MATH 4581 Lecture 9 Tuesday Sept. 21, 2021

- $\boxed{\text{Calculus of Variations ("derivation" of Laplace's PDE)}}$
- Calculus Review – directional derivatives and normal derivatives
- separation of variables
  - Laplace's equation on a disk.
- Properties of solutions of Laplace's PDE
  - mean value property
  - maximum principle(s)
- Some technical definitions (epsilon-delta, advanced calculus)
  - continuity
  - open sets and closed sets
  - closure and boundary

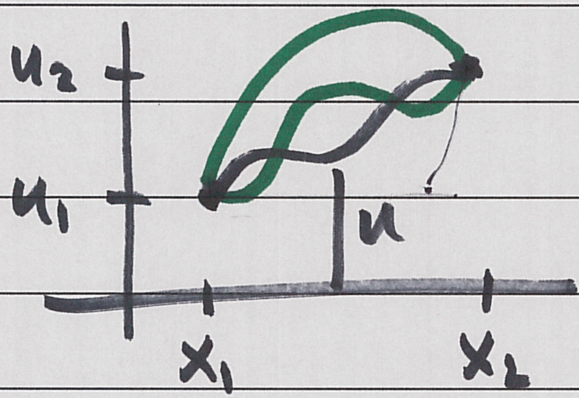
[redacted]

Calculus of Variations:

This is about finding minima of real valued functions whose domain is an infinite dimensional set of functions.

Simple Examples:

$$\{u : u(x_1) = u_1,\ u(x_2) = u_2\}$$



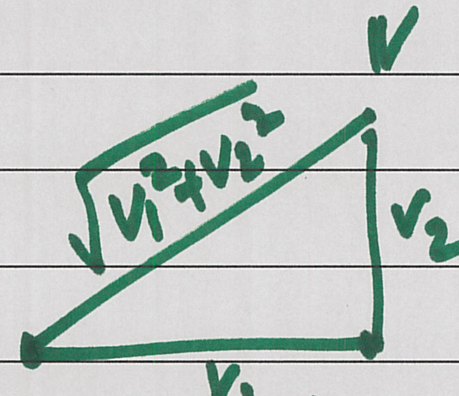
$$L[u] = \int_{x_1}^{x_2} \sqrt{1 + u'(x)^2}\, dx$$

length "functional"

[redacted]

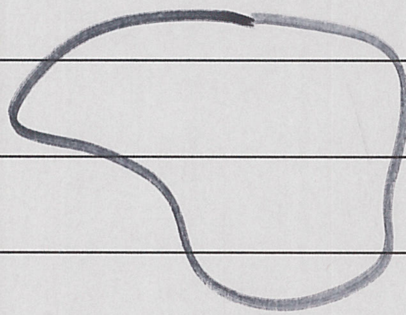
$$D[u] = \int_{x_1}^{x_2} u'(x)^2 \, dx$$

↑ Dirichlet "Energy"



$\underline{\underline{n=2}}$ (or higher)

