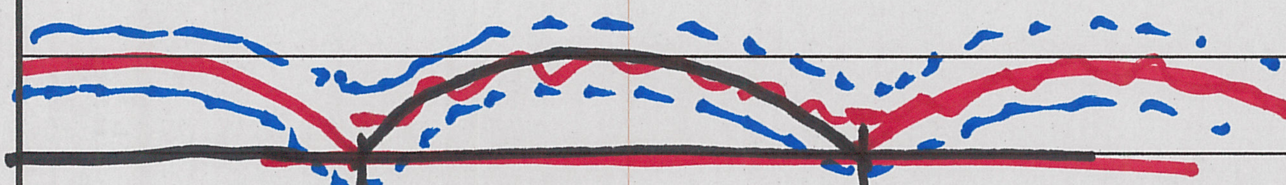
MATH 4581 Lecture 19 Thursday Oct. 28, 2021

Assignment 4 Problem 4

$$\begin{cases} u_t = u_{xx} & \text{on } (0,\pi)\times(0,\infty) \\ u_x(0,t) = 0 = u_x(\pi,t) \\ u(x,0) = \sin x \end{cases}$$

$\leftarrow$ cosine basis

$\{e^{-j^2 t} \cos jx\}_{j=0}^{\infty}$



0 $\pi$

$$\sin x = a_0 + \sum_{j=1}^{\infty} a_j \cos jx$$

[redacted]

$u_t = u_{xx}$ $\qquad u = A(x)B(t)$

$$\begin{cases} u_x(0,t) = 0 \\ u_x(\pi,t) = 0 \end{cases} \qquad \begin{array}{l} A'(0)B(t) = 0 \\ A'(\pi)B(t) = 0 \end{array} \qquad \begin{array}{l} AB' = A''B \\ \dfrac{A''}{A} = \dfrac{B'}{B} = \lambda \end{array}$$

$$\begin{cases} A'' = \lambda A \\ A'(0) = 0 \\ A'(\pi) = 0 \end{cases} \qquad \lambda > 0,\ \underline{\underline{\lambda = 0}},\ \lambda < 0$$

$\uparrow A_0 B_0 = \text{const.} = a_0$

$\lambda < 0 = -\mu^2$ $\qquad A(x) = a\cos\mu x + \underline{\underline{b\sin\mu x}}$

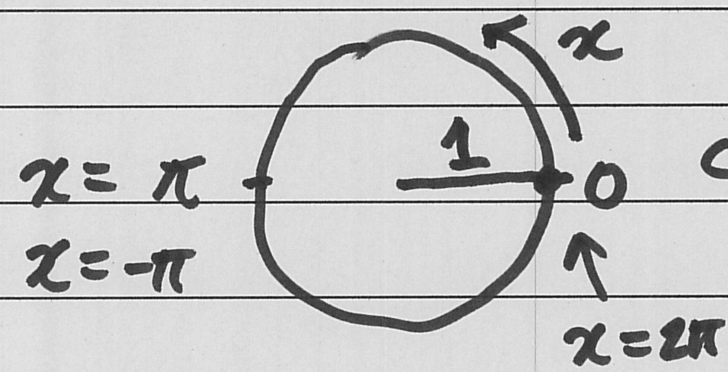
$A'(x) = -a\mu\sin\mu x + b\mu\cos\mu x$

$A'(0) = 0 \Longrightarrow b = 0.$

$A'(\pi) = 0 \Longrightarrow \boxed{\sin\mu\pi = 0}$

[redacted]

Assignment 4 Problem 5 (typo)




circular wire

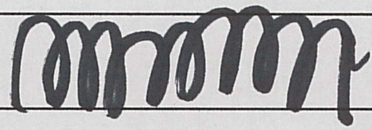
Correct "boundary" conditions

$$\begin{cases} u(0,t) = u(2\pi,t), & t > 0 \\ u_x(0,t) = u_x(2\pi,t), & t > 0 \end{cases}$$

periodicity conditions

[redacted]

