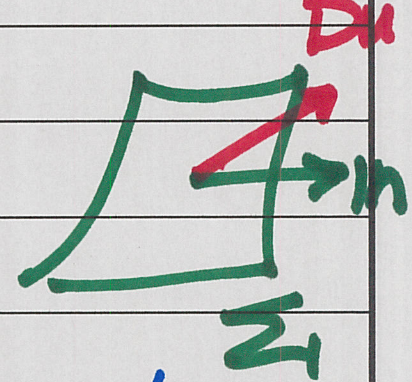
MATH 4581 Lecture 13 Tuesday Oct. 5, 2021

✓ ∘ calculus of variations

✓ ∘ directional derivatives



Fourier's Law: $\vec{\phi} = -K\, Du$

heat energy crossing Σ_1 per time

$$\int_{\Sigma_1} \vec{\phi}\cdot n = -K \int_{\Sigma_1} Du\cdot n$$

$D_n u$

✓ ∘ $\Delta u = 0$ on $B_a(0)$

∘ MEAN VALUE PROPERTY

MAXIMUM PRINCIPLE

∘ MATHY DEFINITIONS "OPEN", "CLOSE", etc.

[redacted]

→ THE RATE OF CHANGE OF u (TEMPERATURE) in the direction [redacted] to Σ_1

Superposition :

$L : A \to B$ operator (linear)

$L(u+v) = Lu + Lv$

$L(au+bv) = a\,Lu + b\,Lv$

If $Lu = 0$ and $Lv = 0$, then

$L(au+bv) = 0.$

APPLICATION If $\Delta u_j = 0$ for $j = 1, 2, 3, \ldots$

Then (maybe) $\Delta\left(\sum_{j=1}^{\infty} a_j u_j\right) = 0$

Another kind of superposition:

If $L u_j = f_j$

Then (maybe) $L\left(\sum_i a_j u_j\right) = \underline{\underline{\sum_i a_i f_j}}$

f

inhomogeneous linear problem

[redacted]

MEAN Value Property:

$$\Delta u = 0 \quad \text{on} \quad B_a(0) \subseteq \mathbb{R}^2 \qquad (n=2)$$

$$B_a(0) = \{(x,y) : x^2+y^2<a^2\}.$$

(ball of radius a in $\mathbb{R}^2$)

$$u(r,\theta) = a_0 + \sum_{j=1}^{\infty} a_j r^j \cos j\theta + \sum_{j=0}^{\infty} b_j r^j \sin j\theta$$

$$u(0,\cdot) = a_0 = \frac{1}{2\pi}\int_0^{2\pi} u(a,\theta)\,d\theta$$

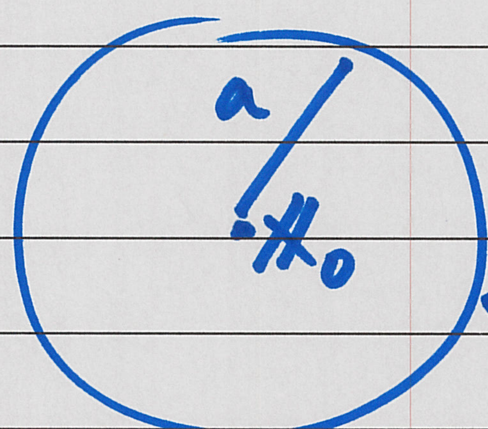
$$u(0,\cdot) = u\big|_{r=0} \qquad u(a,\theta) = u\big|_{\partial B_a(0)}$$

[redacted]

<u>In general</u>: If $\Delta u = 0$ on $\mathcal{U}$ and $\overline{B_a(x_0)} \subseteq \mathcal{U}$, then

$$u(x_0) = \frac{1}{2\pi a} \int_{\partial B_a(x_0)} u$$

$\uparrow$ average of u on $\partial B_a(x_0)$



$\leftarrow \partial B_a(x_0)$

MEAN VALUE PROPERTY

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Integration on $\partial B_a(x_0)$ (integration on a circle)

$$\int_{\partial B_a(x_0)} u = \int_0^{2\pi} \underbrace{u \circ \gamma(t)}\ \underbrace{\sigma}_{\text{scaling factor}}\ dt$$

