Transport Equations

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We have discussed a model for heat flow, also called heat **conduction** and/or heat **diffusion** based in part on a **flux field** $\vec{\phi}$ with physical dimensions

$$[\vec{\phi}] = \frac{[\text{energy}]}{L^{n-1} \cdot T} \tag{1}$$

where n is the (mathematical) dimension of the model for the medium in which the flow takes place. In this model, the medium is generally considered rigid (non-deformable) and non-moving. For example, we consider a rod ($n = 1$), a rigid plate ($n = 2$), and a solid non-deformable, non-moving rigid material filling a fixed volume ($n = 3$). The key property of the flux field is that calculation of a flux integral

$$\int_{\mathcal{S}} \vec{\phi} \cdot \vec{n}$$

over an $(n-1)$-dimensional submanifold $\mathcal{S}$ with normal field $\vec{n}$ gives the **rate of flow** of heat energy across $\mathcal{S}$ in the direction $\vec{n}$. Most often the submanifold $\mathcal{S}$ is taken to be the boundary ∂R of an n-dimensional region within the solid medium.

It is also possible to consider a moving medium (for example a liquid or gas) with motion modeled by a **velocity field** $\mathbf{v} : \mathcal{U} \to \mathbb{R}^n$. In this case one usually has and may consider the **motion of mass** determined by a mass density ρ with physical dimensions

$$[\rho] = \frac{M}{L^n}.$$

Such a density is often considered to have a spatial character, so that $\rho = \rho(x, t)$ depends on position x, in some n-dimensional (universal) space $\mathcal{U}$ containing the medium, and the total mass within a region $R \subset \mathcal{U}$ at time t is given by

$$\int_R \rho.$$

An important special case is when ρ is constant; the medium is sometimes called **incompressible** in this case. Associated with a velocity field and a mass density is a natural flux field given by the product

$$\vec{\phi}_m = \rho \mathbf{v}.$$

This is called the **natural mass flux field**. Notice that since $[\mathbf{v}] = L/T$ the physical dimensions of the mass flux field are given by

$$[\vec{\phi}_m] = \frac{M}{L^{n-1} \cdot T}.$$

This dimensional assertion should be compared carefully to the analogous expression for the energy flux field given in (1). Accordingly, it is natural to model the rate of mass flow through a submanifold $\mathcal{S}$ by

$$\int_{\mathcal{S}} \vec{\phi}_m \cdot \vec{n},$$

and (in the absense of creation or destruction of mass) one models the **conservation of mass** by

$$\frac{d}{dt}\int_R \rho = -\int_{\partial R} \vec{\phi}_m \cdot \vec{n} = -\int_{\partial R} \rho\, \mathbf{v}\cdot\vec{n}.$$

Differentiation under the integral sign and application of the divergence theorem (when these things are possible) allows us to write this conservation law in the form

$$\int_R \left(\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \mathbf{v})\right) = 0. \tag{2}$$

We recall the **fundamental lemma of vanishing integrals**:

Lemma 1 *If $f : \mathcal{U} \to \mathbb{R}$ is a continuous real valued function on an open subset $\mathcal{U}$ of $\mathbb{R}^n$ and*

$$\int_R f = 0 \qquad \text{for all regions } R \subset \mathcal{U},$$

then $f(x) = 0$ for every $x \in \mathcal{U}$.

When the fundamental lemma applies to the integral conservation law (2) we obtain from it the **differential conservation law**

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\, \mathbf{v}) = 0. \tag{3}$$

This is also called the **equation of conservation of mass**. In the special case of fluid mechanics (3) is called the **continuity equation**. We are not going to say much about this equation, but notice that in the incompressible case (3) simplifies to the partial differential equation

$$\operatorname{div} \mathbf{v} = 0$$

for the n components of the velocity field $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ which is called **Euler's conservation equation** or simply **Euler's equation**.

