Assignment 10: Complex Analysis
Due Tuesday April 26, 2022

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Problem 1 *(open mapping theorem) Recall that the open mapping theorem of complex analysis (Theorem 4.4 of Chapter 3 in S&S) asserts that if* $f : \Omega \to \mathbb{C}$ *is a non-constant holomorphic function on an open connected set* $\mathcal{U} \subset\subset \Omega$*, then*

$$f(\mathcal{U}) = \{f(z) : z \in \mathcal{U}\} \qquad \text{is an open subset of } \mathbb{C}.$$

There is also an **open mapping theorem of functional analysis** *which asserts that certain images of open subsets are open. Look up the open mapping theorem of functional analysis and compare it to the open mapping of complex analysis. For example, are there any special cases when one implies the other? (It goes without saying that in order to do this problem properly you should write down the statement of the functional analytic result.)*

Problem 2 *(maximum modulus principle Theorem 4.5 of Chapter 3 in S&S) Recall that the maximum modulus principle states:*

> *If* f *is a non-constant holomorphic function, there cannot exist an interior point at which* $|f(z)|$ *attains a maximum value.*

In particular, if $f : \Omega \to \mathbb{C}$ *with* f *non-constant on* $D_r(z_0) \subset\subset \Omega$*, then*

$$|f(z_0)| < \max_{z \in \overline{D_r(z_0)}} |f(z)|.$$

(a) *Consider* $\psi(z) = (z+1)/(z-1)$ *and* $f : D_1(0) \to \mathbb{C}$ *by*

$$f(z) = e^{\psi(z)}.$$

- **(i)** *Show* f *satisfies the hypotheses of the maximum modulus principle.*
- **(ii)** *Show* f *has a continuous extension to* $\partial D_1(0)\backslash\{1\}$ *and an extension to* $\partial D_1(0)$ *with* $|f|$ *lower-semicontinuous.*[1]
- **(iii)** *Find*

$$\sup_{z \in \partial D_1(0)} |f(z)|.$$

(b) *(Chapter 3 Corollary 4.6 in S&S) Assume* $f : \Omega \to \mathbb{C}$ *is holomorphic and* $\overline{\Omega}$ *is compact. Show that if* f *extends continuously to* $\partial\Omega$*, then*

$$\sup_{z \in \Omega} |f(z)| = \max_{z \in \partial\Omega} |f(z)|.$$

(c) *Consider* $f : D_1(0) \to \mathbb{C}$ *by*

$$f(z) = e^{-\psi(z)}$$

with ψ *given in part* **(a)** *above.*

- **(i)** *Find the image* $\{\psi(z) : z \in D_1(0)\}$.
- **(ii)** *Find*

$$\sup_{z \in D_1(0)} |f(z)| \qquad \text{and} \qquad \sup_{z \in \partial D_1(0)\backslash\{1\}} |f(z)|.$$

- **(iii)** *Critique the assertion*

$$\lim_{t \to \infty} e^{-\psi(it)} = f(1).$$

(d) *Can you state and prove a version of Stein and Shakarchi's Corollary 4.6 that applies to the function* $f : D_1(0) \to \mathbb{C}$ *by* $f(z) = e^{\psi(z)}$ *considered in part* **(a)** *above?*

[1] A function $g : A \to \mathbb{R}$ like $|f|$ is **lower-semicontinuous** on a set $A \subset \mathbb{C}$ if for each $z_0 \in A$ the following holds: For any $\epsilon > 0$, there is some $\delta > 0$ for which $z \in A \cap D_\delta(z_0)$ implies $g(z) > g(z_0) - \epsilon$.

Problem 3 *(S&S Chapter 3 Exercise 21; simply connected domains)*

(a) *State a/the definition of what it means for an open set $\Omega \subset \mathbb{C}$ to be* **simply connected**.

(b) *Give two different proofs that if $f : \Omega \to \mathbb{C}$ is holomorphic and α parameterizes any loop in any simply connected subdomain of Ω, then*

$$\int_\alpha f = 0.$$

Hint: For one proof define a primitive and for the other use Lang's theorem.

(c) *An open set $\Omega \subset \mathbb{C}$ is said to be* **star shaped** *if there is some $z_0 \in \Omega$ for which*

$$(1-t)z_0 + tz \in \Omega \qquad \text{for } 0 \le t \le 1 \text{ whenever } z \in \Omega.$$

Show every star shaped set is simply connected.

(d) *An open set $\Omega \subset \mathbb{C}$ is said to be* **convex** *if*

$$(1-t)z_1 + tz_2 \in \Omega \qquad \text{for } 0 \le t \le 1 \text{ whenever } z_1, z_2 \in \Omega.$$

Show every convex set is star shaped (and hence simply connected).

