Wave Equation

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I'm going to present here a derivation of the wave equation, with emphasis on the case of one spatial dimension, as a model for a physical system. This equation

$$u_{tt} = \sigma^2 u_{xx} \tag{1}$$

in one spatial dimension or more generally

$$u_{tt} = \sigma^2 \Delta u \tag{2}$$

in n spatial dimensions is often presented as the model for the vertical vibrations of an n-dimensional elastic membrane. In one dimension in particular (1) is viewed as a model for the vertical motion of a violin or guitar string, and in two dimensions ($n = 2$) the equation (2) is said to model the vertical motion of a drum head. While this almost universal approach has certain advantages, I find the presentation of this model objectionable in various respects. Fortunately, I have what I consider a nice alternative. I will consider a rather different physical system. To be precise, I wish to consider the inhomogeneous "internal" extension and compression of an elastic one-dimensional continuum. Most of you are familiar with a kind of special case in which a spring is homogeneously extended or compressed, and I will begin with a review of certain aspects of that model/system. In fact, we will use the so called "mass-spring" system, and the modeling of the spring in particular, to obtain several of the physical relations used in our derivation.

1 The One-dimensional Oscillator

Figure 1 illustrates the physical system typically modeled by the second order linear oscillator equation/ODE

$$m\ddot{x} = -k(x - \ell). \tag{3}$$

There are several physical relations we wish to derive from this simple model system for our model, but there are some differences as well. We are not particularly interested in the mass attached to the end of the spring, but we also wish to consider the case where the spring itself has mass. Above all, we wish to consider **inhomogeneous** deformation of the one-dimensional continuum (spring) which is strictly ruled out in the ODE model.

Let us start by considering some basic aspects of what an inhomogeneous deformation would look like and how such a thing might be modeled. Geometrically, we still begin with reference to an **equilibrium configuration** though we will consider some equilibrium configurations that are somewhat different from that illustrated at the bottom of Figure 1. Given that particular equilibrium configuration, however, and imagining the ends are both fixed at $x = 0$ and $x = L = \ell$, an inhomogeneous deformation is illustrated on the bottom of Figure 2. We can represent the geometric deformation of a one-dimensional continuum[1]

[1]It may be noted at this point the we have used the symbol x in two distinct ways. The symbol x as it appears in (3) represents a function of time t giving the time dependent coordinate of the end of a homogeneously deformed spring. In the geometric deformation model being introduced here (and in the wave equation) the symbol x represents an independent spatial variable in the interval $[0, L]$ upon which the deformation $w = w(x)$ (or $w = w(x, t)$) depends.

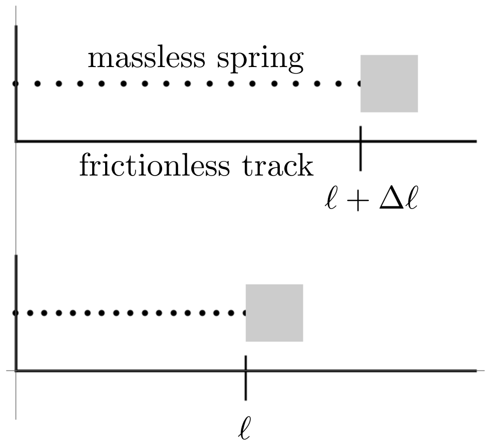


Figure 1: Homogeneous extension of a massless spring with a mass m attached to one end and the other end held stationary. Equilibrium (bottom) and stretched (top). This system is often modeled with an ordinary differential equation given by Newton's second law with a restoring force modeled by Hooke's law $F = -k(x - \ell)$ where k is called the Hooke's constant of the spring and the configuration in which the spring occupies a length ℓ is considered to be at equilibrium.

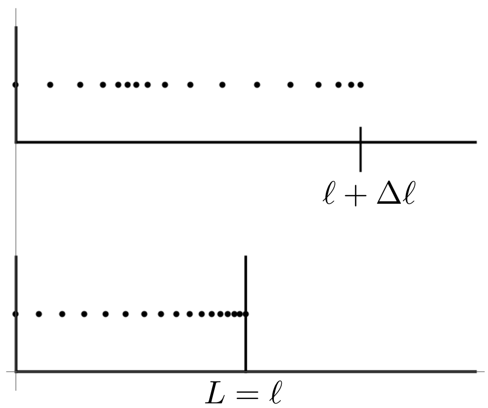


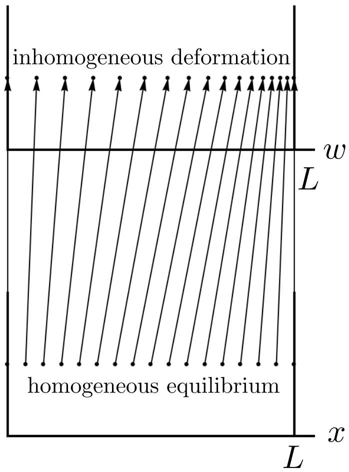
Figure 2: Inhomogeneous deformations of a one-dimensional continuum from an equilibrium represented by the image of a function $w : [0, L] \to \mathbb{R}$. Fixed ends (bottom) and with net extension (top).

modeled on the interval $[0, L]$ by a function $w : [0, L] \to \mathbb{R}$ where $w(x)$ gives the (deformed) position corresponding to the point located at $x \in [0, L]$ before the deformation. Using a function $w : [0, L] \to [0, L]$ to model the bottom deformation in Figure 2 is illustrated with additional detail in two different ways in Figure 3.

We are representing positions in our continuum/interval by a collection of plotted points in our illustrations. When we say, for example, that the bottom left illustration in Figure 3 represents a **homogeneous equilibrium**, it should be noted that **the spacing** of the plotted points, in addition to representing positions, may be used to represent **density variations** even in equilibrium. More examples of this illustration technique will be given below, but this possibility may be appreciated without further discussion.

Exercise 1 *(Haberman Problem 4.2.1) Calculate and plot the equilibrium position of a one-dimensional continuum/spring having horizontal/zero gravity density ρ_0 (constant) and length L at equilibrium under the following conditions:*

(a) *The spring has one end fixed at a height H and hangs vertically in a gravity field of acceleration g with the other end free. Hint(s): This is more complicated than you might guess because a spring*



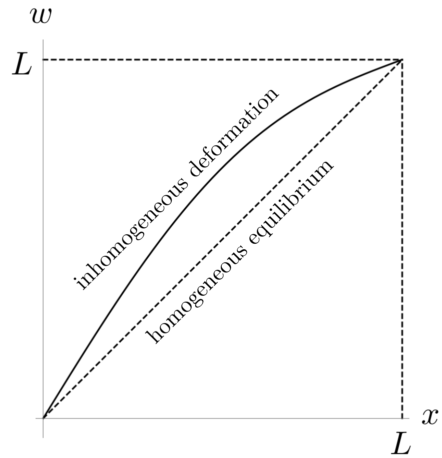


Figure 3: Inhomogeneous deformation of a one-dimensional continuum from a homogeneous equilibrium with fixed endpoints. Represented by a mapping $w : [0, L] \to [0, L]$ (left) and by the graph of the function w (right).

