MATH 4581 Lecture 10 Thursday Sept. 23, 2021

TO DO LIST:

- Calculus of variations
- directional and normal derivatives
- Laplace's PDE on a disk
- Mean Value Property and Maximum Principle
- open and closed sets, continuity, closure, boundary, ...

[redacted]

Last time Calculus of Variations

$u : \mathcal{U} \to \mathbb{R}$, $\mathcal{U} \subseteq \mathbb{R}^n$

General Set Up:

$u|_{\partial \mathcal{U}} = g$

admissible class $A \subseteq \{ \underline{\underline{u}} : \cancel{u(a) = c,\ u(b) = d} \}$

Problem: Minimize $\mathcal{J} : A \to \mathbb{R}$ by

$$\mathcal{J}[u] = \int_{x \in \mathcal{U}} F(x, \overset{u}{Du}, Du)$$

↑ Lagrangian

Lagrangian integral functional.

$u : \mathcal{U} \to \mathbb{R}$

Minimize $\mathcal{J}[u] = \int_{\mathcal{U}} F(x, u, Du)$

over $u \in A$.

1^{st} variation

(1^{st} order necessary condition)

$\mathcal{V} \leftarrow$ class of admissible perturbations

Fix $\phi \in \mathcal{V}$, $\underline{u + \varepsilon\phi} \in A$

$$\boxed{\delta \mathcal{J}_u[\phi] = \frac{d}{d\varepsilon} \mathcal{J}[u + \varepsilon\phi] \Big|_{\varepsilon=0} = 0}$$

$$\frac{d}{d\varepsilon}\mathcal{J}[u+\varepsilon\phi]\Big|_{\varepsilon=0} = 0 \quad \underline{\text{for all}}\ \phi \in \mathcal{V}$$

$u : \mathcal{U} \to \mathbb{R}$

$\mathcal{U} \subseteq \mathbb{R}^n$

$$\frac{d}{d\varepsilon}\int_{x\in\mathcal{U}} F(x, u+\varepsilon\phi, \underline{Du+\varepsilon D\phi})$$

<u>calc 1</u>

$f = f(x)$

$$F = F(x, z, p)$$

chain rule (+ integrate under $\int$)

<u>calc 3</u>

$f = f(x,y,z)$

$$\int_{\mathcal{U}}\left[\frac{\partial F}{\partial z}\cdot\phi + \sum_{j=1}^{n}\frac{\partial F}{\partial p_j}\cdot\frac{\partial \phi}{\partial x_j}\right]$$

[redacted]

$$Du+\varepsilon D\phi = \Big(\underbrace{\frac{\partial u}{\partial x_1}+\varepsilon\frac{\partial \phi}{\partial x_1}}_{p_1}, \dots, \underbrace{\frac{\partial u}{\partial x_n}+\varepsilon\frac{\partial \phi}{\partial x_n}}_{p_n}\Big)$$

$D(u+\varepsilon\phi)$

In the p_j slot we find $\frac{\partial u}{\partial x_j} + \varepsilon \frac{\partial \phi}{\partial x_j}$

so $$\frac{\partial}{\partial \varepsilon}\left(\frac{\partial u}{\partial x_j} + \varepsilon \frac{\partial \phi}{\partial x_j}\right) = \frac{\partial \phi}{\partial x_j}$$ and

$$\underline{\delta \mathcal{F}_u[\phi]} = \underbrace{\int_U \left[\underset{\uparrow\, \frac{\partial F}{\partial z}(x,u,Du)}{\frac{\partial F}{\partial z}} \phi + \sum_{j=1}^{n} \underset{\nwarrow\, \frac{\partial F}{\partial p_j}(x,u,Du)}{\frac{\partial F}{\partial p_j}} \cdot \frac{\partial \phi}{\partial x_j} \right]}_{\boxed{\frac{d}{d\varepsilon} \int_U F(x, u+\varepsilon\phi, Du+\varepsilon D\phi)}}$$

$$\underline{\delta \mathcal{F}_u[\phi]} = \frac{d}{d\varepsilon} \mathcal{F}[u+\varepsilon\phi] \Big|_{\varepsilon=0}$$

[redacted]

If u is a minimizer of $\mathcal{F} : \mathcal{A} \to \mathbb{R}$, then

$$\delta \mathcal{F}_u[\phi] = \int_{\mathcal{U}} \left[\frac{\partial F}{\partial z} \phi + \sum_{j=1}^{n} \frac{\partial F}{\partial p_j} \cdot \frac{\partial \phi}{\partial x_j} \right] = 0$$

for all $\phi \in \mathcal{V}$.

<u>End of last lecture:</u> What does this tell you about the minimizer u.

