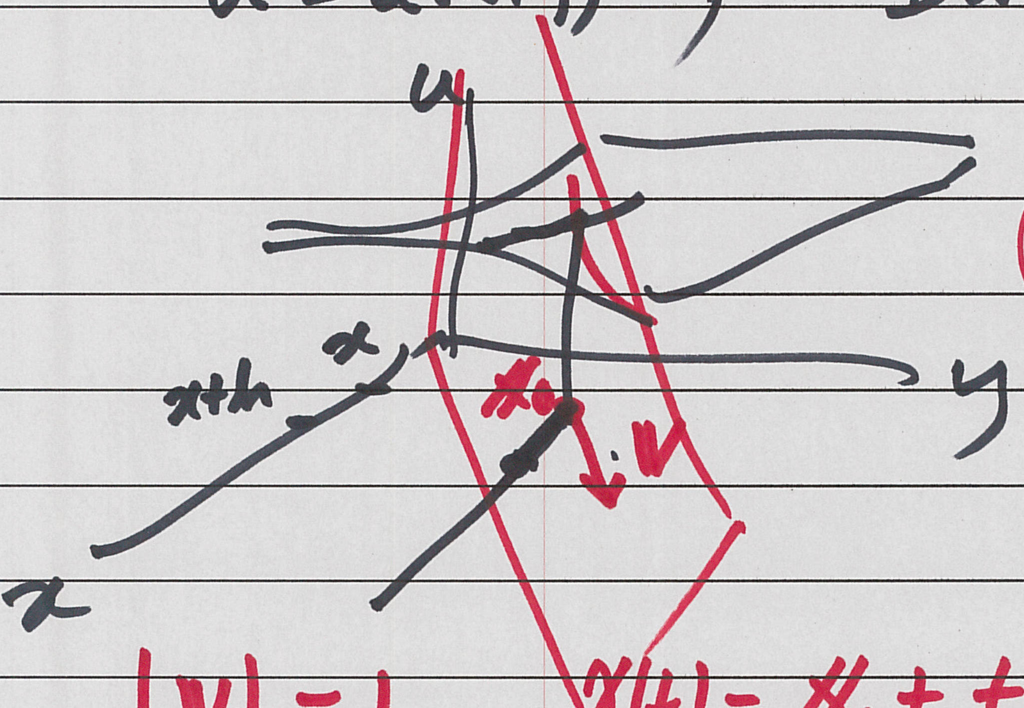
MATH 4581 Lecture 12 Thursday Sept. 30, 2021

- Calculus of Variations ✓
- directional derivatives ✓
- Laplace's PDE on a disk
- Mean Value Property and the Maximum Principle(s)
- open and closed and closure, ...

[redacted]

Calculus of multivariable functions

$u = u(x,y)$, $\quad Du = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$



$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(x+h,y) - u(x,y)}{h}$

difference quotient

$|\mathbf{v}| = 1$, $\quad \gamma(t) = \mathbf{x}_0 + t\,\mathbf{v}$

Directional Derivative

$$D_{\mathbf{v}} u(\mathbf{x}_0) = \lim_{h \to 0} \frac{u(\mathbf{x}_0 + h\mathbf{v}) - u(\mathbf{x}_0)}{h}$$

[redacted]

<u>In calculus</u> $|\mathbb{v}| = 1$

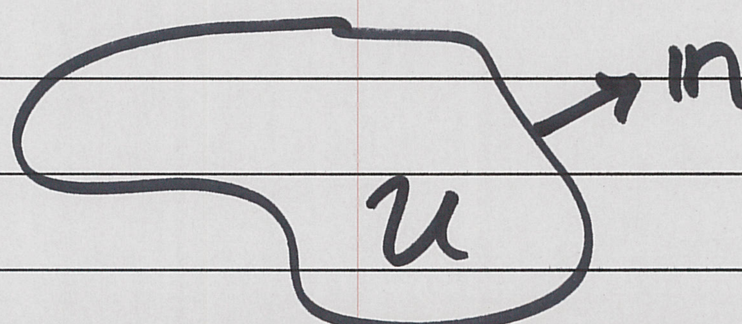
$$\underline{D_{\mathbb{v}} u(x)} = \underline{Du \cdot \mathbb{v}}$$

<u>NOW</u>: Maybe $|\mathbb{v}| \neq 1$

$$= \left(Du \cdot \frac{\mathbb{v}}{|\mathbb{v}|}\right) \cdot |\mathbb{v}|$$

<u>Main use for us</u>:

$$\underline{\underline{Du \cdot n}} = D_n u$$



Appears in Fourier's Law $\vec{\Phi} = \underline{\underline{K Du}}$

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$\underline{D_w u} : S^1 \longrightarrow \mathbb{R}$ has a max, a min, and a $\underline{\underline{\text{zero}}}$