(Hooke's) constant k does not only depend on the stretching/elastic properties of the spring but also on the length of the spring. For the given spring define a materially dependent constant **elasticity** *by $\epsilon = kL$. Model the tension force in the spring in terms of the elasticity. You may take MKS units with $g = 9.8$ m/s^2 and any specific values $\rho_0 > 0$ and $k > 0$ for your plots.*

(b) *The spring has one end at a height H and the other end fixed at a height $H - M$ and thus "sags" due to the gravitational acceleration. Note: You may have to adjust the physical constants ρ_0, L, H, M, and k to get something physically reasonable.*

1.1 Elasticity model

Restricting to homogeneous extensions and compressions we can replace[2] the ODE (3) with

$$m\ddot{\xi} = -\epsilon\left(\frac{\xi}{\ell} - 1\right).$$

If the elasticity ϵ is a constant, this gives undamped oscillatory motion

$$\xi(t) = \ell + [\xi(0) - \ell]\cos\left(\sqrt{\frac{\epsilon}{m\ell}}\, t\right) + \dot{\xi}(0)\sqrt{\frac{m\ell}{\epsilon}}\ \sin\left(\sqrt{\frac{\epsilon}{m\ell}}\, t\right)$$

in agreement with the ODE analysis of (3).

More generally, the force relation

$$F = -\epsilon\left(\frac{\xi}{L} - 1\right) \tag{4}$$

may be interpreted to correspond to a homogeneous deformation function $w : [0, L] \to \mathbb{R}$ with $\xi = w(L)$. In this case,

$$w(x) = \frac{\xi}{L}\, x$$

and (4) can be written in the suggestive form

$$F = -\epsilon(w_x - 1). \tag{5}$$

[2]Here we have changed the symbol used for the end position of the spring.

Note that (5) is apparently applicable at any point $w(x) = w(x,t)$ along the homogeneous extension or compression giving (at least with the neglect of inertial forces, i.e., forces arising from nonzero velocities) the tension force in the direction to the left due to the spring. In particular, if we take this force to be opposite in direction and equal in magnitude to the weight of the spring to the right of $w(x)$ with respect to a right pointing (constant gravitational) acceleration even for inhomogeneous deformations, then we obtain a solution for part (a) of Exercise 1 from a force balance. More precisely, assuming a constant mass density ρ_0 for the equilibrium position of the spring we can write

$$\epsilon(w_x - 1) = m(x)g = \rho_0(L - x)g.$$

Thus the corresponding inhomogeneous extension to the right is given by,

$$w = w(x) = \left(1 + \frac{\rho_0 g}{\epsilon} L\right) x - \frac{\rho_0 g}{2\epsilon} x^2 \tag{6}$$

as illustrated on the left in Figure 4. In order to obtain a more natural orientation, we can write $y = -w(x)$ and plot the parametric representation of position given by $\gamma(x) = (0, y(x))$ as indicated on the right in Figure 4. Notice that the formula (6) gives the final length of the hanging spring

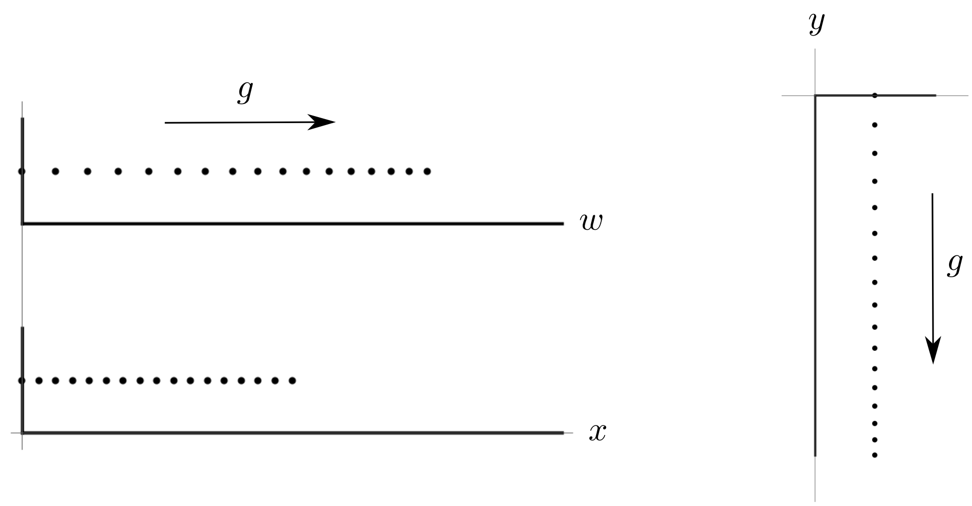


Figure 4: Constant elasticity model of a spring hanging under its own weight. Horizontal coordinates (left). Vertical coordinate (right).

$$y(L) = -L - \frac{\rho_0 g}{2\epsilon} L^2.$$

The solution of part (b) of Exercise 1 is not as straightforward using a force balance as we have done here for the case of a free end. We will introduce an alternative method to solve part (a) below, and this approach will give a straightforward solution for part (b).

1.2 Density and density relations

We have already used above a constant lineal density model for an inhomogeneously deformed one-dimensional elastic continuum (i.e., spring). In this model, the mass of the deformed spring between $w(a)$ and $w(b)$, corresponding to the equilibrium interval $[a,b] \subset [0,L]$, is simply $\rho_0(b-a)$. We also mentioned the essential independence of forces arising from the elastic properties of the continuum and the forces related to mass, either from a field force like gravity or from inertia/motion. More generally, for homogeneous extensions (and possibly compressions) as illustrated in Figure 1 it is natural to attach a mass density to a particular extension or compression as follows: Beginning with the equilibrium configuration modeled on the interval $[0, L]$, we assume a lineal density function $\rho : [0, L] \to [0, \infty)$ with the mass between a and b (in equilibrium) given by

$$m = \int_a^b \rho(x)\, dx. \tag{7}$$

Assuming homogeneous extension, the density $\delta : [0, w(L)] \to [0, \infty)$ along the deformed spring should simply be proportional to the equilibrium density:

$$\delta_{\text{hom}}(p) = \frac{L}{w(L)}\, \rho(w^{-1}(p)).$$

When $w(L) > L$, corresponding to extension, then the density in the image at $w(x)$ is smaller than the corresponding equilibrium density $\rho(x)$, and when $w(L) < L$, corresponding to compression, then the density value $\delta(w(x))$ is greater than $\rho(x)$.

