MATH 4581 **Lecture 7**, Tuesday Sept. 14, 2021

Last time: The divergence

The divergence Theorem.

$$\operatorname{div} \mathbb{V}(\mathbb{P}) = \lim_{R \to \{\mathbb{P}\}} \frac{1}{\mu(R)} \int_{\partial R} \mathbb{V} \cdot \mathbb{n}$$

$$\int_{U} \operatorname{div} \mathbb{V} = \int_{\partial U} \mathbb{V} \cdot \mathbb{n}$$

not $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}$

- Intrinsic Mathematics.
- Assignment 1 Problem 4. ?

[redacted]

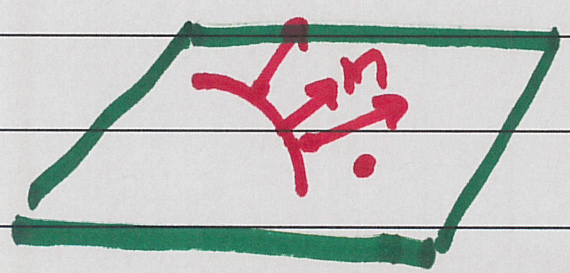
Classical numbers

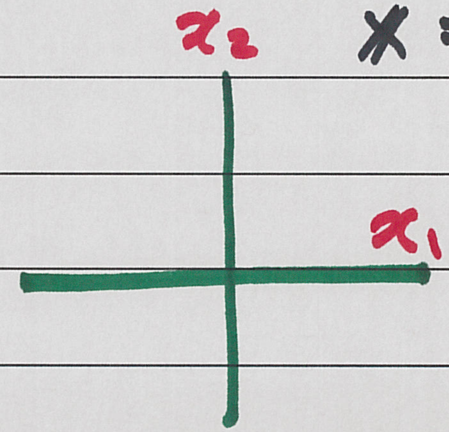
0 1

Intrinsic Mathematics:

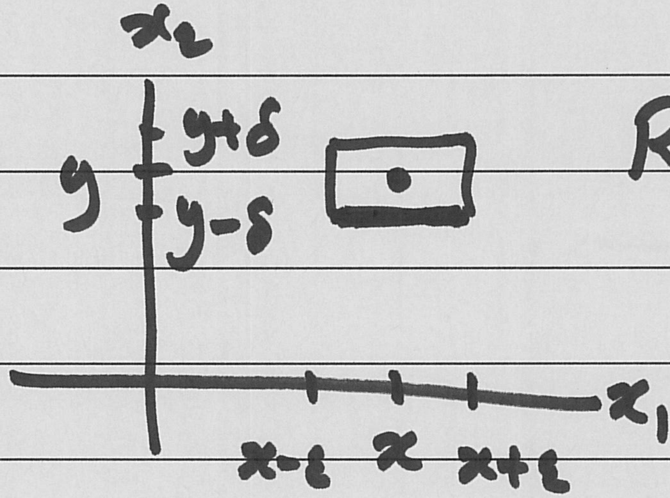
Hermann Weyl: The introduction of numbers as coordinates is an act of violence.

Euclid	Descartes
a point A	(2,3)
P	$X = (x_1, x_2)$





[redacted]