# Wave Equation:

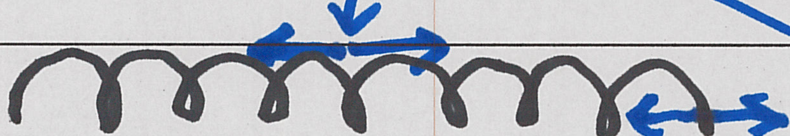


equilibrium

0 $\ell$

tension force

homogeneous extension



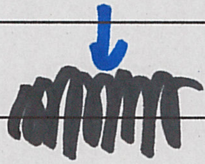
force $|F| = k(\ell' - \ell)$

0 $\ell'$

MAPPING FUNCTION

$w(x) = \frac{\ell'}{\ell} x$

homogeneous compression



force $|F| = k(\ell - \ell')$

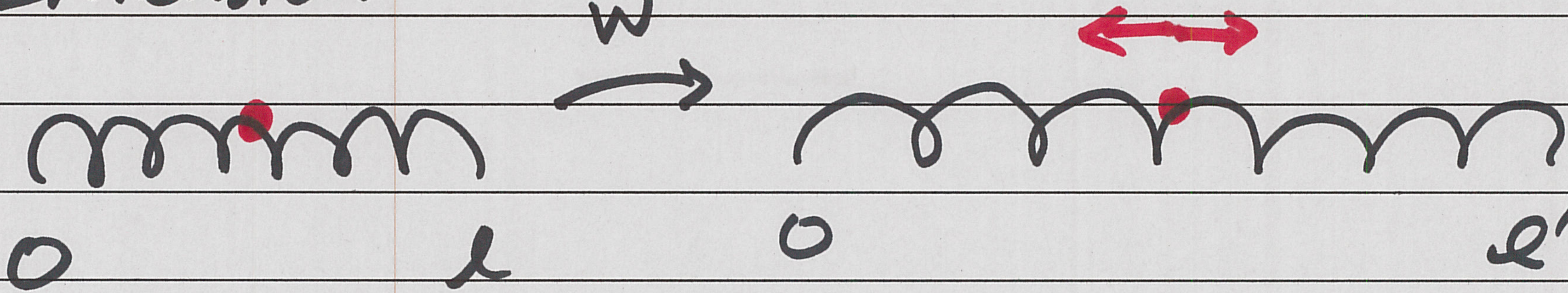
0 $\ell'$

MAPPING FUNCTION

$w(x) = \frac{\ell'}{\ell} x$

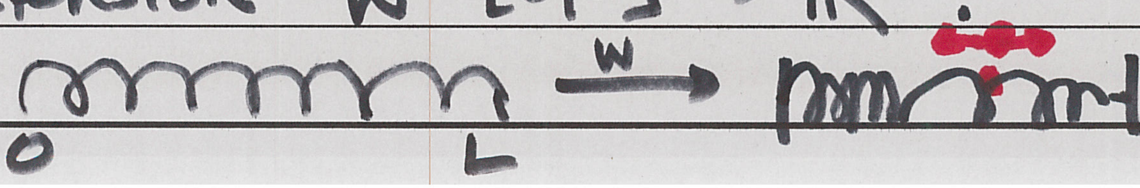
[redacted]

