ w(0.046730) &= 1.31445 \\ w(0.053405) &= 1.33509. \end{aligned}$$

(d) *Go back to the energy $E = E[w]$ and assume the elasticity ϵ is* **not constant** *but depends on the extension w'. Show that the Euler-Lagrange ODE can be written in the form*

$$\frac{d}{dx}\left[\frac{\epsilon'(w')}{2}(w'-1)^2 + \epsilon(w')\ (w'-1)\right] = -\rho g \tag{2}$$

where

$$w' = \frac{dw}{dx} \qquad \text{but} \qquad \epsilon' = \frac{d\epsilon}{dp}.$$

(e) *Integrate equation (2) from x to L again assuming $w'(L) = 1$ to obtain*

$$\frac{\epsilon'(w')}{2}(w'-1)^2 + \epsilon(w')\ (w'-1) = \rho g(L-x). \tag{3}$$

The big question at this point is the following:

What is the form of the elasticity $\epsilon = \epsilon(w')$?

Careful consideration of the measured data strongly suggests

1. *ϵ is not constant, and*
2. *the basic shape of the deformation is* **not** *quadratic.*

After careful consideration of the dependence of ϵ of w' as determined by the data and partially by the ODE (3), I suggest the following form:

$$\epsilon(p) = \frac{\mu}{\sqrt{p-1}} + \sigma$$

where μ and σ are constants.

(f) *Use this form of the elasticity to write (3) as a quadratic equation for* $\sqrt{w' - 1}$.

(g) *Solve for* w' *in terms of* x *and integrate from* 0 *to* x *assuming* $w(0) = 0$ *to find* w.

(h) *Go back to part (f) and note that the quadratic equation for* $\sqrt{w' - 1}$ *is a linear relation between* μ *and* σ. *Use the data to estimate* $w'(x^*)$ *at some point* $x^* \in [0, L]$, *and then find a formula/linear relation for* $\sigma = \sigma(\mu)$. *Hint: You can use* $w'(0)$.

(i) *Substituting the formula for* σ *from part (h) into the formula for* w *from part (g), you should obtain a formula for* w *depending on* x *and* μ. *Use the measured values of* L *and* $w(L)$ *to find a value of* μ *around* $\mu = 0.06$.

(j) *With your values of* μ *and* σ *compare the model function from part (g) with the measured data from part (c).*

(k) *(Bonus) Give a physical explanation for the following:*

- **(i)** *Why is it true that the elasticity for a slinky satisfies*

$$\epsilon(w') \longrightarrow +\infty \qquad \text{as} \qquad w' \searrow 1?$$

- **(ii)** *Why is the particular form*

$$\epsilon(w') = \frac{\mu}{\sqrt{w' - 1}} + \sigma$$

physically reasonable? Note: I think I understand how to do this problem. If you have a good answer, it's probably a publishable result.

Problem 6 *(Bonus: Transverse oscillations of a hanging slinky)*

This problem requires the numerical techniques mentioned in Lecture 27 among other things. You will only get a numerical solution, so you'll also need to solve ODEs numerically.

We have a model for the non-homogeneous extension of a slinky hanging in gravity. In fact we have two models, one in which the elasticity is constant and an improved model in which the elasticity depends on the tension in the spring (which in turn depends on the local magnitude of extension). You can approach this problem with either of those models, but it is much easier with the assumption of constant elasticity, so let's stick with that. Use your solution from part (b) of Problem 5.

We also have an approximate model of the transverse oscillations of a hanging chain based on a simple tension (Problem 4 of Assignment 7). That model looks like

$$\rho u_{tt} = (T u_x)_x$$

where ρ *is the density and* T *is the tension. In this case, we have for the slinky an* x *interval* $[0, L]$ *and a constant density function* ρ_0*, and we are looking for the transverse displacement* u *on the image/deformed interval* $[0, w(L)] = [0, 1.3367]$*. Let's use the variable* ξ *on this interval. Then the PDE becomes*

$$\rho u_{tt} = (T u_\xi)_\xi$$

with the boundary condition $u(0, t) = 0$*.*

Find $\rho = \rho(\xi)$ *and* $T = T(\xi)$*, separate variables, and model the small transverse oscillations of the hanging slinky. Animate some fundamental modes. You can do it.*

Here are some hints:

(i) *Study the hanging chain problem carefully. Some aspects of it apply to this problem; others do not. Of course, you cannot change variables to express solutions in terms of a Bessel function, but you still get a linear second order ODE with non-constant coefficients from separation of variables. That ODE also has the same kind of singular behavior at one endpoint. That is, there is a regular-singular ODE somewhere in the background. It is not a standard one, so we cannot expect to find it using a change of variables.*

(ii) *Separate variables and change variables* $\eta = w(L) - \xi$ *in the spatial ODE/Sturm-Liouville problem to put the singular point* $\xi = w(L)$ *at the origin and obtain an unknown function* $y = y(\xi)$ *on* $[0, w(L)]$ *(along with an unknown eigenvalue).*

(iii) Numerically *solve the spatial ODE subject to the initial condition* $y(0) = 1$ *and* $y'(0) = p$*.*