Again, we note/recall that for a homogeneous extension

$$w(x) = \frac{w(L)}{L}\, x,$$

so the scaling factor for the density may be expressed in terms of the derivative:

$$\delta_{\text{hom}}(p) = \frac{\rho(w^{-1}(p))}{w'(w^{-1}(p))}. \tag{8}$$

Essentially the same reasoning applies to an inhomogeneous deformation of a spring with inhomogeneous equilibrium density ρ by a change of variables. The mass of the deformed spring between $w(a)$ and $w(b)$ is still given by the integral (7) and can be expressed as an integral over the deformed portion by a change of variables:

$$m = \int_{w(a)}^{w(b)} \rho \circ w^{-1}(p)\, \frac{1}{w' \circ w^{-1}(p)}\, dp. \tag{9}$$

Notice the scaling factor $1/w' = 1/w' \circ w^{-1}(p)$ associated with the change of variables. When $w' > 1$, corresponding to extension, the integrand in (9) decreases, and when $w' < 1$, corresponding to compression, the integrand in (9) increases. You should convince yourself that this is as it should be. In view of the change of variables formula (9) for the mass we can define a density $\mu : [0, L] \to [0, \infty)$ by

$$\mu(x) = \frac{\rho(x)}{w'(x)}$$

corresponding to the density in the image at $w(x)$ with respect to the deformation w. The density μ may be considered a kind of **material density**. To make this terminology more precise let is imagine the deformation w is obtained as the terminal deformation of a (smooth) family of deformations

$$W : [0, L] \times [0, T] \to [0, \infty)$$

starting from equilibrium. This means

(i) For each fixed $t \in [0, T]$, the function $\phi : [0, L] \to [0, \infty)$ by $\phi(x) = W(x, t)$ is an admissible deformation, i.e., $W(0, t) = 0$ and $W_x(x, t) > 0$,

(ii) $W(x, 0) = x$, and

(iii) $W(x, T) = w(x)$.

Now we can consider the image of each intermediate deformation

$$\Gamma = \Gamma(t) = \{\phi(x) : x \in [0, L]\} = [\phi(0), \phi(L)] = \{W(x, t) : x \in [0, L]\} = [W(0, t), W(L, t)]$$

and a density function $\delta : \Gamma \times [0, T] \to [0, \infty)$ with $\delta(p) = \delta(p, t)$ giving the density in the material image Γ at the point $p = \phi(x) = W(x, t)$. We have then

$$\mu(x, t) = \frac{\rho(x)}{W_x(x, t)} = \delta_{\text{inhom}}(\phi(x), t) = \delta_{\text{inhom}}(W(x, t), t).$$

Thus, $\mu = \mu(x, t)$ gives the density as x moves along the material flow. We have used t as the symbol for the parameter with respect to which a one-dimensional continuum "flows" from equilibrium to a given deformation w on an interval $[0, T]$. This all suggests, of course, that t represents a time variable and T is a terminal time. In fact, many other parameters can be used for a continuous family of deformations. One of the most useful and common alternative choices for such a variable is the parameter in a convex combination.

Exercise 2 *Let*

$$\rho : [0, 2] \to [0, \infty) \qquad \text{by} \qquad \rho(x) = 2 - (x - 1)^1$$

represent the equilibrium density of a spring which is heavier in the middle than on the ends. Consider the inhomogeneous deformation

$$w : [0, 2] \to [0, 4] \qquad \text{by} \qquad w(x) = x^2.$$

Realize the deformation w as the terminal state of a continuous deformation

$$W : [0, 2] \times [0, 1] \to [0, \infty) \qquad \text{by} \qquad W(x, s) = (1 - s)x + sw(x).$$

Use the plotting of appropriately spaced points to represent the following:

(a) *The mass density of the equilibrium configuration.*

(b) *The mass density δ in the image under the deformation w.*

(c) *The deformation and corresponding mass density at each "time" s as an animation.*

Calculate and plot the material density $\mu : [0, 2] \times [0, 1] \to [0, \infty)$ by

$$\mu(x, s) = \frac{\rho(x)}{W_x(x, s)} = \delta((1 - s)x + sw(x), s).$$

1.3 Elastic force and potential energy

We return to the suggestive comparison between (4) and (5) above and adopt the latter as a general model for inhomogeneous deformations. Perhaps a simple way to view the overall objective of this section is in the following terms:

1. We wish to give an alternative solution of Exercise 1 part (a) based on potential energy and which adapts easily to part (b) of the same exercise.

2. The underlying idea is that elastic potential energy may be computed by integration of an appropriate **potential energy density** along a deformed spring.

We begin with a brief review of the elastic force relations with some change of notation: If a simple spring with Hooke's constant k and length ℓ is deformed homogeneously to have length ℓ', then the force (at the end and the tension force throughout) is given by

$$F_{\text{hom}} = -k(\ell' - \ell) = -k\ell\left(\frac{\ell'}{\ell} - 1\right) = -\epsilon\left(\frac{\ell'}{\ell} - 1\right).$$

If this homogeneous deformation is realized by $w : [0, \ell] \to [0, \ell']$, then

$$w(x) = \frac{\ell'}{\ell}\, x,$$

and we can write

$$F_{\text{hom}} = -\epsilon(w' - 1). \tag{10}$$

We henceforth assume for any deformation that the tension force (to the left) along a deformed spring determined by $w : [0, L] \to [0, \infty)$ with $w(0) = 0$ and $w' > 0$ is given by

$$F_{\text{hom}} = -\epsilon(w' - 1) \tag{11}$$

where $\epsilon > 0$ is a material constant with units of force called the **elasiticy**. In this definition, the elasticity may also be assumed to depend on $x \in [0, L]$ or also other quantities, but for the moment we will keep things simple by assuming constant elasticity.

Recall that potential energy is generally determined by "force $\times$ distance" with respect to an "assembling" motion, and a length integral required if the force varies along that motion. For example, in extending the end of a spring from a position $x > \ell$ to a position $x + \Delta x$, with $\Delta x > 0$ considered small, the work required (and the accumulated potential energy) is approximately

$$\Delta E_{\text{hom}} = k(x^* - \ell)\Delta x$$

where x^* is some (or any) point with $x < x^* < x + \Delta x$. Thus, if we extend the same spring from length ℓ to length $\ell' > \ell$, we can approximate the potential energy of the extended spring by taking a partitiion

$$\ell = x_0 < x_1 < x_2 < \cdots < x_m = \ell'$$

with points $x_j^* \in (x_j, x_{j+1})$ for $j = 0, 1, 2, \ldots, m-1$ and adding up the approximate energies for each small step in the (assemblying) motion from length ℓ to length ℓ':

$$\Delta E_j \approx k(x_j^* - \ell)(x_{j+1} - x_j), \qquad E_{\text{hom}} \approx \sum_{j=0}^{m-1} k(x_j^* - \ell)(x_{j+1} - x_j).$$

This Riemann sum limits to

$$E_{\text{hom}} = k\int_{\ell}^{\ell'} (x - \ell)\, dx = \frac{k}{2}\,(\ell' - \ell)^2.$$

Notice that the value is positive even if $\ell' < \ell$ corresponding to a compression.