$f = f(x)$

$f'(x) = 0$

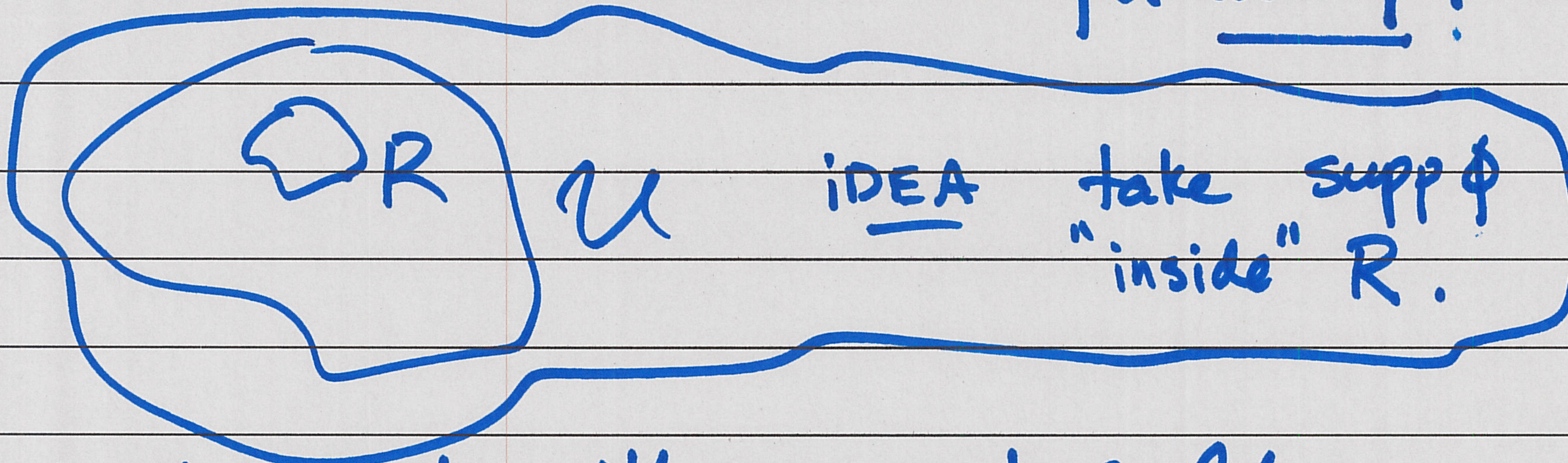
← algebraic eqn. for $x = x_{min}$

[redacted]

$$\int_{\mathcal{U}} \left[\frac{\partial F}{\partial z} \phi + \sum_{j=1}^{n} \frac{\partial F}{\partial p_j} \frac{\partial \phi}{\partial x_j} \right] = 0$$

dot product with $D\phi$.

for <u>all</u> ϕ:



i<u>DEA</u> take supp ϕ "inside" R.

1. Take ϕ with supp $\phi \subseteq \mathcal{U}$.

$$\Rightarrow \phi\big|_{\partial \mathcal{U}} = 0.$$

2. Let $\mathbb{W} = \left(\frac{\partial F}{\partial p_1}, \frac{\partial F}{\partial p_2}, \dots, \frac{\partial F}{\partial p_n} \right)$

$$\operatorname{div}(\phi \mathbb{W}) = D\phi \cdot \mathbb{W} + \phi \operatorname{div} \mathbb{W}$$

[redacted]

$$\int_{\mathcal{U}} \operatorname{div}(\phi \mathbb{W}) = \underbrace{\int_{\mathcal{U}} D\phi \cdot \mathbb{W}} + \int_{\mathcal{U}} \phi \operatorname{div} \mathbb{W}$$

$$= \boxed{\int_{\partial \mathcal{U}} \phi \mathbb{W} \cdot n}$$

(divergence Theorem)

← This term vanishes $\phi\big|_{\partial \mathcal{U}} = 0$

$$\int_{\mathcal{U}} \frac{\partial F}{\partial z} \phi + \int_{\mathcal{U}} \underline{\sum} \frac{\partial F_j}{\partial p_j} \underline{\frac{\partial \phi}{\partial x_j}},$$

$$= \int_{\mathcal{U}} \frac{\partial F}{\partial z} \phi - \int_{\mathcal{U}} (\operatorname{div} \mathbb{W}) \phi$$

[redacted]

$1^{\underline{st}}$ order necessary cond.:

$$\int_{\mathcal{U}} \left[\frac{\partial F}{\partial z} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial p_j}\right) \right] \phi$$

$$= 0 \quad \text{for all } \phi$$

with $\operatorname{supp}\phi \subseteq \mathcal{U}$

$$\frac{\partial}{\partial x_j}\left(\frac{\partial F}{\partial p_j}(x, u, Du) \right)$$

$1^{\underline{st}}$ and $2^{\underline{nd}}$ partials of $\underline{u}$

[redacted]

$$\int_{\mathcal{U}} \left[\frac{\partial F}{\partial z} - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left(\frac{\partial F}{\partial p_j} \right) \right] \phi = 0$$

for all ϕ with $\operatorname{supp} \phi \subseteq \mathcal{U}$.

Fundamental Lemma of the Calculus of variations:

If f is continuous and $\int_{\mathcal{U}} f\phi = 0$ for all ϕ with $\operatorname{supp} \phi \subseteq \mathcal{U}$, then $f(x) = 0$ for $x \in \mathcal{U}$.

[redacted]

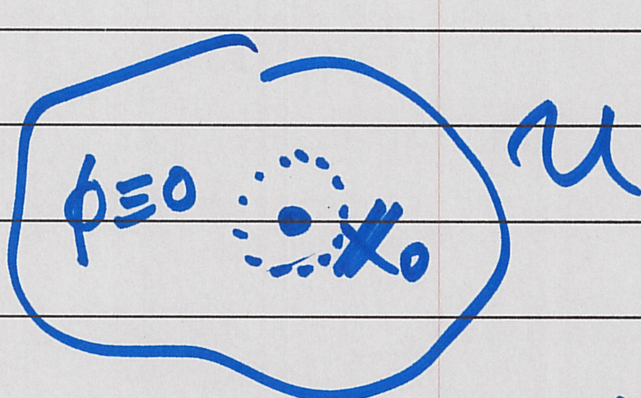
Fundamental Lemma (CV)

If $\int f\phi = 0$ for all ϕ

then $f \equiv 0$.

Proof: If $f(x_0) \neq 0$ for some $x_0 \in \mathcal{U}$.

Say $f(x_0) > 0$.



Take ϕ with $\phi \geq 0$
and supp ϕ close to x_0

[redacted]