1 Other Flows

Generalizing the mass flow considered above one may take any **distributed quantity** $\mathcal{N}$ determined by a density δ so that the total amount of $\mathcal{N}$ in a region R (at time t) is given by

$$\int_R \delta.$$

It is then natural to form the flux field for $\mathcal{N}$ given by

$$\vec{\phi}_n = \delta\, \mathbf{v}.$$

Naturally, $[\delta] = [\mathcal{N}]/L^n$ so that

$$[\vec{\phi}_n] = \frac{[\mathcal{N}]}{L^{n-1}\cdot T}$$

and the discussion of conservation of $\mathcal{N}$ within the space $\mathcal{U}$ parallels that of conservation of mass:

$$\frac{d}{dt}\int_R \delta = -\int_{\partial R} \delta\, \mathbf{v}\cdot\vec{n} \tag{4}$$

and (with sufficient regularity and applicability of the condition (4) to enough regions R)

$$\frac{\partial \delta}{\partial t} + \operatorname{div}(\delta\ \mathbf{v}) = 0. \tag{5}$$

These are expressions of the conservation of $\mathcal{N}$ within the medium under consideration; you can think of $\mathcal{N}$ as energy, or concentration of a pollutant, or many other distributed quantitites. Equation (5) is the general form of a **transfer equation** for the quantity $\mathcal{N}$.

It is sometimes the case that one introduces a law of **specific $\mathcal{N}$ capacity** into the model having a form

$$\delta = \gamma \rho u$$

where u is some more easily measured (scalar) quantity, ρ is mass density, and γ is a materially dependent proportionality "constant" which may also depend on position (and time and other things). In this case, the transfer equation may be written as

$$\frac{\partial}{\partial t}(\gamma \rho u) + \operatorname{div}(\gamma \rho u\ \mathbf{v}) = 0. \tag{6}$$

If γ and ρ are constants, then (6) simplifies to

$$\frac{\partial u}{\partial t} + \operatorname{div}(u\ \mathbf{v}) = 0$$

which may also be written as

$$\frac{\partial u}{\partial t} + Du \cdot \mathbf{v} + u \operatorname{div} \mathbf{v} = 0. \tag{7}$$

When the velocity field $\mathbf{v}$ is considered as given and u is considered unknown, this is the basic form of a **transfer equation** which is a specific kind of first order linear partial differential equation. There is a well-developed theory for solution of these partial differential equations, and we will consider a couple special cases in the next section.

On the other hand, it is perhaps worth remembering something you know (or should have learned) about ordinary differential equations:

> A linear ordinary differential equation as long as the order is two or higher, or a system of two or more linear ordinary differential equations, is usually impossible to solve explicitly (in closed form) unless the coefficients are **constant**.

The coefficients of the first order spatial derivatives in (7) are given by the coordinate functions of the velocity field $\mathbf{v}$, and in a similar way unless these functions are constant it is extremely unlikely one can write down any kind of explicit solution for the (linear first order) partial differential equation (7).

2 Heat Transfer

Returning to the quantity of thermal or heat energy, we can imagine that heat energy flows or moves in a medium **only due to the flow determined by a velocity field v**. Note carefully: There is no heat flow as considered before (in the sense of heat conduction or heat diffusion) in this case. This is what Haberman (and others) call **convection**, though often the term convection is reserved for a motion of heat energy driven by temperature itself. We are not considering that properly because in order to do so one would have to consider properly the motion of mass and the velocity field as unknown as well, which is much more complicated. For this reason, I will refer to the **motion of heat energy by the mechanism of moving mass in the medium**, especially when the velocity field is assumed given, simply as **heat transfer**.

The mathematical discussion is unchanged; we simply use the heat energy density θ in place of the general density δ and the law of specific heat capacity:

$$u_t + Du \cdot \mathbf{v} + u \operatorname{div} \mathbf{v} = 0$$

where u is the temperature and we have assumed the specific heat capacity and the mass density are constant.

Let us consider an even more special case in which the velocity field has the form

$$\mathbf{v} = v\mathbf{e}_1 = v(1, 0, \dots, 0) \in \mathbb{R}^n$$

and v is a real valued function which is **only a function of the first spatial variable**: $v = v(x_1)$. In rectangular coordinates, our heat transfer equation becomes

$$u_t + vu_x + uv' = 0$$

where

$$v' = \frac{dv}{dx_1} = \frac{dv}{dx}.$$

Our assumption(s) essentially reduce us to the case of one spatial dimension, but things can still be quite complicated. To get a very specific (and even simpler) problem, let us assume v' is a constant. More specifically, let us take $v_0 > 0$ (constant) and $v = v_0 + ax$ and consider three cases: $a < 0$, $a = 0$, and $a > 0$.