$|w| = 1$

$\underline{n=2}$ a circle of directions  S^1

$$S^{n-1} = \partial B_1 \subseteq \mathbb{R}^n$$

$$B_1(p) = \{ x : |x - p| < 1 \}$$

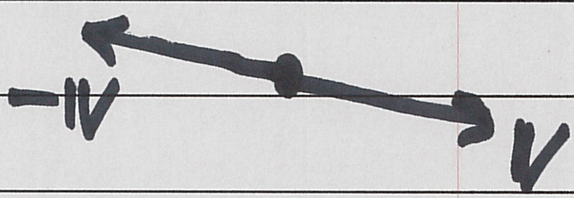
r r

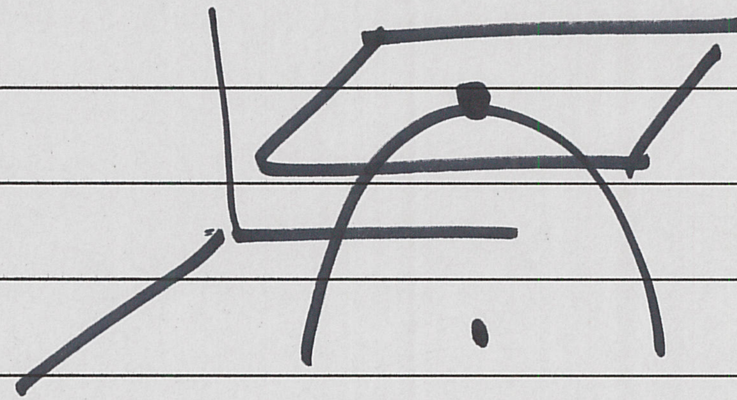
"ball of radius r"

"open ball of radius r"

[redacted]

$$D_{-\mathbb{v}} u = - D_{\mathbb{v}} u$$





$$D_{\mathbb{v}} u = Du \cdot \mathbb{v}$$

$$\left(|\mathbb{v} \cdot \mathbb{w}| \leq \|\mathbb{v}\| \, \|\mathbb{w}\| \right)$$

Cauchy-Schwarz

$$\underline{\underline{|D_{\mathbb{v}} u| \leq |Du|}}$$

Q: Is there some $\mathbb{v}$ with

$$D_{\mathbb{v}} u = |Du| \text{ ?}$$

A: $\mathbb{v} = Du/|Du|$.

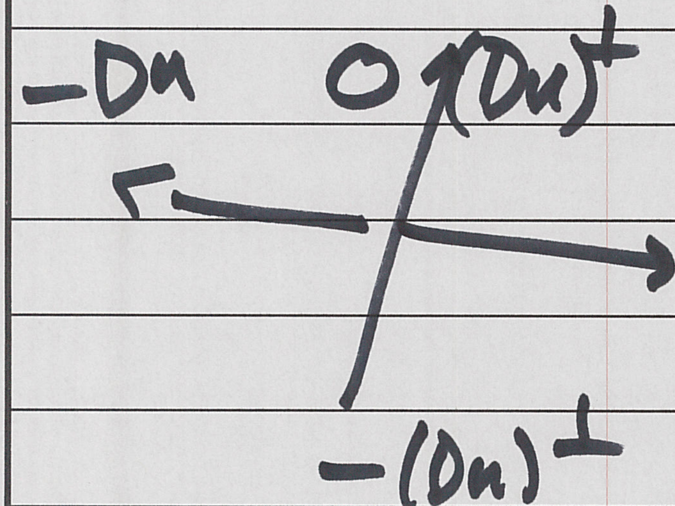
[redacted]

$\underline{n=2}$

zero directions

$$(Du)^{\perp} = \left(-\frac{\partial u}{\partial y}, \frac{\partial u}{\partial x}\right)$$

↑ rotation by 90° counter-clockwise



$u(\mathbf{x}) = c$

$\{\mathbf{x} \in \mathcal{U} : u(\mathbf{x}) = c\}$

This means Du is orthogonal to the <u>level curves</u> of u.