parameterize the circle

$$\gamma(t) = (a\cos t, a\sin t) + x_0$$

$$\gamma : \underset{\text{parameter domain}}{[0, 2\pi]} \to \mathbb{R}^2$$

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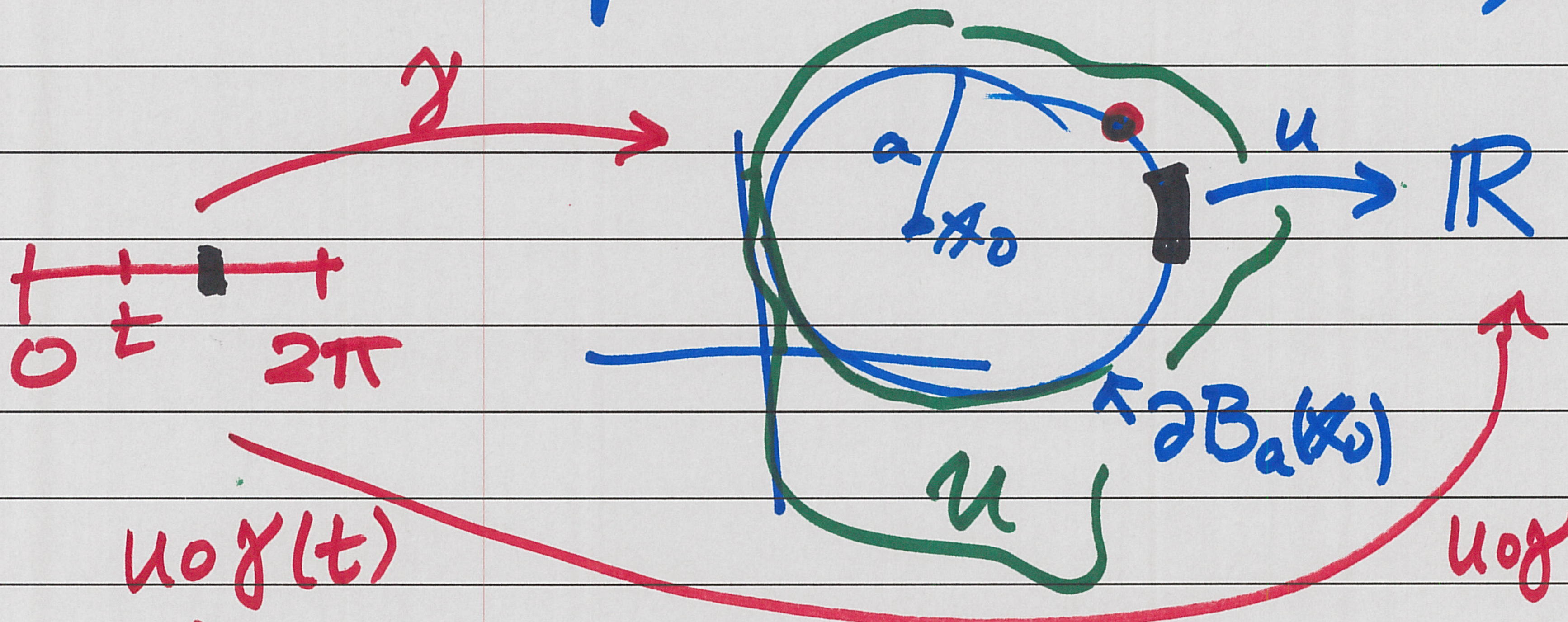
Function Composition (in this case):



The composition of u with/on γ

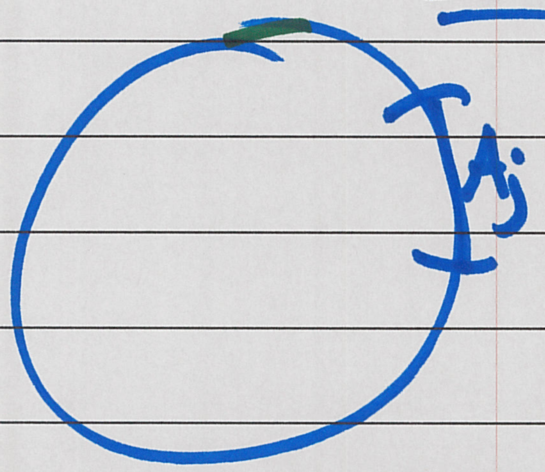
[redacted]

$$\int_{\partial B_a(x_0)} u = \int_0^{2\pi} u \circ \gamma(t)\ \sigma\ \underline{\underline{dt}}$$

$$u(a\cos t, a\sin t)$$

$$\wr \qquad\qquad \wr$$

$$\sum u(p_j^*)\ \underline{\underline{\ell(A_j)}} \qquad \sum u \circ \gamma(t_j^*)\ \Delta t_j$$



What is the scaling factor σ?

For curves:

$$\sigma = |\gamma'(t)|$$

$\gamma(t) = x_0 + (a\cos t, a\sin t)$, $\underline{\dot\gamma(t) = (-a\sin t, a\cos t}$ [redacted]

$$\int_{\partial B_a(x_0)} u = \int_0^{2\pi} u\left(x_0 + (a\cos t, a\sin t)\right) \cdot \underset{\sigma}{\underset{\|}{a}}\, dt$$

<u>MV Prop</u>:

$$u(x_0) = \frac{1}{2\pi a} \int_{\partial B_a(x_0)} u$$

"In coordinates"

$$u(x_0) = \frac{1}{2\pi} \int_0^{2\pi} u\left(x_0 + (a\cos t, a\sin t)\right) dt$$

[redacted]

Strong Maximum Principle:

If $\Delta u = 0$ on $\mathcal{U}^{\text{open}}$ and $\mathcal{U}$ is bounded and connected

And say u is continuous to $\partial\mathcal{U}$

Then

$$u(x_0) < \max_{x \in \partial\mathcal{U}} u(x) \quad \text{for } x_0 \in \mathcal{U}$$

unless $u \equiv$ constant. "=" only $u|_{\partial B_a(x_0)} \equiv M$.