CASE 1. $a = 0$

In this case, $u_t + v_0 u_x = 0$ for some constant $v_0 > 0$. The equation is quite easy to solve in terms of an initial temperature distrubution $u_0 = u_0(x) = u(x, 0)$ for $x \in \mathbb{R}$ as follows:

Consider a curve in the (x, t) plane given by $\xi = x_0 + v_0 t$. (This curve is called a **characteristic line** with propagation speed v_0.) Notice that if we have a solution u, then

$$\frac{d}{dt} u(\xi, t) = u_x(\xi, t) v_0 + u_t(\xi, t) = 0.$$

This means, u must be constant along the characteristic line and have constant value $u(x_0, 0) = u_0(x_0)$. The value $u_0(x_0)$ is known, or at least would be known in terms of the inital temp u_0 if we knew the starting position x_0 for the characteristic. Thus, we ask the question: Given a point (x, t), is there a unique starting point x_0 for which $\xi(t) = x_0 + v_0 t = x$? Of course, you can see there is exactly one such point, namely,

$$x_0 = x - v_0 t.$$

This means

$$u(x, t) = u_0(x - v_0 t).$$

And, it will be noted, this is a formula for a solution. Before we move on to the next case, note that the characteristic lines constitute a collection of parallel lines in the x, t-plane. These lines happen to point into the first quadrant in that plane. In the other cases, the situation is not so simple, but this is a good picture to keep in mind.

Also, the temperature profile along the model line remains unchanged but simply translates to the right at constant velocity v_0 with the medium.

CASE 2. $a > 0$

The equation is now

$$u_t + (v_0 + ax)u_x + au = 0, \tag{8}$$

and the coefficient of u_x is not constant, but let's see if our "method of characteristics" might still tell us something. Again we start at a point $(x_0, 0)$, but instead of a line, let us imagine a curve $\xi = \xi(t)$ with $\xi(0) = x_0$ propagating out into the $t > 0$ region of the x, t-plane. We calculate:

$$\frac{d}{dt}u(\xi(t), t) = \xi' u_x + u_t. \tag{9}$$

Comparing this to our PDE (8) suggests that we might arrange to have

$$\xi'(t) = v_0 + a\xi(t). \tag{10}$$

This is a first order ODE for ξ, and we can solve it with initial value $\xi(0) = x_0$ as follows: We first observe that

$$[e^{-at}\xi]' = e^{-at}[\xi' - a\xi] = v_0 e^{-at}.$$

Remember the factor e^{-at} is called an integrating factor. Integrating both sides from time 0 to time t gives

$$e^{-at}\xi - x_0 = \int_0^t v_0 e^{-a\tau}\, d\tau = -\frac{v_0}{a}[e^{-at} - 1]$$

or

$$\xi(t) = \left(x_0 + \frac{v_0}{a}\right) e^{at} - \frac{v_0}{a}.$$

So then we have a curve parameterized by $\gamma(t) = (\xi(t), t)$, and we can try to find the value of a solution u along this curve. For this curve, the calculation (9) gives

$$\frac{d}{dt}u(\xi(t), t) = \xi' u_x + u_t = (v_0 + a\xi)u_x + u_t = -au(\xi(t), t).$$

That is, the propagating value $w(t) = u(\xi(t), t)$ satisfies an ODE $w' = -aw$. Again, this is a simple first order linear ODE. It is even homogeneous, and the solution with $w(0) = u_0(x_0)$ is

$$w(t) = u_0(x_0)e^{-at}. \tag{11}$$

Again, we are faced with the question: Is there a unique starting point $(x_0, 0)$ for which our characteristic propagates to (and through) the point (x, t)? For this we need

$$\xi(t) = \left(x_0 + \frac{v_0}{a}\right) e^{at} - \frac{v_0}{a} = x.$$

That is,

$$x_0 = \left(x + \frac{v_0}{a}\right) e^{-at} - \frac{v_0}{a}.$$

Again, we get a solution

$$u(x, t) = w(t) = u_0\left(\left(x + \frac{v_0}{a}\right) e^{-at} - \frac{v_0}{a}\right) e^{-at} \tag{12}$$

from (11).

CASE 3. $a < 0$

The same analysis, and formula (12) applies in this case. Note that when $a < 0$ the temperature at all points along the line decreases in the sense that the characteristic equation (10) also represents to motion of the media initially located at $x = x_0$ and moving according to the velocity field. Thus, when $a < 0$ the formula (12) says that the portion of material starting at position $x = x_0$ at time $t = 0$ has cooled from the positive temperature $u_0(x_0)$ to $u_0(x_0)e^{-at}$ at time t. The opposite is true when $a > 0$. Does this make sense?