Small Transverse Oscillations of Hanging Chain and Hanging Slinky Bessel Functions

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November 22, 2021

1 Hanging Chain

A standard problem presented in courses on the classical mathematical methods of engineering is that of finding the fundamental modes of the small oscillations of a hanging chain. The wave equation in the form

$$\rho u_{tt} = (T u_x)_x$$

is used where $\rho > 0$ (assumed constant) models the linear density of a hanging chain of length L, the function $T : [0, L] \to (0, \infty)$ models the tension due to gravity along the chain, and u measures the horizontal transverse displacement. The tension at each point, furthermore, is assumed to be given by

$$T(x) = \rho g(L - x)$$

where g is a gravitational acceleration constant and $\rho(L - x)$ is the mass of the portion of chain "below" x, though we have chosen coordinates so that positive increasing x corresponds to "downward." It is also usually assumed that the "top" point is fixed corresponding to the boundary condition

$$u(0, t) = 0.$$

An alternative is, of course, to take z on the interval $[-L, 0]$ so that the tension becomes $T(z) = \rho g(L+z)$ and the PDE is

$$u_{tt} = g[(z + L)u_z]_z.$$

Expanding our equation in x, we get

$$u_{tt} = g[(L - x)u_x]_x = g(L - x)u_{xx} - g u_x.$$

It will be noted immediately that the equation is spatially singular at $x = L$ where the coefficient of the leading order term u_{xx} vanishes. It is customary to choose coordinates

$$\xi = L - x, \qquad U(\xi, t) = u(L - \xi, t)$$

so that this singularity occurs at the left endpoint $\xi = 0$ of the interval $[0, L]$. The boundary value problem then becomes

$$\begin{cases} U_{tt} = g\xi U_{\xi\xi} + gU_\xi, & \text{on } (0, L) \times (0, \infty) \\ U(L, t) = 0, & t > 0. \end{cases} \tag{1}$$

Exercise 1 *Start with $u = u(z, t)$ on $[-L, 0] \times [0, \infty)$ and change variables to obtain the appropriate boundary value problem starting with vertical coordinates. Hint: You still want a singular PDE with singular point at the left endpoint $\zeta = 0$ on $[0, L]$.*

Separation of variables in (1) with $U = A(\xi)B(t)$ gives

$$AB'' = g\xi A''B + gA'B = 0 \qquad \text{or} \qquad \frac{B''}{B} = g\,\frac{\xi A'' + A'}{A} = -\lambda = -\mu^2.$$

We anticipate here in the B problem obtaining separated variables solutions with regular oscillation determined by

$$B(t) = a\cos\mu t + b\sin\mu t$$

though we have no boundary conditions to justify this anticipation. For the A problem we have at least one boundary condition:

$$\begin{cases} \xi A'' + A' + \mu^2 A/g = 0, & \text{on [0,L]} \\ A(L) = 0. \end{cases}$$

We may add the implied and physically reasonable requirement that $A(0)$ is finite at the singular point $\xi = 0$. It may not be immediately obvious what to do with an ODE like this one, but here is one approach (the one that is usually presented): Write the ODE as

$$A'' + \frac{1}{\xi}A' + \frac{\mu^2}{g\xi}A = 0 \tag{2}$$

and compare to the **standard Bessel equation** of order ν

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0. \tag{3}$$

The two equations are not immediately comparable, but there is a standard procedure which we now describe/present according to which a connection can be made. We consider the function

$$f(\xi) = \xi^p y(a\xi^q)$$

for arbitrary exponents p and q and an arbitrary coefficent a (all constant) with, of course, y a solution of the Bessel equation (3). Notice that

$$f' = aq\xi^{p+q-1}\,y' + p\xi^{p-1}\,y = \xi^{p-1}\left[aq\xi^q\,y' + p\,y\right]. \tag{4}$$