Proof: $u(x_0) = \frac{1}{2\pi a} \int_{\partial B_a(x_0)} u \leq M$

$\boxed{M = \max_{\overline{\mathcal{U}}} u} = \max_{\partial\mathcal{U}} u$

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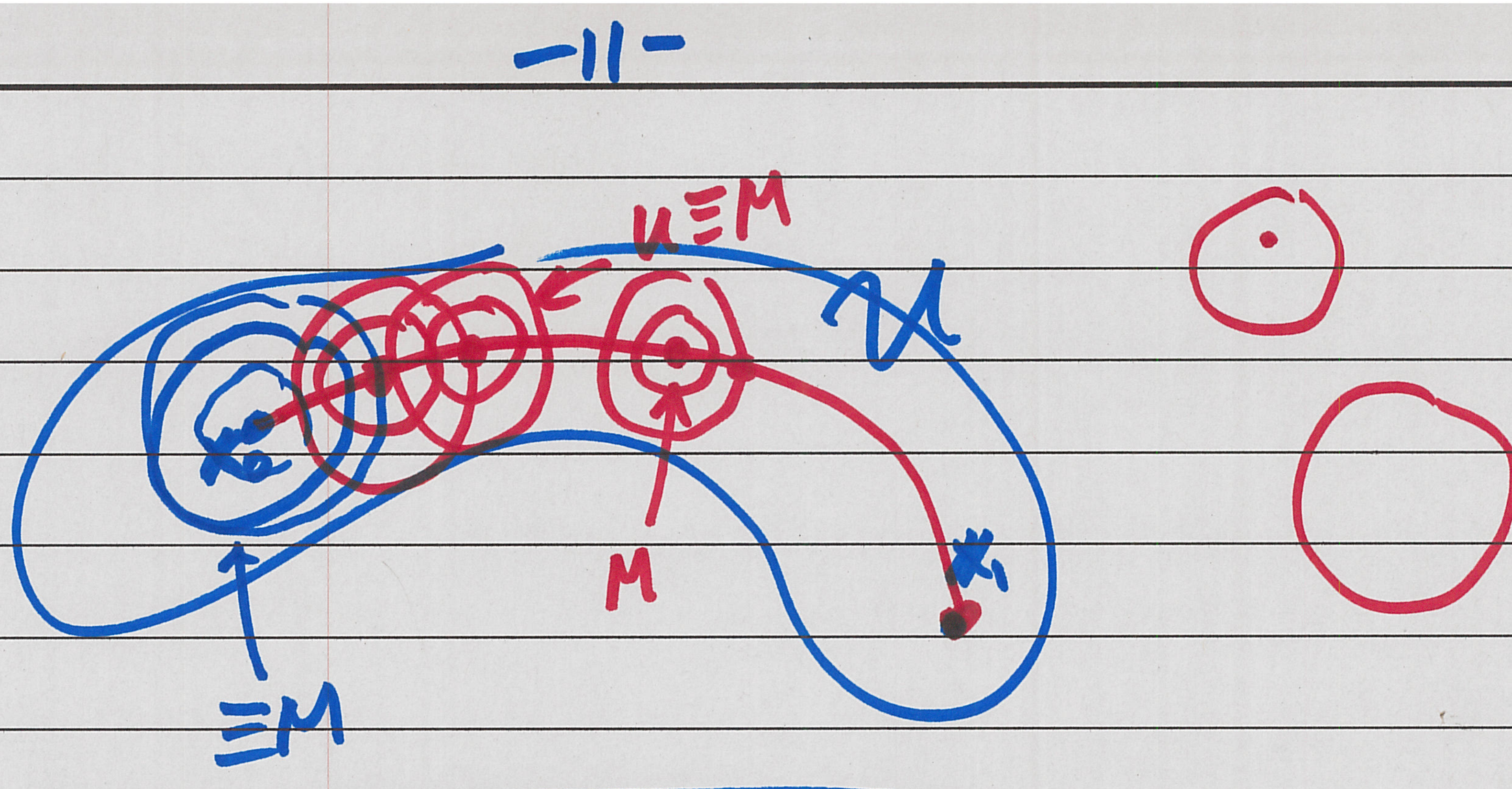

$$\boxed{u(x_0) = M \Rightarrow u\big|_{\partial B_a(x)} \equiv M}$$

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