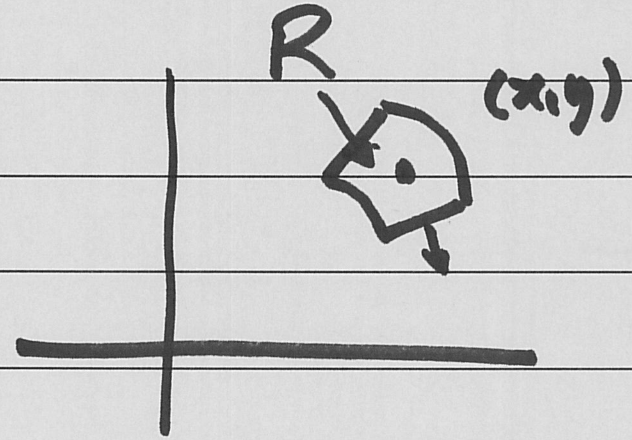


$$R = (x-\varepsilon, x+\varepsilon) \times (y-\delta, y+\delta)$$

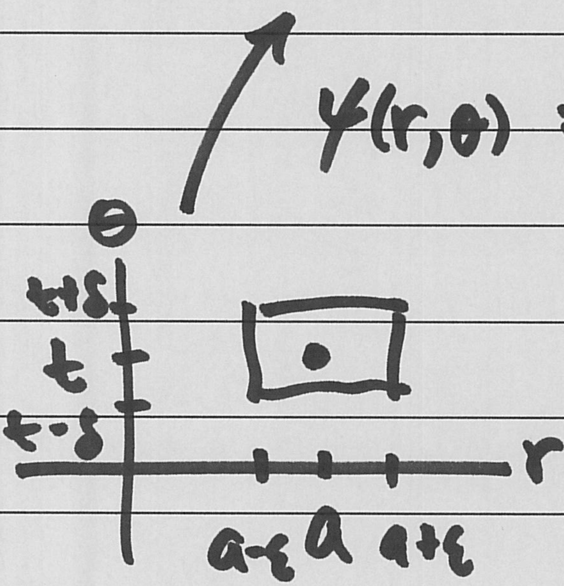
<u>Exercise</u>: Show $\displaystyle \lim_{\substack{R \to \{\mathbf{x}\} \\ \mathbf{x} = (x,y)}} \frac{1}{4\varepsilon\delta} \int_{\partial R} \mathbb{V} \cdot \mathbb{n}$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2}.$$

[redacted]



$$\mathbf{W} = W_1(r,\theta)\,\mathbf{e}_1 + W_2(r,\theta)\,\mathbf{e}_2$$



$$\psi(r,\theta) = (r\cos\theta, r\sin\theta)$$

$$\lim_{\varepsilon,\delta \to 0} \frac{1}{\mu(R)} \int_{\partial R} \mathbf{W}\cdot\mathbf{n}$$

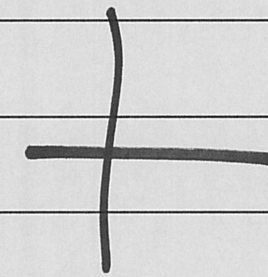
= a very different [redacted]
coordinate
formula

$$\frac{\partial W_1}{\partial r}, \frac{\partial W_1}{\partial \theta}, \ldots$$

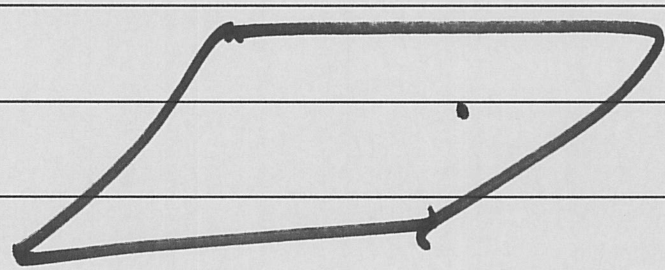
Intrinsic Laplacian ?

$n=2$ $\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \text{div}\, Du$

$Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$



You need an intrinsic gradient.



[redacted]

Homework Assignment 1 #6

(b)

$$\begin{cases} u_t = u_{xx} \text{ on } (0,L) \times (0,T) \\ u(0,t) = u_0 , \quad t > 0 \\ u(L,t) = u_1 , \quad t > 0 \end{cases}$$

$$\boxed{u(x,t) = \frac{u_1(t) - u_0(t)}{L} x + u_0(t)}$$

(a) $u_t = u_{xx}$?

ADVICE : Start with something you know really well — and learn stuff from the[redacted]

$\boxed{u_t = 0}$ ← using the form and calculating u_{xx}

↖ 1st order PDE

$$\Rightarrow \boxed{u = g(x)}$$

$$\Rightarrow \underbrace{\frac{u_1(t) - u_0(t)}{L} x + u_0(t)} = g(x)$$

$$\Rightarrow \frac{u_1'(t) - u_0'(t)}{L} x + u_0'(t) = 0$$

↑ affine polynomial in x

[redacted]

<u>Claim</u>: $\ell(x) = mx + b$ is not linear unless $b = 0$.

<u>Proof</u>: If ℓ is linear (in x), then

$$\ell(x_1 + x_2) = \ell(x_1) + \ell(x_2)$$

$$mx_1 + b \qquad\qquad mx_1 + b \qquad mx_2 + b$$

$$mx_1 + mx_2 + b = mx_1 + mx_2 + 2b$$

$$b = 2b$$

$$b = 0.$$

[redacted]

<u>ℓ is affine</u>

$$\Rightarrow \quad \frac{u_1'(t) - u_0'(t)}{L} = 0 \quad \text{and} \quad u_0'(t) = 0.$$

$\rightarrow u_0 = \text{const.}$

So $u_1'(t) = 0$.

So $u_1 = \text{const.}$

[redacted]

(b) What if $u_1(t) - u_0(t)$ is constant

but $u_0(t)$ is not constant.

(0) Then $u(x,t) = \dfrac{u_1(t) - u_0(t)}{L} x + u_0(t)$

does not satisfy $u_t = u_{xx}$.

(1) What PDE does u satisfy?