(e) *Let θ be fixed with $0 \le \theta < 2\pi$ and consider a set*

$$\Omega = \mathbb{C} \backslash \{re^{i\theta} \in \mathbb{C} : r \ge 0\}.$$

This domain may be called a **branch cut plane**. *Show Ω is star shaped (and simply connected).*

(f) *Give a precise definition of a "keyhole domain" based on the annulus $D_r(z_0)\backslash\overline{D_\epsilon(z_0)}$ and show such a set is simply connected. Hint: For the definition, consider the cut plane Ω in part* **(e)** *above.*

(g) *Give an example of what Stein and Shakarchi mean by a "nonconvex sector" in Exercise 21 part(b) of Chapter 3. If you can figure that out, prove such a thing is simply connected.*[2]

[2] I don't really know what they have in mind. Update: It was pointed out to me that any sector determined by an angle θ with $\pi < \theta < 2\pi$ is nonconvex.

Problem 4 *(S&S Chapter 3 Exercise 22) Given a holomorphic function* $f : D_1(0) \to \mathbb{C}$ *which extends continuously to* $\partial D_1(0)$ *and the specific holomorphic function* $g : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ *by* $g(z) = 1/z$, *is it possible that*

$$f\Big|_{|z|=1} = g\Big|_{|z|=1}?$$

Problem 5 *(complex logarithm) Let* Ω *be a cut plane as in part* **(e)** *of Problem 3 above, and assume* $1 \in \Omega$. *Given* $z \in \Omega$ *let* α *be a path connecting* 1 *to* z *and define*

$$g(z) = \int_\alpha \frac{1}{z}.$$

(a) *By taking a branch of the logarithm (to get a branch of* $\operatorname{Arg}(z) = \operatorname{Im}\log(z)$ *on* Ω*) explicitly write down a path* α *connecting* 1 *to a specific point* x *on the real axis and then connecting* x *to* z *along an arc of a circle. Compute the value of* g *explicitly.*

(b) *Show that on the basis of general principle, i.e., Cauchy's theorem or Lang's theorem, that* g *is well-defined.*

(c) *Use a difference quotient to compute* $g'(z)$ *and hence verify that* g *is holomorphic on* Ω.

(d) *Let* W *be a subdomain of* Ω *with* $1 \in W$ *and* $f : W \to \mathbb{C}$ *be a holomorphic function satisfying*

(i) $e^{f(z)} \equiv z$ *for* $z \in W$, *and*

(ii) $f(x) = \ln(x)$ *for* $x \in W \cap (0, \infty)$ *where* $\ln : [0, \infty) \to \mathbb{R}$ *is determined by*

$$\ln(1 + x) = \int_1^x \frac{1}{\xi}\, d\xi.$$

Can you show $f \equiv g$?

Problem 6 *(complex logarithm and complex powers* z^α*) Let*

$$\mathcal{L} = \cup_{j=-\infty}^{\infty} \mathcal{L}_j$$

be the Riemann surface for e^z *with* $\operatorname{Arg}(z) \in (2\pi j - \pi, 2\pi j + \pi]$ *on the* j*-th sheet* $\mathcal{L}_j$ *with* $\log : \mathcal{L} \to \mathbb{C}$ *well-defined and the restriction*

$$\log\Big|_{\mathcal{L}_j} : \mathcal{L}_j \to \mathbb{C}$$

denoted by $\log_j$*. Given* $\alpha \in \mathbb{C}\backslash(-\infty, 0]$*, what is the difference between*

$$e^{\alpha \log_0(z)} \qquad \text{and} \qquad e^{\alpha \log_j(z)} \quad \textit{for } j \neq 0?$$

Hint: There are two different ways to interpret the phrase "difference between" in this problem, a mathematical way and a "general" way, which also might mean "quotient of."

Problems 7 through 10

The remaining problems 7-10 are devoted to aspects of the following vague assertion:

Conjecture 1 *There is a "natural isometric embedding" of a Riemann surface in the neighborhood of a* **branch point** *into* $\mathbb{C} \times \mathbb{C} = \mathbb{C}^2$*. If* $\mathcal{R}$ *is the Riemann surface associated with* $f : \mathbb{C} \to \mathcal{R}$*, and there is a branch point at* p^**, then the isometric embedding* $\mathcal{J} : \mathcal{R}^* \to \mathbb{C} \times \mathbb{C}$ *may be defined on* $\mathcal{R}^* = \mathcal{R} \cup \{p^*\}$ *so that*

(i) *The surface* $\mathcal{R}^*$ *is nonsingular, and*

(ii) *The function* $f^{-1} : \mathbb{C} \to \mathcal{R}^*$ *is "desingularized."*

If you are willing to wade through these problems, then I hope you will have (1) a pretty good idea of what **(i)** involves, (2) at least some idea of what **(ii)** involves/intends, and (3) (perhaps most importantly) some insight into the mysterious nature of branch points.

What are **branch points**? Are they singularities? Are they regular points in some sense? If these questions interest you, then I encourage you to read on.