Differentiating again we have

$$\begin{aligned} f'' &= a^2q^2\xi^{p+2q-2}\,y'' + \left[aq(p+q-1)\xi^{p+q-2} + apq\xi^{p+q-2}\right]\,y' + p(p-1)\xi^{p-2}y \\ &= a^2q^2\xi^{p+2q-2}\,y'' + aq(2p+q-1)\xi^{p+q-2}\,y' + p(p-1)\xi^{p-2}\,y \\ &= \xi^{p-2}\left[a^2q^2\xi^{2q}y'' + aq(2p+q-1)\xi^q\,y' + p(p-1)y\right]. \end{aligned} \tag{5}$$

Putting $x = a\xi^q$ in the Bessel equation (3) we can write

$$y'' = -\frac{1}{a\xi^q}\,y' - \left(1 - \frac{\nu^2}{a^2\xi^{2q}}\right)\,y$$

or

$$a^2q^2\xi^{2q}\,y'' = -aq^2\xi^q\,y' - q^2\left(a^2\xi^{2q} - \nu^2\right)\,y.$$

Making this substitution in (5) we have

$$f'' = \xi^{p-2}\left[aq(2p-1)\xi^q\,y' + \left[p(p-1) + q^2\nu^2 - a^2q^2\xi^{2q}\right]\,y\right]$$

We also know from (4) that

$$aq\xi^{p+q-1}\,y' = f' - p\xi^{p-1}\,y = f' - \frac{p}{\xi}\,f$$

since $\xi^p y = f$ or

$$aq(2p-1)\xi^q\, y' = \frac{2p-1}{\xi^{p-1}}\, f' - p(2p-1)\, y.$$

Therefore,

$$\begin{aligned} f'' &= \xi^{p-2}\left[\frac{2p-1}{\xi^{p-1}}\, f' - p(2p-1)\, y + \left[p(p-1) + q^2\nu^2 - a^2q^2\xi^{2q}\right] y\right] \\ &= \xi^{p-2}\left[\frac{2p-1}{\xi^{p-1}}\, f' + \left[q^2\nu^2 - p^2 - a^2q^2\xi^{2q}\right] y\right] \\ &= \frac{2p-1}{\xi}\, f' + \left(\frac{q^2\nu^2 - p^2}{\xi^2} - a^2q^2\xi^{2q-2}\right) f. \end{aligned}$$

Finally then we have obtained a somewhat different ODE (or family of ODEs depending on the real parameters a, p, and q) also sometimes called Bessel ODEs:

$$f'' + \frac{1-2p}{\xi}\, f' + \left(\frac{p^2 - q^2\nu^2}{\xi^2} + a^2q^2\xi^{2q-2}\right) f = 0.$$

Let us compare this ODE to our original hanging chain ODE (2)

$$A'' + \frac{1}{\xi}\, A' + \frac{\mu^2}{g\xi}\, A = 0.$$

If $1 - 2p = 1$ or $p = 0$, then the order one coefficients agree. We need then to arrange for the zero order coefficent that

$$a^2q^2\xi^{2q-2} - \frac{q^2\nu^2}{\xi^2} = \frac{\mu^2}{g\xi}.$$

For this we must have $2q - 2 = -1$ or $q = 1/2$ and $\nu = 0$. Finally, we need

$$a^2q^2 = \frac{a^2}{4} = \frac{\mu^2}{g}$$

or $a = 2\mu/\sqrt{g}$.