$$\boxed{\{u : u|_{\partial\mathcal{U}} = g\}}$$



$\mathcal{U} \subseteq \mathbb{R}^2$

$$Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

$$D[u] = \int_{\mathcal{U}} |Du|^2$$

$$= \int_{\mathcal{U}} \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$$

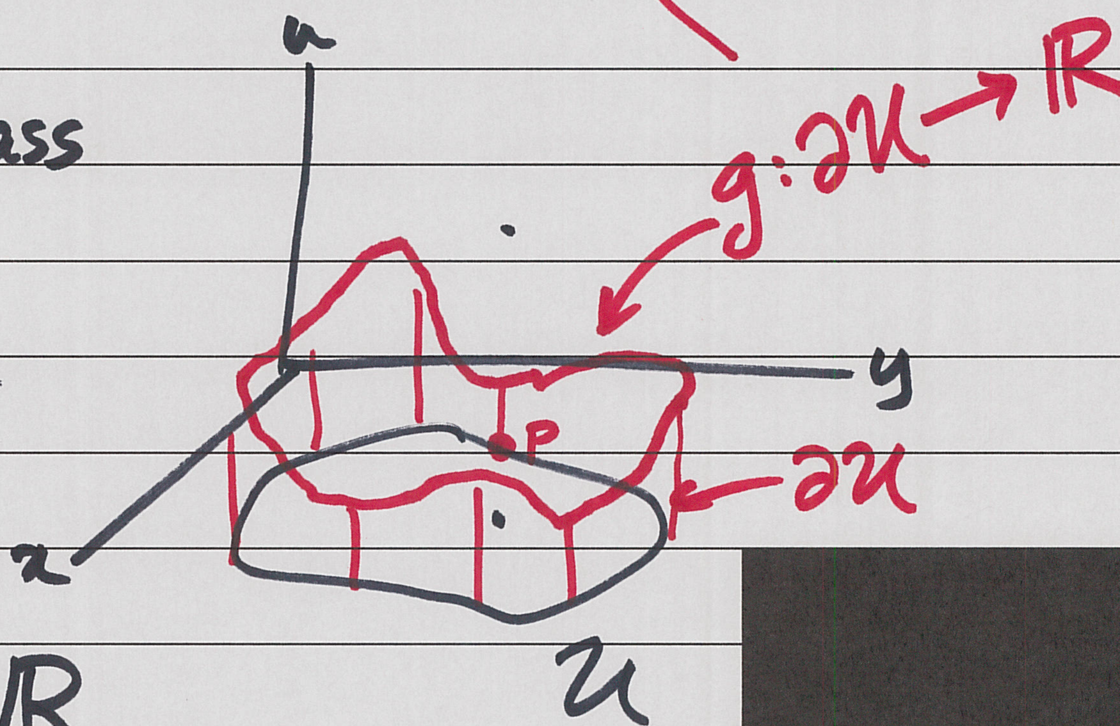
[redacted]

The domain of Dirichlet energy :

$$A = \{ u : u|_{\partial \mathcal{U}} = g \}$$

admissible class

$n=2$



closure

$u : \overline{\mathcal{U}} \to \mathbb{R}$,

$u|_{\partial \mathcal{U}}$ ← the restriction of u to $\partial \mathcal{U}$

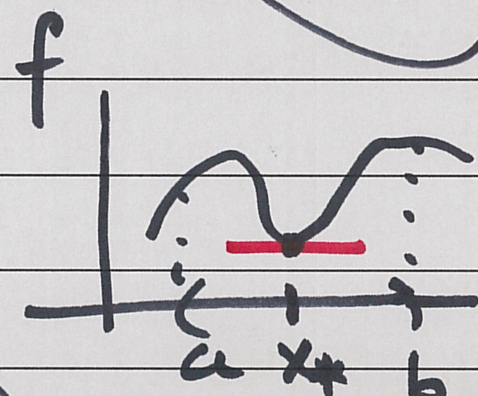
[redacted]

$$D[u] = \int_{\mathcal{U}} |Du|^2$$

$$\text{Area}[u] = \int_{\mathcal{U}} \sqrt{1 + |Du|^2}$$

A <u>minimizer</u> is a function u_* for which

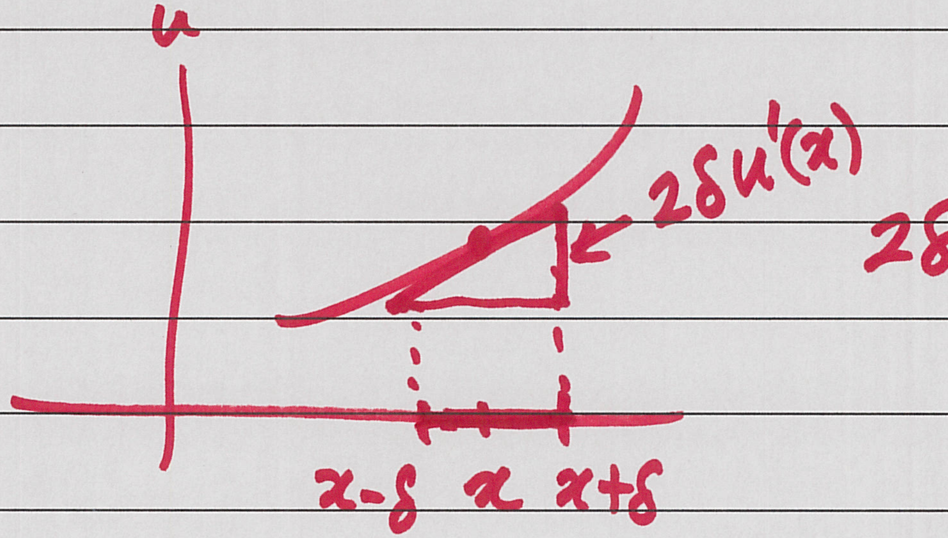
$$D[u_*] \leq D[u] \quad \text{for all } u \in \mathcal{A}.$$



$$f(x_*) \leq f(x)$$
$$\text{for all } a \leq x \leq b$$

[redacted]