Extension



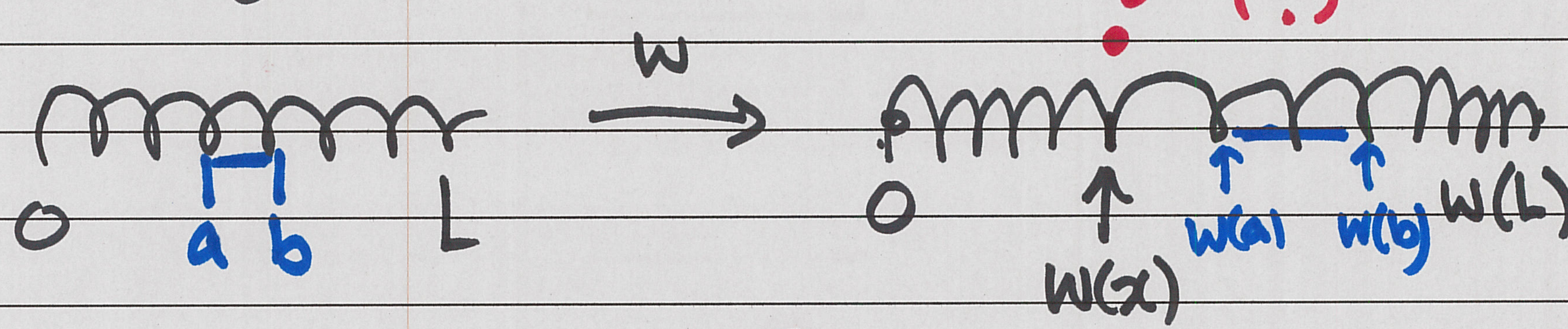

$$\begin{cases} F = -k(\ell' - \ell) = -\underbrace{k\ell}_{\epsilon}\left(\dfrac{\ell'}{\ell} - 1\right) \\ w(x) = \dfrac{\ell'}{\ell}\,x \end{cases}$$

What should be the tension force at each point of an inhomogeneous extension $w : [0, L] \to \mathbb{R}$ ?




[redacted]

# Inhomogeneous deformation




hom: $F_{hom} = -\varepsilon \left( \frac{\ell'}{\ell} - 1 \right)$ ← how is this ratio captured in an inhomogeneous extension?

$W_{hom} = \frac{\ell'}{\ell} x$

$$F = -\varepsilon \left( w'(x) - 1 \right)$$

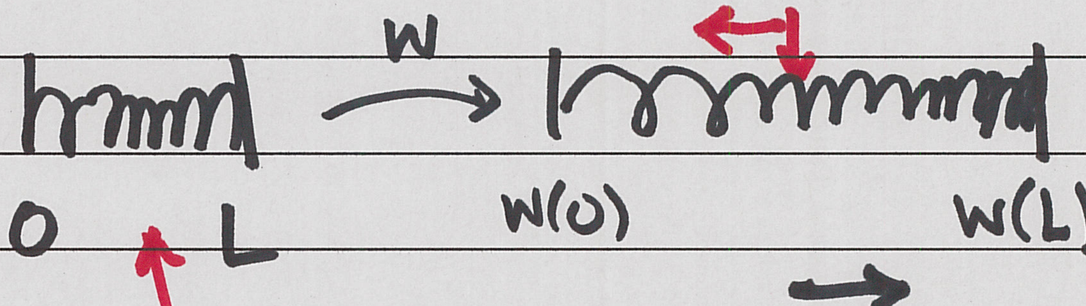
$$w'(x) = \lim_{a,b \to x} \frac{w(b) - w(a)}{b-a}$$

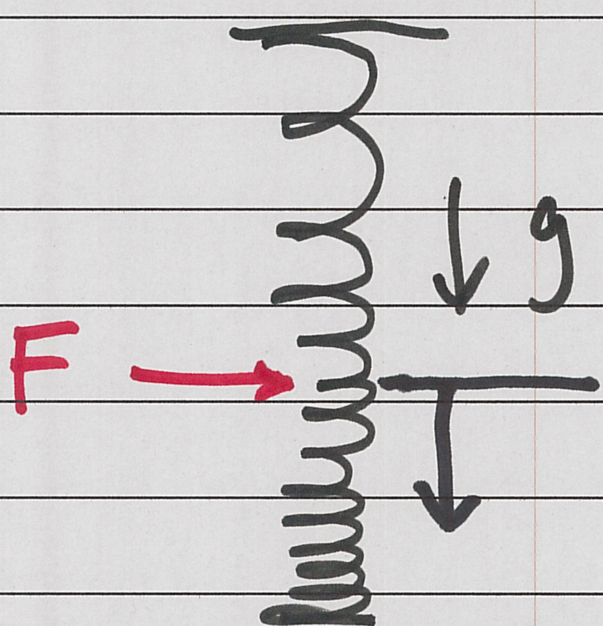
[redacted]