The same computation (in the homogeneous case) may be executed using the elasticity $\epsilon = k\ell$:

$$\Delta E_{\text{hom}} \approx \epsilon\left(\frac{x^*}{\ell} - 1\right)\Delta x,$$

$$\Delta E_j = \epsilon\left(\frac{x_j^*}{\ell} - 1\right)(x_{j+1} - x_j), \qquad E_{\text{hom}} \approx \sum_{j=0}^{m-1} \epsilon\left(\frac{x_j^*}{\ell} - 1\right)(x_{j+1} - x_j),$$

and

$$E_{\text{hom}} = \epsilon \int_{\ell}^{\ell'} \left(\frac{x}{\ell} - 1\right) dx = -\frac{\epsilon\ell}{2}\left(\frac{\ell'}{\ell} - 1\right)^2.$$

Finally, we introduce the crucial idea: We wish to express this potential energy as an integral, either over the equilibrium interval $[0, \ell]$ or over $[0, \ell']$ the deformation interval, of a **potential energy density**. Thus, we can write for example

$$E_{\text{hom}} = \frac{\epsilon}{2}\int_0^{\ell} \left(\frac{\ell'}{\ell} - 1\right)^2 dx$$

or

$$E_{\text{hom}} = \frac{\epsilon}{2}\int_0^{\ell'} \left(\frac{\ell'}{\ell} - 1\right)^2 \frac{\ell}{\ell'}\, dp.$$

Notice that the potential energy density is given on $[0, \ell]$ by

$$\mu_E = \frac{\epsilon}{2}\left(\frac{\ell'}{\ell} - 1\right)^2$$

and on the image $[0, \ell']$ by

$$\delta_E = \frac{\epsilon}{2}\left(\frac{\ell'}{\ell} - 1\right)^2 \frac{\ell}{\ell'}$$

At this point, we can jump to the assumption that these potential energy densities for the homogeneous case adapt to the inhomogeneous case in the form(s)

$$\mu_E = \frac{\epsilon}{2}(w' - 1)^2$$

and on the image $[0, \ell']$ by

$$\delta_E = \frac{\epsilon}{2}(w' - 1)^2 \frac{1}{w'}$$

where $w : [0, L] \to [0, \infty)$ gives an inhomogeneous deformation and in the expression for δ_E in the image $w' = w' \circ w^{-1}$. If these expressions are accepted, the elastic potential energy in a spring configuration may be written as

$$E_{\text{inhom}} = \frac{\epsilon}{2}\int_0^L (w' - 1)^2\, dx$$

or (by a change of variables)

$$E_{\text{inhom}} = \frac{\epsilon}{2}\int_0^{w(L)} (w' \circ w^{-1}(p) - 1)^2 \frac{1}{w' \circ w^{-1}(p)}\, dp$$

Exercise 3 *Obtain the expression(s) for E_{inhom} from the beginning using a Riemann sum approximation involving the approximate potential energy increment*

$$\Delta E_{\text{inhom}} \approx \epsilon\,(W_x(x^*, t) - 1)\,\Delta x$$

for an appropriate assemblying motion $W = W(x, t)$.

In the presence of an ambient field of acceleration each configuration of the spring, including the equilibrium configuration, may be assigned a potential energy. The computation of this energy, as above, is obtained by considering assembly of the configuration from small pieces at some reference location. In the case of gravity, we can imagine each incremental piece $[x_j, x_{j+1}]$ or $[w(x_j), w_{j+1}]$ of the spring is located at some reference position $x = B$ and is translated, against the acceleration field g, to the position $x = w(x_j^*)$. The incremental mass is approximately

$$\Delta M_j \approx \rho(x_j^*)\Delta x_j = \rho(x_j^*)(x_{j+1} - x_j)$$

or

$$\Delta M_j = \delta(w(x_j^*))\Delta w_j = \delta(w(x_j^*))(w(x_{j+1}) - w(x_j)) \approx \frac{\rho(x_j^*)}{w' \circ w(x_j^*)}(w(x_{j+1}) - w(x_j)).$$

The corresponding force is constant, so the incremental potential energy is easy to compute:

$$\Delta E_g \approx -\int_B^{w(x_j^*)} \rho(x_j^*)(x_{j+1} - x_j)g = -(w(x_j^*) - B)\rho(x_j^*)g(x_{j+1} - x_j).$$

Notice we have taken the sign in accordance with our deformations w with $w' > 0$ so that larger extensions to the right lower the energy. This corresponds to a gravitational acceleration g in the positive w direction.

The Riemann sum approximating the potential energy of the entire final configuration is

$$E_g \approx -g \sum_{j=0}^{m-1} \rho(x_j^*)(w(x_j^*) - B)(x_{j+1} - x_j).$$

and the limiting value is

$$E_g = -g \int_0^L \rho(w - B)\, dx = -g \int_0^L \rho w\, dx + B \int_0^L \rho = -g \int_0^L \rho w\, dx + BM \tag{12}$$

where M is the total mass of the spring.

Exercise 4 *Express E_g as an integral over the image interval $[0, w(L)]$ and the linear mass density δ of the deformation.*

Exercise 5 *Express E_g as an integral over the equilibrium interval $[0, L]$ in terms of the material density $\mu = \mu(x)$ of the deformation.*

Assuming $\rho = \rho_0$ is constant at equilibrium, then the expression (12) for the gravitational potential energy of a deformed configuration simplifies to

$$E_g = -g\rho_0 \int_0^L w\, dx + BM.$$

We now claim that an alternative solution of Exercise 1 part (a) may be obtained under the assumption that the observed configuration is that which minimizes the total potential energy

$$E = E_{\text{inhom}} + E_g.$$

Recognizing $E = E[w]$ as a functional on $\mathcal{W} = \{w \in C^2[0, L] : w(0) = 0, w' > 0\}$ we may use the calculus of variations to find the minimizer. Furthermore, the constant term MB from E_g makes no difference in the minimization problem, so we can proceed to compute the first variation of the functional

$$\int_0^L \left[\frac{\epsilon}{2}(w' - 1)^2 - \rho_0 g w\right] dx.$$

Recall that the relevant computation is

$$\frac{d}{d\alpha} \int_0^L \left[\frac{\epsilon}{2}(w' + \alpha\phi' - 1)^2 - \rho_0 g(w + \alpha\phi)\right] dx = \int_0^L [\epsilon(w' + \alpha\phi' - 1)\phi' - \rho_0 g\phi]\, dx$$