Thus, if y is a solution of the Bessel equation of order $\nu = 0$, then

$$f(\xi) = y\left(\frac{2\mu}{\sqrt{g}}\sqrt{\xi}\right)$$

is a solution of the separated variables equation (2) for A. We know furthermore that $\sqrt{\xi}$ tends to zero as $\xi \searrow 0$, but the standard Bessel function Y_0 (of order zero of the second kind) tends to $-\infty$ logarithmically. In particular,

$$\ln\sqrt{\xi} = \frac{1}{2}\ln\xi \searrow -\infty \qquad \text{as } \xi \searrow 0.$$

We conclude that we should use a multiple of J_0, the Bessel function of order zero of the first kind for y. Thus, we consider solutions of the ODE

$$A(\xi) = J_0\left(\frac{2\mu}{\sqrt{g}}\sqrt{\xi}\right).$$

Finally, we need $A(L) = 0$ for the other boundary condition which means

$$2\mu\sqrt{\frac{L}{g}} = Z_{0j}$$

must be a zero of J_0. This (finally) tells us our eigenvalues $\lambda = \mu^2$:

$$\mu_j = \frac{Z_{0j}}{2}\sqrt{\frac{g}{L}}$$

where Z_{0j} is the j-th positive zero of J_0. In terms of the original variables we can write

$$U_j(\xi, t) = \cos\left(\frac{Z_{0j}t}{2}\sqrt{\frac{g}{L}}\right) J_0\left(Z_{0j}\sqrt{\frac{\xi}{L}}\right)$$

and

$$u(x, t) = \cos\left(\frac{Z_{0j}t}{2}\sqrt{\frac{g}{L}}\right) J_0\left(Z_{0j}\sqrt{1-\frac{x}{L}}\right).$$

2 A Second Approach

In order to adapt the analysis above to the case of a hanging slinky the main obstacle is, first of all, that there is no change of variable to reduce the problem to some standard ODE. As a consequence, probably the most expedient approach is to solve (or attempt to approximate solutions of) the ODE numerically. One can obtain a good deal of intuition from the problem above. For this reason, let us reconsider the problem above without the use of the Bessel function J_0. If we didn't know about that, what would we do?

Accordingly, we return to the equation (2)

$$A'' + \frac{1}{\xi}A' + \frac{\mu^2}{g\xi}A = 0$$

and consider it directly as a Sturm-Liouville type equation on the interval $[0, L]$ with a singularity at $\xi = 0$. We already know there is a solution of this equation which is finite valued at $\xi = 0$. Also, (2) is a linear equation, so given any solution A the scaled function cA is also a solution. This means we can essentially find out everything about the solutions which are finite at $\xi = 0$ by considering only solutions with, for example, $A(0) = 1$. Of course, we can't numerically solve this ODe starting with $\xi = 0$ and $A(0) = 1$ because the equation is singular; the coefficients are not well-defined at $\xi = 0$. On the other hand, we could guess (and also check) that the derivative $A'(0)$ for the solution with $A(0) = 1$ is also well-defined and finite.

You may remember or observe that $J_0'(0) = 0$, but

$$\frac{d}{d\xi}J_0(\sqrt{\xi}) = \frac{J_0'(\sqrt{\xi})}{2\sqrt{\xi}} = -\frac{J_1(\sqrt{\xi})}{2\sqrt{\xi}}. \tag{6}$$

The identity $J_0' = -J_1$ is a little bit like $(d/d\theta)\cos\theta = -\sin\theta$. On the other hand, $J_1(0) = 0$, so the expression in (6) is an indeterminate form. In fact, $J_1'(0) = [J_0(0) - J_2(0)]/2 = 1/2$, so the value of $A'(0)$ is finite and nonzero. The point of this discussion is to suggest there is something special about $A'(0)$. You can't just specify that value arbitrarily due to the singularity. And you can see this numerically. If you start relatively close to the singularity $\xi = 0$, say at $\xi = 0.001$ and solve the ODE (2) numerically down to $\xi = 0.00000001$ subject to the initial conditions $A(0.001) = 1$ and $A'(0.001) = p$ takes various values, you will find that for some values of p the value you get $A(0.00000001)$ is very large and positive—much greater than 1. For other values of p the value you get $A(0.00000001)$ is very large and negative. there is a narrow range for p in which you can get $A(0.00000001)$ close to 1. This suggests, and in fact it is true, that there is only one value $A'(0) = p$ for which (2) has a solution with $A(0) = 1$.