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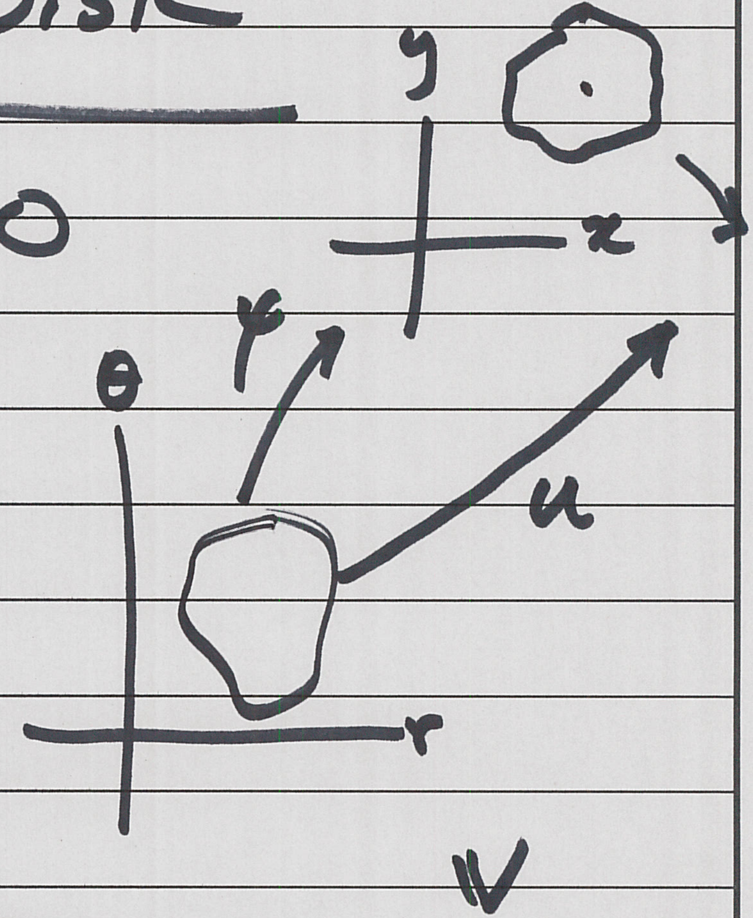
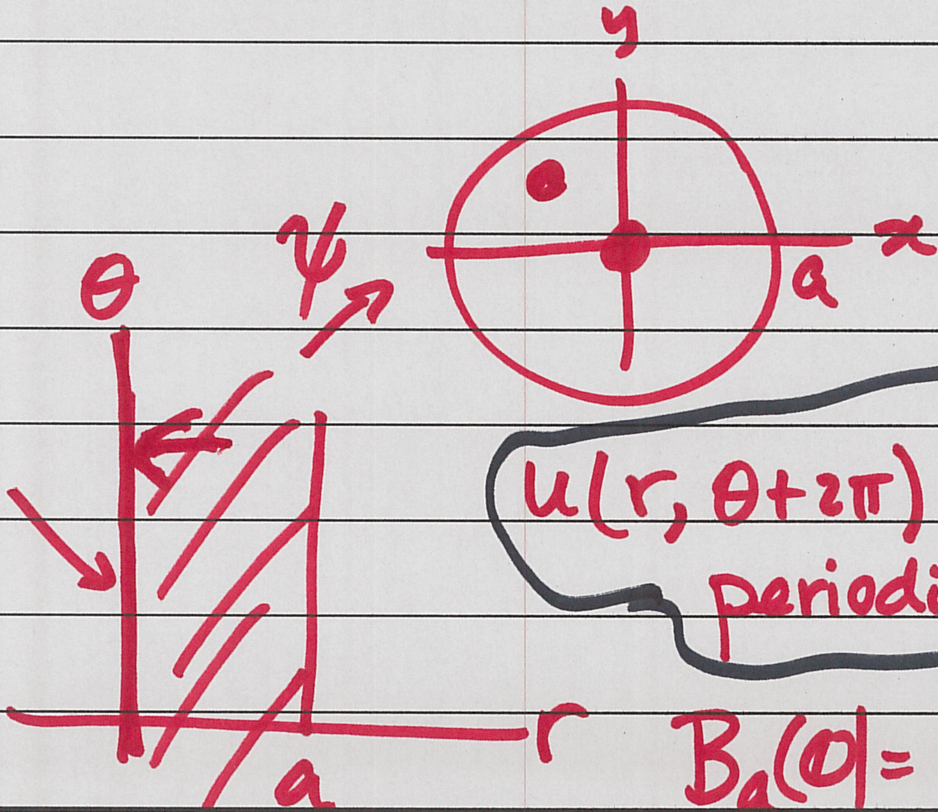
Laplace's Equation on a Disk

$$\begin{cases} \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0 \\ u(a,\theta) = f(\theta) \end{cases}$$