$$\mathcal{W} = \{ w \in C^1[0,L] : w' > 0 \}$$

$\uparrow$ admissible class.

$F = -\epsilon(w'-1)$




density function

$\rho : [0,L] \to (0,L)$

$\uparrow$ constant ?

or $\rho = \rho(x)$

$$\boxed{+\epsilon(w'(x)-1) = \rho(L-x)g}$$

[redacted]

$$w' = \frac{\rho g}{\epsilon}(L-x) + 1$$

$\nwarrow$ w is quadratic!

$$\boxed{w(x) = -\frac{\rho g}{2\epsilon}x^2 + \left(\frac{\rho g}{\epsilon}L + 1\right)x}$$

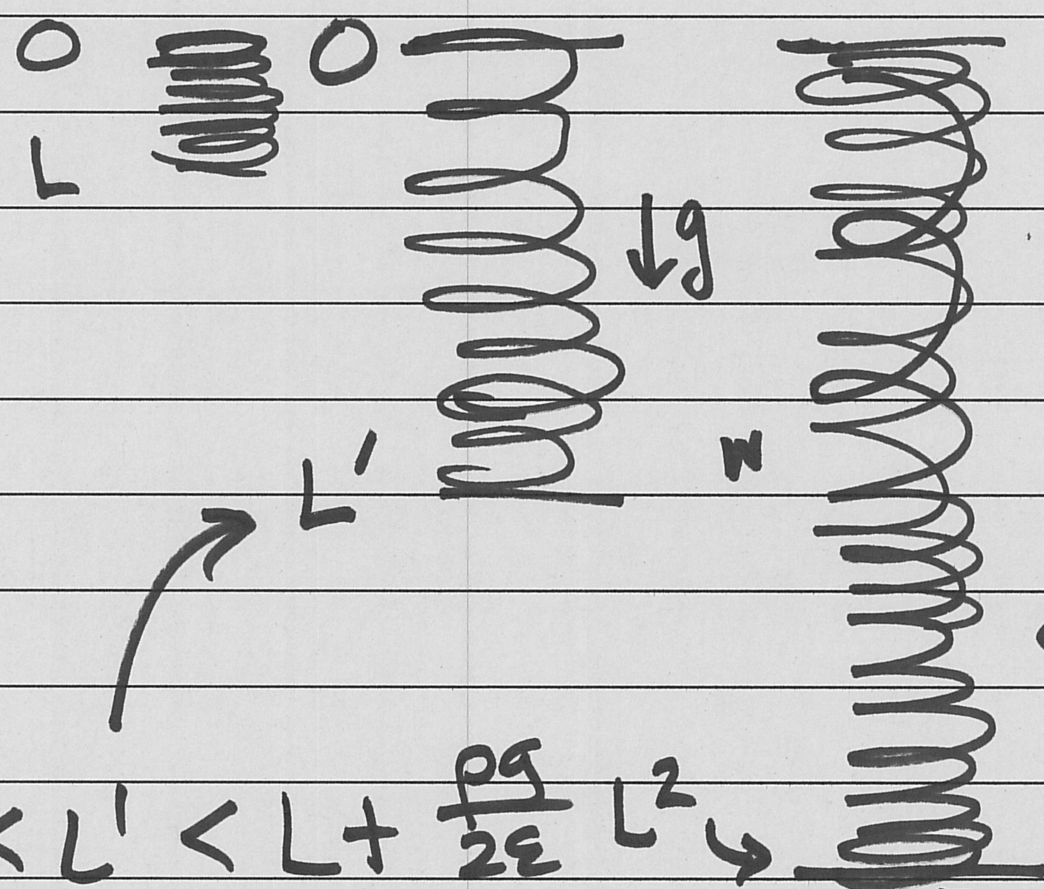
$(w(0) = 0)$

SLINKY MODELERS: THIS IS A LITTLE OFF.