$n=1$



length of tangent segment: $\sqrt{(2\delta)^2+(2\delta u'(x))^2}$

$$= 2\delta\sqrt{1+u'(x)^2}$$

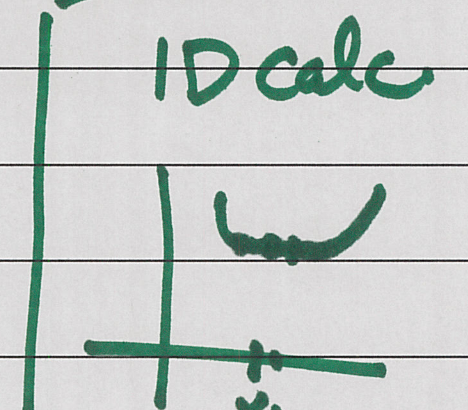
$$\sum \sqrt{1+u'(x_j^*)^2}\,(2\delta_j) \to \int\sqrt{1+u'^2}$$

[redacted]

Find a minimizer u_* of

$$D[u] = \int_U |Du|^2 .$$

1D calc

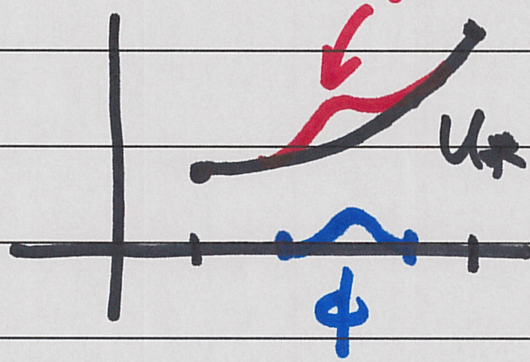


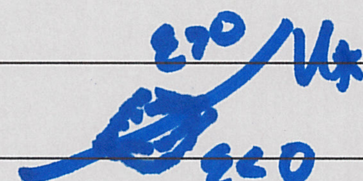
Q: What is the derivative of D at u_*?

$\underline{n=1}$

some admissible $u = u_* + \varepsilon\phi$

variation of u_*





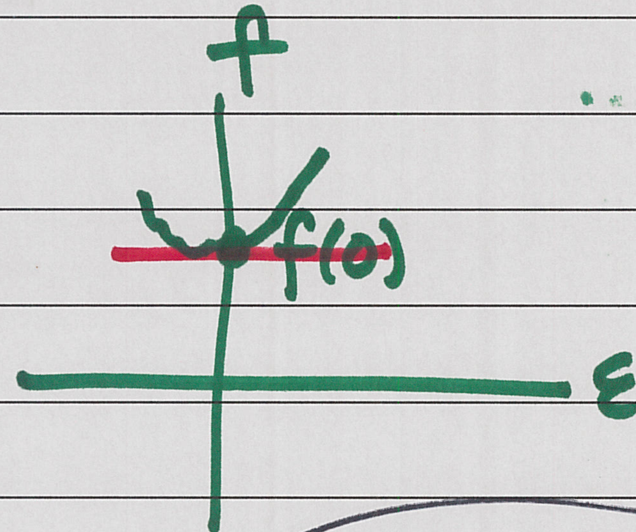
$$D[u_* + \varepsilon\phi] \geq D[u_*]$$