$u(r, \theta + 2\pi) = u(r,\theta)$
periodicity

$B_a(0) = \{(x,y) : x^2 + y^2 < a^2\}$

coordinates are singular at $r=0$

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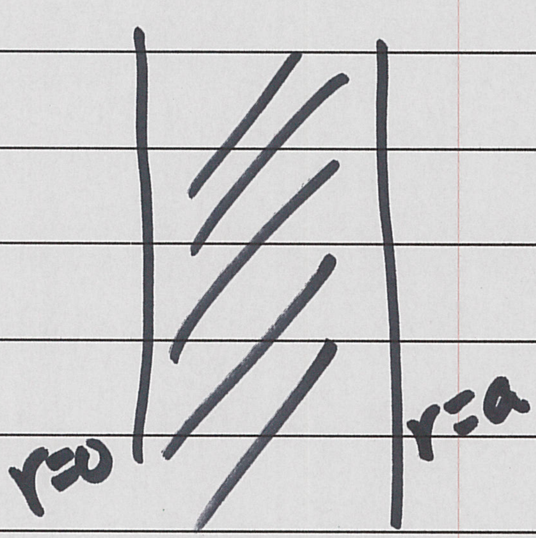
$$\begin{cases} \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0 & \text{on } (0,a)\times\mathbb{R} \\ u(a,\theta) = f(\theta) & \text{periodic. } (2\pi) \end{cases}$$



$$u = A(r)\,B(\theta)$$

$$\frac{1}{r}(rA')'B + \frac{1}{r^2} A B'' = 0.$$

$$-\frac{B''}{B} = \frac{r(rA')'}{A} = \lambda$$

$$A(r)\,B(\theta+2\pi) = A(r)\,B(\theta) \quad \text{for all } \theta.$$

$$\Rightarrow \boxed{\begin{array}{l} B(\theta+2\pi) = B(\theta) \\ B'(\theta+2\pi) = B'(\theta) \quad \text{for all } \theta\in\mathbb{R} \end{array}}$$

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$$\left\{\begin{array}{l} B_0 = a_0 \\ B_j = a_j \cos j\theta \quad , j = 1, 2, \dots \\ \tilde{B}_j = b_j \sin j\theta \quad , j = 1, 2, \dots \end{array}\right.$$

$$\left(-\frac{B''}{B} = \lambda \right), \quad \lambda = 0$$

$$\boxed{\begin{array}{l} B(\theta) = m\theta + b \\ B(\theta + 2\pi) = B(\theta) \end{array}}$$

$$\lambda = j^2$$

$$(r A_0')' = 0 \implies r A_0' = \alpha_0$$

$$A_0' = \frac{\alpha_0}{r}$$

singular at $r=0$ unless $\alpha_0 = 0$.

$A_0' = 0 \implies A_0 = \text{const}.$

$$r(r A_j')' = r^2 A_j'' + r A_j' = j^2 A_j$$

$\leftarrow$ ODE for A_j

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$$\boxed{r^2 A_j'' + r A_j' - j^2 A_j = 0}$$

homogeneous 2^{nd} order linear ODE (non-constant coefficient).

Euler Equation

$$A_j = r^p, \quad A_j' = p r^{p-1}, \quad A_j'' = p(p-1) r^{p-2}$$

$$p(p-1) + p - j^2 = 0$$

$$p = \pm j.$$

$$\boxed{A_j = \alpha_j r^{-j} + \beta_j r^j}$$

singular at $r=0$ unless $\alpha_j = 0$.

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$$\begin{cases} u_0 = a_0 \quad (\text{const.}) \\ u_j = a_j \underset{\substack{\uparrow \\ A_j}}{r^j} \underset{\substack{\uparrow \\ B_j}}{\cos j\theta}, \quad \tilde{u}_j = b_j r^j \sin j\theta \end{cases}$$

$j = 1, 2, 3, \ldots$

separated variables "solutions"

$$\begin{cases} \frac{1}{r}(r u_r)_r + \frac{1}{r^2} u_{\theta\theta} = 0 \quad \text{on } (0,a) \times \mathbb{R} \\ u(a,\theta) = f(\theta) \leftarrow \end{cases}$$

Try

$$u(r,\theta) = a_0 + \sum_{j=1}^{\infty} a_j r^j \cos j\theta + \sum_{j=1}^{\infty} b_j r^j \sin j\theta$$

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$$u(r,\theta) = a_0 + \sum_{j=1}^{\infty} r^j \left(a_j \cos j\theta + b_j \sin j\theta \right)$$

Hope: By choosing the coefficients a_0, a_j, and b_j, $j = 1, 2, 3, \ldots$ appropriately, u will satisfy $u(a,\theta) = f(\theta)$.

$$\boxed{a_0 + \sum_{j=0}^{\infty} a^j \left(a_j \cos j\theta + b_j \sin j\theta \right) = f(\theta).}$$

Basis: 1, $\cos j\theta$, $\sin j\theta$ $\quad j = 1, 2, 3, \ldots$

$$2\pi a_0 = \int_0^{2\pi} f(\theta)\, d\theta$$

[redacted]