where ϕ is an admissible perturbation. Restricting to smooth perturbations ϕ with compact support in $(0, L)$, we find

$$\delta E_w[\phi] = \int_0^L [-\epsilon w'' - \rho_0 g]\, \phi.$$

An extremal for the problem, therefore, satisfies the ODE

$$w'' = -\frac{\rho_0 g}{\epsilon}. \tag{13}$$

This implies, subject to admissiblity considerations, that

$$w(x) = -\frac{\rho_0 g}{2\epsilon}\, x^2 + w'(0)\, x. \tag{14}$$

At this point, we may observe that the leading order term agrees with our solution (6) derived from force balance at each point using the inhomogeneous deformation force formula and the mass hanging below. We may, furthermore, apply the force balance condition at the single point $w(0) = 0$, namely

$$\epsilon(w'(0) - 1) = Mg = \rho_0 g L$$

to obtain

$$w'(0) = 1 + \frac{\rho_0 g}{\epsilon}\, L$$

so that our variational solution (14) is in complete agreement with (6).

The second order ODE (13) arising in our variational approach has other solutions. In particular, if we restrict to admissible solutions

$$\mathcal{W}_0 = \{w \in C^2[0, L] : w(0) = 0, w(L) = L, w' > 0\}$$

with fixed endpoints, then the condition $w(L) = L$ applied to (14) implies

$$-\frac{\rho_0 g}{2\epsilon}\, L^2 + w'(0)\, L = L$$

so that

$$w'(0) = 1 + \frac{\rho_0 g}{2\epsilon}\, L$$

and (14) becomes

$$w(x) = -\frac{\rho_0 g}{2\epsilon}\, x^2 + \left(1 + \frac{\rho_0 g}{2\epsilon}\, L\right)\, x. \tag{15}$$

This function will not always be admissible. Note that

$$w'(x) = -\frac{\rho_0 g}{\epsilon}\, x + 1 + \frac{\rho_0 g}{2\epsilon}\, L.$$

Therefore,

$$w'(L) = 1 - \frac{\rho_0 g}{2\epsilon}\, L.$$

This quantity is negative, in violation of the admissibility condition $w' > 0$, when

$$\frac{\rho_0 g}{2\epsilon}\, L > 1.$$

In particular, if the product of the density and the length of the spring is too large, or the elasticity is too small, then the model breaks down with the point of greatest deformation to the right falling beyond $w(L) = L$. Translating this observation to the vertical coordinate $y = -w(x)$, we obtain that the lowest point of the deformation is a point from the interior of the spring which falls below the level $y = -L$. This is not entirely unexpected.

We make three further observations about the solution of part (b), the "sagging" spring. First, if one differentiates w twice one obtains

$$w_{xx} = -\frac{\rho_0 g}{\epsilon}$$

which marks w as an equilibrium solution of the forced wave equation

$$\frac{\rho_0}{\epsilon} w_{tt} = w_{xx} + \frac{\rho_0 g}{\epsilon}$$

or

$$w_{tt} = \frac{\epsilon}{\rho_0} w_{xx} + g$$

with constant positive forcing of magnitude g. As a matter of fact, w is also an equilibrium solution for the forced heat equation

$$w_t = w_{xx} + \frac{\rho_0 g}{\epsilon}$$

with the same forcing, though that is not immediately relevant here.

To put our physical problem in more familiar coordinates, we set $y = -w$ and have

$$y(x) = \frac{\rho_0 g}{2\epsilon} x^2 - \left(1 + \frac{\rho_0 g}{2\epsilon} L\right) x$$

so that $y'' = \rho_0 g/\epsilon$ and y is an equilibrium solution of the forced PDE

$$\rho_0 \, y_{tt} = \epsilon \, y_{xx} - \rho_0 g. \tag{16}$$

Finally, setting $u_E(x) = y(x) - x$, we get the sagging solution Haberman has in mind in his Problem 4.2.1. This gives a third method to solve part (b) of Exercise 1: We can recognize the solution as an equilibrium solution for a forced wave equation (16) with boundary values $y(0) = 0$ and $y(L) = L$. Of course, we do not yet have a derivation of the wave equation nor any clear reason to believe this PDE is related to the inhomogeneous deformation of a spring, but we will have those things soon.

Exercise 6 *Use the equilibrium approach (equilibrium for the wave equation) to solve part (a) of Exercise 1. See Haberman's Problem 4.4.1.*

2 Derivation(s) of the wave equation

We have now developed an adequate model of the elastic tension force in a one-dimensional continuum to apply Newton's second law and obtain an equation of motion. One additional ingredient is required for this derivation which we will introduce momentarily.

2.1 Kenematics

Kinematics refers generally to Isaac Newton's theory of "particle motion" in which a "particle" or point mass of (fixed) mass M moves under the influence of any forces applied to the particle according to the well known relation/equation

$$\mathbf{f} = M\mathbf{a} = m\ddot{\mathbf{x}}$$

where $\ddot{\mathbf{x}}$ is the acceleration vector for the motion given by $\mathbf{x} : (0, T) \to \mathbb{R}^n$ having $\mathbf{x} = \mathbf{x}(t)$ (with respect to the **time** parameter t) and $\mathbf{f}$ is the vector sum of all the forces acting on the particle which may depend directly on time t through, for example, field forces and indirectly on time through the motion itself in the form of the position $\mathbf{x}$ and the velocity $\dot{\mathbf{x}}$, but presumably not on higher order derivatives of $\mathbf{x}$ nor indirectly on time through change in the mass of the particle. This is, of course, known as **Newton's second law**, and most derivations of the wave equation invoke this relation in some form.[3] It is usually not pointed out that the relation cannot be properly used in this context directly but some fundamentally modified form is required. We will call such a modified form a **kinematical continuum assumption** or just a continuum assumption for short.