[redacted]

$$\mathcal{D}[u_* + \varepsilon\phi] \geq \mathcal{D}[u_*]$$

$$\| \quad f(\varepsilon)$$

$$f(\varepsilon) \geq f(0)$$



$$\Longrightarrow \quad \boxed{f'(0) = 0}$$

Calculus $df_p(v)$

$$\boxed{\delta\mathcal{D}_{u_*}[\phi] = \frac{d}{d\varepsilon}\mathcal{D}[u_* + \varepsilon\phi]\Big|_{\varepsilon=0}}$$

first variation

[redacted]

$$\delta \mathcal{D}_u[\phi] = \frac{d}{d\varepsilon} \mathcal{D}[u+\varepsilon\phi] \Big|_{\varepsilon=0}$$

$$\mathcal{D}[u+\varepsilon\phi] = \int_{\mathcal{U}} \left[|Du|^2 + 2\varepsilon\, Du \cdot D\phi + |D\phi|^2 \varepsilon^2 \right]$$

$$\frac{d}{d\varepsilon} \mathcal{D}[u+\varepsilon\phi] = \int_{\mathcal{U}} \left[2\, Du \cdot D\phi + 2\varepsilon |D\phi|^2 \right]$$

$$\boxed{\delta \mathcal{D}_u[\phi] = 2 \int_{\mathcal{U}} Du \cdot D\phi}$$

For a minimizer u_*

$$\int_{\mathcal{U}} Du_* \cdot D\phi = 0 \quad \text{for all } \phi.$$

[redacted]

$$\delta \mathcal{D}_{u_*}[\phi] = \frac{d}{d\varepsilon} \mathcal{D}[u_* + \varepsilon \phi] \Big|_{\varepsilon = 0}$$

<u>Compute</u>: $\mathcal{D} = \int_U |Du|^2$

$$\mathcal{D}[u + \varepsilon\phi] = \int_U |D(u+\varepsilon\phi)|^2$$

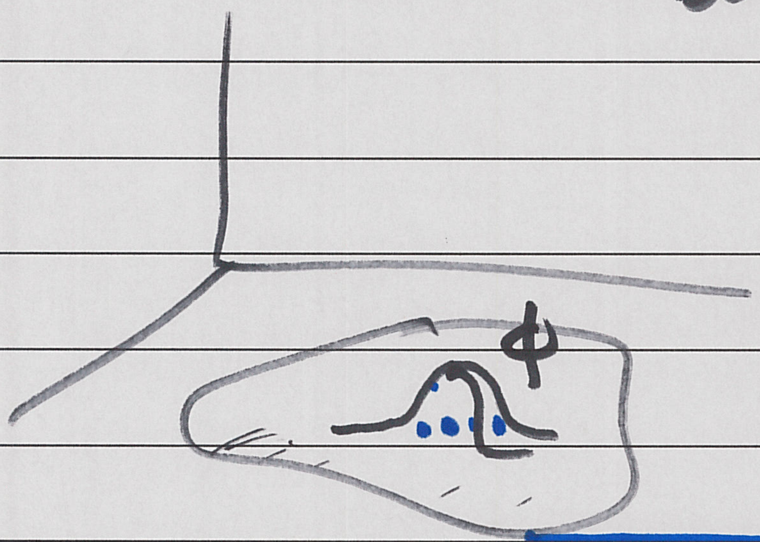
$$= \int_U |Du + \varepsilon D\phi|^2$$

$$= \int_U \Big[|Du|^2 + 2\varepsilon\, Du \cdot D\phi + \varepsilon^2 |D\phi|^2 \Big]$$

non-negative homogeneity for a norm $|cv| = |c||v|$ [redacted]

$A = \{ u : u|_{\partial U} = g \}$

↑ should not change the boundary values with ϕ.



$u + \varepsilon\phi$, $\phi|_{\partial U} = 0$.

$\{ x \in U : \phi(x) \neq 0 \}$.

$\mathrm{supp}(\phi) = \overline{\{ x \in U : \phi(x) \neq 0 \}}$

<u>support</u>

<u>require</u> $\mathrm{supp}(\phi) \subseteq U$.

$\Rightarrow \phi|_{\partial U} = 0$.

[redacted]

For a minimizer u_* of Dirichlet energy we have

$$\int_{\mathcal{U}} Du_* \cdot D\phi = 0 \quad \underline{\underline{\text{for all } \phi}}$$

What does this tell me about u_*?

$$\operatorname{div}(\phi Du) = D\phi \cdot Du + \phi \operatorname{div}(Du)$$

$$\Rightarrow \quad \int_{\mathcal{U}} Du \cdot D\phi = -\int_{\mathcal{U}} \operatorname{div} Du \, \phi + \int_{\mathcal{U}} \operatorname{div}(\phi D$$ [redacted]

$$= \underline{\underline{-\int_{\mathcal{U}} (\operatorname{div} Du)\phi}}$$