$u_t = u_0'$ (nonzero)

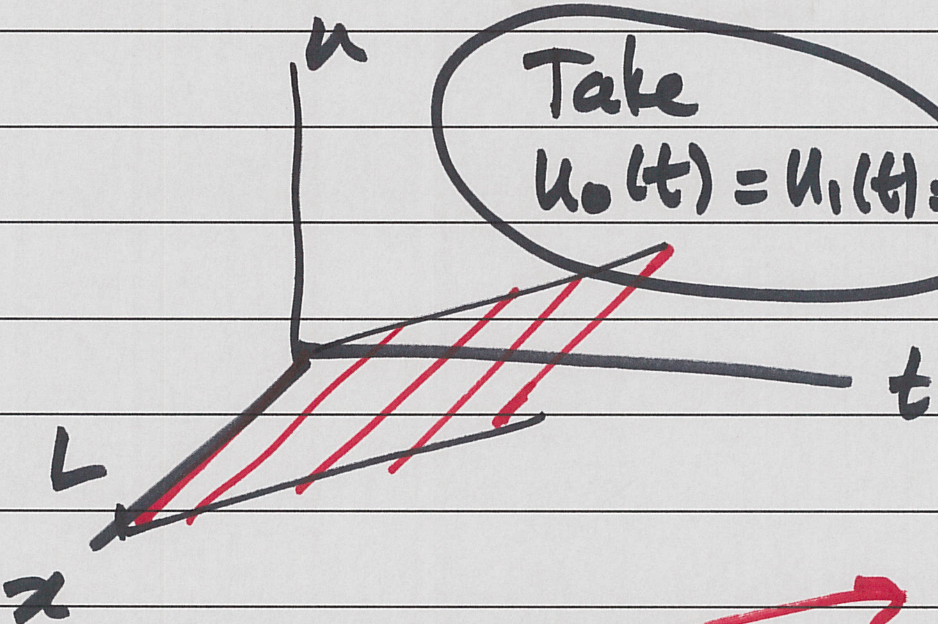
$u_{xx} = 0$

Therefore, u satisfies $u_t = u_{xx} + u_0'$

[redacted]

(2) Can I solve this?

I am interested in the following:



Take
$u_0(t) = u_1(t) = t$

$u_t = u_{xx} + \cancel{t}$ 1

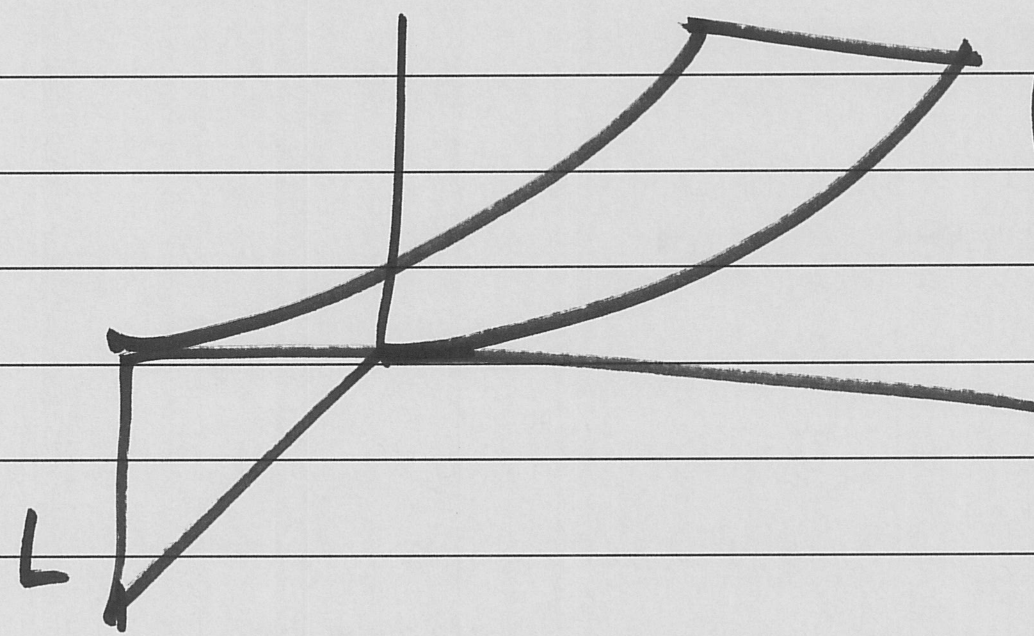
$$\begin{cases} u_t = u_{xx} + 1 \\ u(0,t) = t = u(L,t) \\ u(x,0) = 0 \end{cases}$$

$$u(x,t) = \frac{u_1(t) - u_0(t)}{L} x + u_0(t)$$

This is a reasonable problem — should have a solution.

[redacted]

(c) Give an example.



$$\begin{cases} u_0(t) = t^2 \\ u_1(t) = t^2 + 2 \end{cases}$$

$$\begin{cases} u_t = u_{xx} + 2t \\ u(0,t) = t^2 \\ u(L,t) = t^2 + 2 \\ u(x,0) = \frac{2}{L} x \end{cases}$$

[redacted]