The Abel Limit Theorem (with errors) And Boundary Behavior of Complex Power Series

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Stein and Shakarchi have a simplified version of the Abel limit theorem in the exercises with a hint to use Stein's exercise on partial summation. I had worked through a somewhat more general version of Abel's theorem as presented in Ahlfors in my notes from 2018, but I think I can simplify that presentation a little bit. Any way one does it, the manipulations to prove this result are a bit complicated, so I'm going to present several different approaches and prove both the simplified and the more general version.

Warning: There is an error in the proof(s) in Section 2 below. It is not such a serious error, but it is an obvious and silly error which I'm going to leave there so you can find it and fix it. Of course, I'll fix it too and post a corrected version. There may also be errors in the discussion of the other sections as well; that goes without saying.

1 Boundary Behavior

Recall that the general theorem on convergence of complex power series gives a disk of convergence with absolute convergence inside the disk and divergence outside the disk. For example, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 - \frac{1}{4} z^4 + \cdots \tag{1}$$

has Hadamard radius

$$R = \frac{1}{\lim_{n\to\infty} 1/n^{1/n}} = 1$$

since
$$\lim_{n\to\infty} \ln n^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} \sim \lim_{n\to\infty} \frac{1}{n} = 0.$$
The theorem tells us nothing, however, about the convergence of the series when $|z| = R$. Furthermore, there are interesting cases where it is relatively easy to determine the convergence or divergence of the series when $|z| = R$, but we know nothing apriori about the relation between the values at the boundary and the values in the interior. For example when $z = 1$ in the series (1) we get the **alternating harmonic series**
$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$
of real numbers. This series can easily seen to converge to some number between $1/2$ and 1, between $7/12$ and $5/6$, and so on, but we might like to know what number this is precisely. This is where the Abel limit theorem comes in. It turns out that for this series the function represented on (interior of) the disk $D_1(0)$ is a well-known standard function with a simple formula, tabulated values, and subject to standardized calculation with mathematical software or even a scientific calculator. In particular, for the real number x satisfying $0 < x < 1$, the value
$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots \tag{2}$$
is **known**. In fact, f is known to be well-defined and continuous for $x \in (-1, \infty)$ with well-defined (and known) value $f(1)$. The Abel limit theorem tells us, for example that the value of the alternating harmonic series is $f(1)$.

Theorem 1 (Abel limit theorem version 1) *If the complex power series*
$$\sum_{n=0}^{\infty} a_n z^n$$
has radius of convergence $R = 1$ representing a complex differentiable function $f : D_1(0) \to \mathbb{C}$ on the open unit disk, and
$$\sum_{n=0}^{\infty} a_n = w \in \mathbb{C},$$
then
$$w = \lim_{x\nearrow 1} f(x) = \lim_{x\nearrow 1} \sum_{n=1}^{\infty} a_n x^n.$$

A proof of this theorem is assigned as an exercise in Stein and Shakarchi and is outlined in Problem 4 of Assignment 4 for my complex analysis course Spring 2022. In the same Assignment one should be able to find, relatively, easily the identity of the function defined in (2) and, consequently, the numerical value of the alternating harmonic series. The proof of Stein and Shakarchi using the "summation by parts" formula is a little slick I find. I'm going to offer another proof, which is also tricky and sort of amounts to the same thing, but may conceivably also be considered somehow more straightforward. I imagine (and hope) so.

2 Proof of Theorem 1

The basic idea is simple: The coefficients a_n in

$$\sum a_n x^n$$

can be replaced with differences of the partial sums of the series $\sum a_n$. To be precise,

$$a_n = S_n - S_{n-1} = \sum_{j=0}^{n} a_j - \sum_{j=0}^{n-1} a_j \qquad \text{for } n = 1, 2, 3, \dots \tag{3}$$

where S_k given by

$$S_k = \sum_{n=0}^{k} a_n$$

is the k-th partial sum of the series $\sum a_n$ as usual. With this in mind, we consider the obvious quantity

$$\sum_{n=0}^{N} a_n x^n - \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{N} a_n x^n - w \tag{4}$$

which we wish to show is small when N is large. This is our basic objective. The first coefficient is an exception not subject to (3), we we write (4) as

$$\begin{aligned} a_0 - w + \sum_{n=1}^{N} a_n x^n &= a_0 - w + \sum_{n=1}^{N} x^n (S_n - S_{n-1}) \\ &= a_0 - w \qquad + \qquad x(S_1 - S_0) \\ &\qquad\qquad + x^2 (S_2 - S_1) \\ &\qquad + x^3 (S_3 - S_2) + \cdots + x^N (S_N - S_{N-1}). \end{aligned}$$

Notice that we have written the sum at the end with suggestive spacing so that a, more or less, obvious regrouping of these funitely many terms gives

$$\begin{aligned}
a_0 - w + \sum_{n=1}^{N} a_n x^n &= a_0(1-x) - w + xS_1(1-x) \\
&\qquad\qquad + x^2 S_2(1-x) \\
&\qquad\qquad + x^3 S_3(1-x) + \cdots + x^N S_N \\
&= a_0(1-x) - w + (1-x)\sum_{n=1}^{N-1} x^n S_n + x^N S_N \\
&= (1-x)\sum_{n=0}^{N-1} x^n S_n - w + x^N S_N. \qquad (5)
\end{aligned}$$