2.2 Point masses and center of mass

The following discussion is usually given as a justification for certain aspects of the continuum assumptions below at least in the case of rigid bodies with continuously distributed mass. Let us specialize Newton's law to one dimension and a system of m masses M_0, M_1, M_2,..., M_m located at positions $x_0 < x_1 < x_2 < \cdots < x_m$ with $x_j = x_j(t)$ for $j = 0, 1, 2, \ldots, m$. If the first and last mass experience (external) forces $-f_0$ and f_m respectively and each mass M_j exerts a force F_{ij} on mass M_i with $F_{ij} + F_{ji} = 0$, then

$$M_0\ddot{x}_0 = -f_0 + \sum_{j=1}^{m} F_{0j},$$

$$M_m\ddot{x}_m = f_m + \sum_{j=1}^{m} F_{mj},$$

and

$$M_i\ddot{x}_i \sum_{j\neq i} F_{ij}.$$

Summing these relations we obtain

$$\sum_{i=0}^{m} M_i\ddot{x}_i = f_m - f_0$$

which implies the traditional conclusion

$$M\ \ddot{x}_{\rm cm} = f_m - f_0 \tag{17}$$

where

$$M = \sum_{j=1}^{m} M_j \qquad \text{is the total mass}$$

and

$$x_{\rm cm} = \frac{1}{M}\sum_{j=1}^{m} M_j x_j \qquad \text{is the center of mass.}$$

Exercise 7 *How does the argument/discussion above change if each mass M_j only exerts a force on each mass M_i adjacent to M_j in the line of masses?*

For mass continuously distributed along an interval according to a density function $\rho : [0, L] \to (0, \infty)$, the notion of **center of mass** generalizes is a reasonably straightforward manner: We can consider a partition of the interval $0 = x_0 < x_1 < \cdots < x_m = L$ and the masses

$$M_j = \int_{x_j}^{x_{j+1}} \rho(x)\,dx = \rho(x_j^*)(x_{j+1)} - x_j) \qquad \text{for } j = 0, 1, 2, \ldots, m-1.$$

[3] See Haberman for example, though almost any book on partial differential equations treating the wave equation may be consulted as well.

Here the point x_j^* is given by the mean value theorem for integrals and satisfies $x_j < x_{j+1} < x_j$. We can then say that for any points ξ_j^* with $x_j \le \xi_j^* < x_{j+1}$ for $j = 0, 1, 2, \ldots, m$

$$\sum_{j=0}^{m-1} M_j \xi_j^* = \sum_{j=0}^{m-1} \rho(x_j^*)(x_{j+1)} - x_j)\xi_j^*,$$

and

$$x_{\mathrm{cm}} = \lim \frac{\displaystyle\sum_{j=0}^{m-1} M_j \xi_j^*}{\displaystyle\sum_{j=0}^{m-1} M_j} = \frac{\displaystyle\int \rho(x)\, x\, dx}{\displaystyle\int_0^L \rho(x)\, dx}. \tag{18}$$

Here the limit is taken as $\max\{x_{j+1} - x_j : j = 0, 1, 2, \ldots, m\}$ tends to zero.

Exercise 8 *The usual Riemann sum for*

$$\int x\, \rho(x)\, dx$$

would be

$$\sum_{j=0}^{m-1} x_j^* \rho(x_j^*)(x_{j+1)} - x_j).$$

Show that the sum

$$\sum_{j=0}^{m-1} \xi_j^* \rho(x_j^*)(x_{j+1)} - x_j)$$

gives the same limiting integral.

We can generalize the definition (18) also for an arbitrary deformation of the interval $[0, L]$ as follows: Let $w : [0, L] \to \mathbb{R}$ with $w' > 0$. We define, as above, a material density $\mu : [0, L] \to (0, \infty)$ and a deformation density $\delta : [w(0), w(L)] \to (0, \infty)$ satisfying

$$\mu(x) = \delta(w(x)) = \frac{\rho(x)}{w'(x)}.$$

Then the center of mass of the deformed interval $[w(0), w(L)]$ is

$$p_{\mathrm{cm}} = \frac{1}{M} \int_{w(0)}^{w(L)} p\delta(p)\, dp = \frac{1}{M} \int_0^L \rho(x)\, w(x)\, dx.$$

Note that we have changed variables using $p = w(x)$ so that $dp = w'(x)\, dx$, and here

$$M = \int_0^l \rho(x)\, dx = \int_{w(0)}^{w(L)} p\delta(p)\, dp$$

as usual.

Exercise 9 *Generalize the discussion of center of mass to rigid bodies and deformations in $\mathbb{R}^3$:*

(a) *Begin with point masses M_1, M_2, $\ldots M_m$ in $\mathbb{R}^3$ located at points $\mathbf{x}_1$, $\mathbf{x}_2$, $\ldots \mathbf{x}_m$.*

(b) *Partition a given volume $\mathcal{V} \subset \mathbb{R}^3$ into volumes $\mathcal{V}_1$, $\mathcal{V}_2$, $\ldots$, $\mathcal{V}_m$ and assume mass within $\mathcal{V}$ is determined by integrating a continuous density function $\rho : \mathcal{V} \to (0, \infty)$. Find a formula for the center of mass of $\mathcal{V}$.*

(c) *Consider a deformation* $\mathbf{w} : \mathcal{V} \to \mathbb{R}^3$ *of* $\mathcal{V}$ *to* $\mathcal{W} = \mathbf{w}(\mathcal{V})$ *with* $D\mathbf{w} > 0$*. Define a material density* μ *and a deformation density* δ *and find the formula for the center of mass of* $\mathcal{W}$*.*

I would like to give you a proof, at least for rigid bodies, of the assertion

$$M\ddot{\mathbf{x}}_{\text{cm}} = \mathbf{f}. \tag{19}$$

I would even be happy to obtain such an assertion for rigid intervals in $\mathbb{R}^1$, but I do not know any very convincing way to do it. Usually, (19) is taken as a kind of axiom for rigid body motion in analogy with the formula (17), so I guess no one else knows how to prove (19) from more elementary considerations, like as a limit of Riemann sums, either.

Let us finish this section with the mention (or recollection) of two generalizations of Newton's second law for particle motion. The first applies to particles with changing mass $M = M(t)$ and is sometimes called the **momentum form** of Newton's second law:

$$\frac{d}{dt}[M\mathbf{v}] = \frac{d}{dt}[M\dot{\mathbf{x}}] = M\ddot{\mathbf{x}} + \dot{M}\dot{\mathbf{x}} = \mathbf{f}.$$

This form has been applied to, for example, rockets treated as point masses with mass decreasing in time due to the burning of fuel. Furtherfore, if the force is constant, then the change in momentum corresponding to a time increment $[t, t + \Delta t]$ is given by

$$M\mathbf{v}(t + \Delta t) - M\mathbf{v}(t) = \mathbf{f}\Delta t.$$

Naturally, this equation becomes an approximation if $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$:

$$M\mathbf{v}(t + \Delta t) - M\mathbf{v}(t) \approx \mathbf{f}(\mathbf{x}, t^*)\Delta t.$$

where t^* is some time between t and $t + \Delta t$, and this approximate equation holds in precisely the sense that

$$\frac{d}{dt}[M\mathbf{v}] = \lim_{\Delta t \searrow 0} \frac{M\mathbf{v}(t + \Delta t) - M\mathbf{v}(t)}{\Delta t} = \lim_{\Delta t \searrow 0} \mathbf{f}(\mathbf{x}, t^*) = \mathbf{f}(\mathbf{x}, t).$$