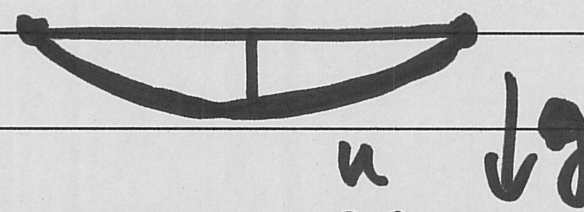
Why? And how should the model be modified to be more exact.

[redacted]

# Fixed Ends sagging




Hakawan
sagging string




$(u_E)_{xx} = \frac{\rho g}{E}$

$u_E''$

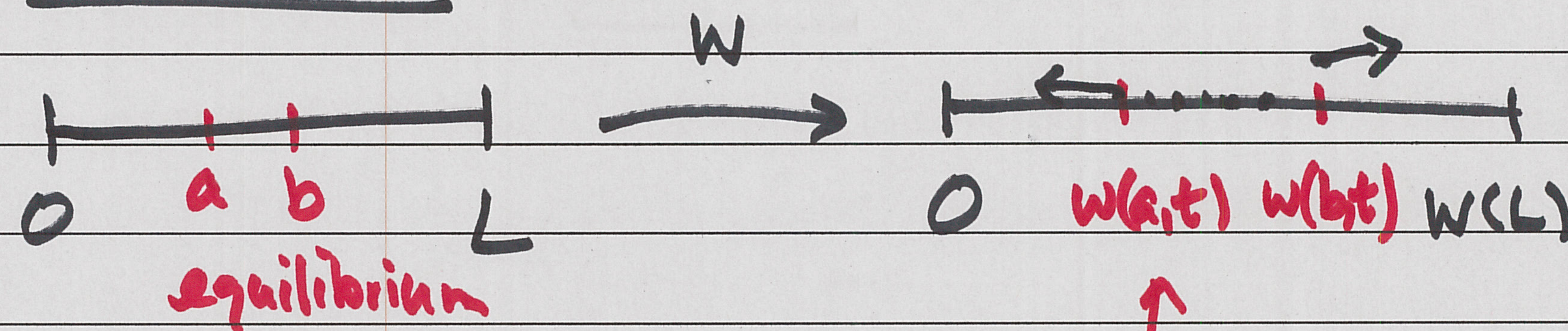
hanging.

$L < L' < L + \frac{\rho g}{2E} L^2$

$w'(L) = 1$

[redacted]

# Derivation of wave PDE - dynamics




deformation interval.

mass $\underline{\rho(b-a)}$

$x^*$

Newton: $\rho(b-a)\,w_{tt}(x^*,t) = -\epsilon\left(w_x(a,t)-1\right) + \epsilon\left(w_x(b,t)-1\right)$

acceleration

[redacted]

$$\rho(b-a)\, W_{tt}(x_*, t) = \epsilon\left(W_x(b,t) - W_x(a,t)\right)$$

$\uparrow$ some/any $x_* \in (a,b)$.

$$\rho\, W_{tt}(x_*, t) = \epsilon\, \frac{W_x(b,t) - W_x(a,t)}{b-a}$$

$\boxed{b, a \to x}$

$$\boxed{\rho\, W_{tt} = \epsilon\, W_{xx}}$$

WARNING: HERE $\rho$ and $\epsilon$ are constant.

[redacted]