At this point, we can go ahead and finish the proof. Notice the last two terms,

$$x^N S_N - w = x^N(S_N - w) + (x^N - 1)w,$$

so that

$$|x^N S_N - w| \le x^N |S_N - w| + (1 - x^N)|w| \le |S_N - w| + (1 - x^N)|w|. \qquad (6)$$

Let $\epsilon > 0$. We can take N large enough so that $|S_N - w| < \epsilon/3$. Having fixed this N, the first term in (5) may be estimated as

$$\begin{aligned}
\left|(1-x)\sum_{n=0}^{N-1} x^n S_n\right| &\le (1-x)\max\{|S_n| : n = 0,1,2,\ldots,N-1\}\sum_{n=0}^{N-1} x^n \\
&\le (1-x)\max\{|S_n| : n = 0,1,2,\ldots,N-1\}\frac{1-x^N}{1-x} \\
&= (1-x^N)M
\end{aligned}$$

where $M = \max\{S_n : n = 0,1,2,\ldots,N-1\}$ is now considered a fixed nonnegative number (fixed with N).

Finally, for N fixed, we can use the continuity of the function $1 - x^N$ at $x = 1$ to take $\delta > 0$ for which $|1 - x| = 1 - x < \delta$ implies

$$1 - x^N < \frac{\epsilon}{3}\min\left\{\frac{1}{|w|+1}, \frac{1}{M+1}\right\}.$$

We get then for $1 - x < \delta$

$$\left|\sum_{n=0}^{N} a_n x^n - w\right| = \left|a_0 - w + \sum_{n=1}^{N} a_n x^n\right| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \qquad \square$$

Looking back at the argument above, there is a kind of "trick" which can be used to avoid the estimation associated with (6). We start by writing the main quantity of interest in (4) as before:

$$\sum_{n=0}^{N} a_n x^n - w = a_0 - w + \sum_{n=1}^{N} x^n (S_n - S_{n-1}),$$

but then we set $T_n = S_n - w$ before the rearrangement so that

$$\begin{aligned}
a_0 - w + \sum_{n=1}^{N} x^n (S_n - S_{n-1}) &= T_0 + \sum_{n=1}^{N} x^n (T_n - T_{n-1}) \\
&= T_0 \qquad + \qquad x(T_1 - T_0) \\
&\qquad\qquad + x^2 (T_2 - T_1) \\
&\quad + x^3 (T_3 - T_2) + \cdots + x^N (T_N - T_{N-1}) \\
&= (1-x) T_0 + (1-x) x T_1 + (1-x) x^2 T_2 + \cdots \\
&\qquad\quad + (1-x) x^{N-1} T_{N-1} + x^N T_N \\
&= (1-x) \sum_{n=0}^{N-1} x^n T_n + x^N T_N.
\end{aligned}$$

Thus, the estimation is simplified as follows:

$$\left|\sum_{n=0}^{N} a_n x^n - w\right| \leq (1-x) \sum_{n=0}^{N-1} |T_n| x^n + |T_N|.$$

Since T_n converges to 0, we can fix N so that $|T_N| < \epsilon/2$ and then take

$$1 - x < \frac{\epsilon}{2} \, \frac{1}{1 + \sum_{n=0}^{N-1} |T_n|}$$

so that

$$\left|\sum_{n=0}^{N} a_n x^n - w\right| \leq (1-x) \sum_{n=0}^{N-1} |T_n| + \frac{\epsilon}{2} < \epsilon. \qquad \square$$

3 Stolz Angle

4 Another Example: Leibniz Series

The complex arctangent function, or a branch of it, may be defined on the open unit disk according to the relation

$$\frac{d}{dz}\tan^{-1} z = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty}(-1)^n z^{2n}. \tag{7}$$

To choose the branch we need to specify $\tan^{-1}(0) = 0$, and technically, we may start directly with the term by term integrated series:

$$\tan^{-1} z = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n+1}. \tag{8}$$

Exercise 1 *Define a function $f : D_1(0) \to \mathbb{C}$ by the series in (8). Use the differentiation theorem for complex power series to conclude*

$$f'(x) = \frac{1}{1+z^2}.$$

Exercise 2 *Given the complex function f from the previous exercise, show the real function $\tan^{-1} : \mathbb{R} \to \mathbb{R}$ with $\tan^{-1}(0) = 0$ satisfies*

$$f(x) = \tan^{-1}(x) \quad \textit{for } \mathbb{R} \ni x \in (-1,1).$$

Hint, restrict f to the real line to obtani a real valued function g of a real variable on $(-1,1)$; show the real derivative g' agrees with the complex derivative f' along the interval $(-1,1)$.

Taking $z = 1$ in the formal power series given in (8) we see the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

which is convergent as an alternating series of real terms with the individual terms tending to 0. The Abel limit theorem tells us that the value may be obtained as the limit

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \lim_{x \nearrow 1} \tan^{-1}(x) = \tan^{-1}(1) = \frac{\pi}{4}.$$