Exercise 10 *Show that for* m *point masses* M_1*,* M_2*,...,* M_m *with time dependent positions* $\mathbf{x}_j$ *and velocities* $\mathbf{v}_j = \dot{\mathbf{x}}_j$ *for* $j = 1, 2, \ldots, m$*, the total momentum satisfies the approximate equation*

$$\sum_{j=1}^{m} [M_j\mathbf{v}_j(t + \Delta t) - M_j\mathbf{v}_j(t)] \approx \sum_{j-1}^{m} \mathbf{f}_j(\mathbf{x}, t_j^*)\Delta t$$

where $\mathbf{f}_j$ *is a force acting on the mass* M_j *and* t_j^* *is some time between* t *and* $t + \Delta t$ *for* $j = 1, 2, \ldots, m$*. Note: You can allow also interaction forces between the masses as long as the are equal in magnitude and opposite in sign. Conclude that*

$$\frac{d}{dt} \sum_{j=1}^{m} M_j\mathbf{v}_j(t) = \sum_{j-1}^{m} \mathbf{f}_j(\mathbf{x}, t).$$

The second is a much more far-reaching generalization sometimes called **Hamilton's action principle** or the **principle of least action**. This principle asserts that physical motions of particles, among all possible motions, are those which are extremal for the **action functional**

$$\mathcal{A}[\mathbf{x}] = \int_0^T \left[\Phi(\mathbf{x}, t) - \frac{1}{2}m\,|\mathbf{v}|^2\right]$$

where Φ is a **potential function** for a (force) field $\mathbf{f} = \mathbf{f}(\mathbf{x}, t)$, that is,

$$\mathbf{f}(\mathbf{x}, t) = D\Phi(\mathbf{x}, t)$$

where D represents the spatial gradient as usual.

Exercise 11 *Derive Newton's second law by taking the first variation of the action functional.*

Exercise 12 *Determine potential functions for*

(a) *a downward constant gravitational acceleration* g*, and*

(b) *an inverse square gravitational force.*

2.3 First derivation

We consider a small portion $[w(x_j,t), w(x_{j+1},t)]$ of the deformed continuum corresponding to an interior equilibrium interval $[x_j, x_{j+1}]$ determined by a partition of $[0, L]$ as above at a particular time t. The elastic force to the left at $w(x_j,t)$ is

$$F_j = -\epsilon(w_x(x_j,t) - 1)$$

and the elastic force to the right at $w(x_{j+1},t)$ is

$$F_{j+1} = \epsilon(w_x(x_{j+1},t) - 1).$$

The sum of these two forces is

$$\Delta F_j = \epsilon(w_x(x_{j+1},t) - w_x(x_j,t)).$$

Due to the fact that our continuum is not rigid, but being deformed in time we need an extension of Newton's second law along the following lines:

Continuum assumption A The sum of the tension forces on the two ends of a deformed interval causes acceleration of some point within the continuum with respect to the total mass. In particular, for a deformed interior increment $[w(x_j,t), w(x_{j+1},t)]$ there exists some $x_j^* \in (x_j, x_{j+1})$ for which

$$M_j\, w_{tt}(x_j^*, t) = \epsilon w_x(x_{j+1},t) - \epsilon w_x(x_j,t) \tag{20}$$

where

$$M_j = \int_{x_j}^{x_{j+1}} \rho(x)\, dx \qquad \text{for } j = 0, 1, 2, \ldots, m-1.$$

Recall that we assumed the sum of the forces on a rigid or deformed continuously distributed body causes acceleration by the sum of the forces on the **center of mass**

$$m\ddot{\mathbf{x}}_{\text{cm}} = \mathbf{f}.$$

This assumption was made in anology with the corresponding result for point masses. The assumption we are making here is made in further analogy with these assertions. Note that it is not required here that $w(x_j^*, t)$ be the center of mass of $[w(x_j), w(x_{j+1})]$ though that conclusion can be proved for motions w that satisfy the wave equation, as we will see later. We could, of course, make this assumption, but the assumption we have taken is more general.

Dividing both sides of (20) by the length increment $x_{j+1} - x_j$ and taking the limit is this increment tends to zero with x_j and x_{j+1} both tending to a common value x, we see the average value of the integral on the left tends to the equilibrium density $\rho(x)$ at x so that

$$\rho(x) w_{tt} = \epsilon w_{xx}.$$

This is, of course, the one-dimensional wave equation for the deformation w.

Exercise 13 *Generalize the derivation above to the case where* $\epsilon = \epsilon(x)$ *is a position dependent elasticity.*

2.4 Second derivation

A different derivation may be given, having the same flavor as the one above, under a different kinematical continuum assumption. Like the center of mass, the notion of **momentum** generalizes in a straightforward manner to continuous distributions. Recall that the momentum of the m point mass system is

$$\sum_{j=1}^{m} M_j \dot{x}_j(t).$$

Thus, we may approximate the momentum of a deformed partition interval $[w(x_j,t), w(x_{j+1},t)]$ by

$$M_j w_t(x_j^*, t)$$

where

$$M_j = \int_{w(x_j,t)}^{w(x_{j+1},t)} \delta(p)\, dp = \int_{x_j}^{x_{j+1}} \rho(x)\, dx = \rho(\xi_j^*)(x_{j+1} - x_j)$$

where again x_j^* and ξ_j^* are some points in (x_j, x_{j+1}). Summing these terms for $j = 0, 1, 2, \ldots, m-1$, the total momentum for the deformed interval $(w(a,t), w(b,t))$ at time t is approximately

$$\sum_{j=0}^{m-1} \rho(\xi_j^*) w_t(x_j^*, t)(x_{j+1} - x_j)$$

so that taking the limit as the maximum partition interval length tends to zero, we get

$$\int_a^b \rho(x)\, w_t(x,t)\, dx$$

for the total momentum.

Continuum assumption B The sum of the tension forces on the two ends of a deformed interval satisfies the momentum form of Newton's second law in the sense that for a deformed increments $[w(x_j, t+\Delta t), w(x_{j+1}, t+\Delta t)]$ and $[w(x_j,t), w(x_{j+1},t)]$ there exist times t^* and t^{**} between t and $t+\Delta t$ for which the change in momentum is given approximately by

$$\int_a^b \rho(x)\, w_t(x, t+\Delta t)\, dx - \int_a^b \rho(x)\, w_t(x,t)\, dx \approx [\ \epsilon[w_x(b,t^*) - 1] - \epsilon[w_x(a,t^{**}) - 1]\ ]\, \Delta t$$

so that

$$\frac{d}{dt} \int_a^b \rho(x)\, w_t(x,t)\, dx = \int_a^b \rho(x)\, w_{tt}(x,t)\, dx = \epsilon[w_x(b,t) - w_x(a,t)]. \tag{21}$$

Dividing by $b-a$ in (21) and letting the interval (a,b) shrink to a particular point x_0, we get

$$\rho(x_0)\, w_{tt}(x_0,t) = \lim_{(a,b)\to\{x_0\}} \frac{1}{b-a} \int_a^b \rho(x)\, w_{tt}(x,t)\, dx = \lim_{(a,b)\to\{x_0\}} \epsilon \frac{w_x(b,t) - w_x(a,t)}{b-a} = \epsilon w_{xx}(x_0,t).$$

That is,

$$\rho w_{tt} = \epsilon w_{xx}.$$

Exercise 14 *Adapt this derivation to the case where the elasticity $\epsilon = \epsilon(x)$ depends on the spatial variable.*

2.5 Third derivation

This is, in a certain sense, the most satisfactory derivation. We need here an expression for the **total action in space and time** for a deforming motion $w = w(x,t)$ as above. We have discussed the total potential energy, and the expression at a given time is

$$\frac{\epsilon}{2}\int_0^L [w_x(x,t)-1]^2\,dx.$$

Thus, the total potential energy in space and time is given by the iterated integral

$$E = \frac{\epsilon}{2}\int_0^T\int_0^L [w_x(x,t)-1]^2\,dx\,dt.$$

The kinetic energy of a deformed incremental interval $[w(x_j,t), w(x_{j+1},t)]$ is approximately,

$$\Delta K_j = \frac{1}{2}\,M_j\,[w_t(x_j^*,t)]^2$$

where

$$M_j = \int_{x_j}^{x_{j+1}} \rho(x)\,dx = \rho(\xi_j^*)(x_{j+1}-x_j)$$

and x_j^* and ξ_j^* are some points in the interval (x_j, x_{j+1}) as usual. Thus, the total kinetic energy at time t is

$$\frac{1}{2}\int_0^L \rho(x)\,[w_t(x,t)]^2\,dx$$

and the total kinetic energy for the motion in space and time is

$$\frac{1}{2}\int_0^T\int_0^L \rho(x)\,[w_t(x,t)]^2\,dx\,dt.$$

The generalization of Hamilton's action to this continuum deformation in space and time is given by

$$\mathcal{A}[w] = E - K = \int_0^T\int_0^L \left\{\frac{\epsilon}{2}[w_x(x,t)-1]^2 - \frac{1}{2}\rho(x)\,[w_t(x,t)]^2\right\}\,dx\,dt.$$

We proceed to find the critical points for this functional (according to Hamilton's prescription for finding the physical motion) in the admissible class

$$\mathcal{W} = \{w = w(x,t) : w_x > 0\}$$

with respect to perturbations $\phi = \phi(x,t)$ depending on both space and time. Specifically, the first variation

$$\delta\mathcal{A}_w[\phi] = \frac{d}{d\alpha}\mathcal{A}[w+\alpha\phi]\Big|_{\alpha=0}$$

is given by

$$\delta\mathcal{A}_w[\phi] = \int_0^T\int_0^L \{\epsilon[w_x(x,t)-1]\phi_x(x,t) - \rho(x)\,w_t(x,t)\,\phi_t(x,t)\}\,dx\,dt.$$

Notice that we have used the full gradient (spatial and time)

$$D^{\text{full}}\phi = (\phi_x, \phi_t)$$

in this calculation, so that

$$\delta \mathcal{A}_w[\phi] = \int_0^T \int_0^L \mathbf{w} \cdot D^{\text{full}} \phi \, dx \, dt = \int_{(0,L)\times(0,T)} \mathbf{w} \cdot D^{\text{full}} \phi$$

where $\mathbf{w} : (0, L) \times (0, T) \to \mathbb{R}^2$ is the vector field given by

$$\mathbf{w} = (\epsilon(w_x - 1), -\rho w_t).$$

Integrating by parts, or more properly applying the divergence theorem in space and time, we find for ϕ with compact support in $(0, L) \times (0, T)$

$$\begin{aligned} \delta \mathcal{A}_w[\phi] &= \int_0^T \int_0^L \{-\epsilon w_{xx}(x,t)\, \phi(x,t) + \rho(x)\, w_{tt}(x,t)\, \phi(x,t)\} \, dx \, dt \\ &= \int_0^T \int_0^L \{-\epsilon w_{xx}(x,t) + \rho(x)\, w_{tt}(x,t)\}\, \phi(x,t) \, dx \, dt. \end{aligned}$$

Thus, by the fundamental lemma of the calculus of variations, stationary points for the action functional satisfy

$$\rho w_{tt} = \epsilon w_{xx}.$$

Thus, we arrive a third time at the wave equation of motion.

If you want to derive an equation of motion, then Hamilton's powerful action principle can be useful:

> **Hamilton's Action Principle:** *If motion occurs in a system over a time interval* $[0, T]$ *involving forces in such a way that the potential energy* $E = E(t)$ *and the kinetic energy* $K = K(t)$ *of the system make sense as functions of time at each point in time during the motion, then* **the observed physical motion is a stationary point***, with respect to some broader class of admissible motions over the same time interval,* **of the action functional**
>
> $$\mathcal{A} = \int_0^T (E - K) \, dt.$$

Exercise 15 *Use Hamilton's action principle to derive the equation of motion for the deformation* $w : (0, L) \times (0, T) \to \mathbb{R}$ *of a one-dimensional elastic continuum in which the elasticity* $\epsilon = \epsilon(x)$ *depends on position in the equilibrium interval* $[0, L]$.

In view of Exercises 13, 14, and 15 (assuming you have done them correctly), you should have realized that the deformation w does not satisfy the simple wave equation when the elasticity ϵ depends on position.

Exercise 16 *Let* $u = w - x$ *where* w *is a time dependent deformation of a one-dimensional elastic continuum according to the model above with elasticity* $\epsilon = \epsilon(x)$. *Show that* u *satisfies the wave equation*

$$\rho u_{tt} = (\epsilon u_x)_x.$$

Exercise 17 *(big jump—potentially difficult) Use Hamilton's action principle to derive a form of the wave equation describing the motion of an elastic continuum of higher spatial dimension. This will require the determination of the potential and kinetic energies associated with a deformation* $w : \mathcal{U} \times (0, T) \to \mathbb{R}^n$ *of an equilibrium manifold* $\mathcal{U} \subset \mathbb{R}^n$. *Hint(s): The identity transformation* $\mathrm{id} : \mathcal{U} \to \mathcal{U}$ *by* $\mathrm{id}(\mathbf{x}) = \mathbf{x}$ *has (full) derivative the identity matrix. The potential energy should measure the amount by which* w *has gradient differing from that of the identity* id. *For the kinetic energy, use (spatial) Dirichlet energy.*

Note: It is very difficult to generalize the first two derivations of the wave equation to higher dimensions. The action principle approach of Exercise 17 is really the "easy" way to obtain such a model.