Preliminary warnings: We will only consider some specific aspects in the direction of the conjecture. Some considerations will be "heuristic," and in particular I will try to outline an approach to proving the following more precise assertion:

Conjecture 2 *If* $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ *denotes the standard Riemann surface for* z^2 *consisting of two sheets* $\mathcal{R}_1$ *and* $\mathcal{R}_2$ *with*

$$\operatorname{Arg}(z) \in [2\pi(j-1), 2\pi j) \quad \textit{for } z \in \mathcal{R}_j,\ j = 1, 2$$

and $\mathcal{R}^* = \mathcal{R} \cup \{p^*\}$ *is the extension of* $\mathcal{R}$ *obtained by adjoining one additional point* p^* *corresponding to the branch point at* 0*, then there is a (natual) metric induced on* $\mathcal{R}$ *from* $\mathbb{C}$ *with*

$$d(p^*, z) = |z| \qquad \text{for } z \in \mathcal{R},$$

and there is a smooth regular surface $\mathcal{S}^* \subset \mathbb{C} \times \mathbb{C}$ *and a smooth bijective isometry*

$$\mathcal{J} : \mathcal{R}^* \to \mathcal{S}^*.$$

This statement is also not entirely complete nor precise because for one thing it is not entirely clear what it means for the isometry $\mathcal{J}$ to be "smooth" at p^*. We have also not addressed the desingularization of the square root at p^*. Nevertheless, Problem 9 gives some indications of how both of these important questions may be addressed. The ultimate highlight/punchline is perhaps the computation of the last part (part **(d)**) of Problem 10. Problems 7 and 8 are especially intended to facilitate the appreciation of this highlight.[3]

For Problems 7 and 8, we will depart from complex analysis and consider some real differential geometry (and smooth geometric measure theory).[4] Here is an impressive sounding title for Problems 7 and 8:

Elementary Besicovich Density

If $\alpha : (t_1, t_2) \to \mathbb{R}^n$ is a smooth and regular ($|\alpha'| \neq 0$) parameterization of a simple/embedded curve Γ, then for each $t_0 \in (t_1, t_2)$ and each $a > 0$, we can consider the length

$$\mathcal{L}(a) = \mathcal{H}^1(\Gamma \cap B_a(\alpha(t_0)))$$

where as usual $B_a(\mathbf{x}_0) = \{\mathbf{x} \in \mathbb{R}^n : |\mathbf{x} - \mathbf{x}_0| < a\}$ and $\mathcal{H}^1$ denotes one-dimensional Hausdorff (length) measure on subsets of $\mathbb{R}^n$. The **Besicovich length density** at $\alpha(t_0) \in \Gamma$ is defined to be

$$\lim_{a \searrow 0} \frac{\mathcal{L}(a)}{2a}.$$

[3]Given that the road from Problem 7 to Part **(d)** of Problem 10 is a bit long, it may make sense to start with Problem 10 Part **(d)** and work back through any concepts which may be unfamiliar, pick up just enough to make the calculation, and then go back through Problems 7-10 carefully to see how the intermediate material fits in.

[4]The parenthetical remark is a little bit of a math joke because geometric measure theory is essentially the study of very non-smooth sets. Nevertheless, we will apply one elementary notion from geometric measure theory to smooth real manifolds.

Problem 7 *(Besicovich length density)*

(a) *(example) Consider a line* $\Gamma \subset \mathbb{R}^n$ *parameterized by* $\alpha : \mathbb{R} \to \mathbb{R}^n$ *by* $\alpha(t) = t\mathbf{e}_j$ *where* $\mathbf{e}_j$ *is a standard unit basis vector in* $\mathbb{R}^n$*. What is* $\Gamma \cap B_a(\alpha(0))$*? What is* $\ell(a) = \mathcal{H}^1(\Gamma \cap B_a(\alpha(0)))$*? What does this tell you about the Besicovich length density?*

(b) *Calculate the Besicovich length density of* **any** *embedded curve* Γ *in* $\mathbb{R}^n$ *at any point* $\mathbf{x}_0 \in \Gamma$*.*

Surfaces and Area Density

A **regular surface patch** in $\mathbb{R}^n$ is a function $\phi : U \to \mathbb{R}^n$ given by $\phi = \phi(\xi, \eta)$ where U is an open subset of $\mathbb{R}^2$ and

$$\left\{ \phi_\xi(\xi, \eta) = \frac{\partial \phi}{\partial \xi}(\xi, \eta), \phi_\eta(\xi, \eta) = \frac{\partial \phi}{\partial \eta}(\xi, \eta) \right\}$$

constitutes a linearly independent pair of vectors in $\mathbb{R}^n$ for each $(\xi, \eta) \in U$. We will generally assume $\phi \in C^\infty(U \to \mathbb{R}^n)$ meaning of course that all partial derivatives of all orders are well-defined and continuous.

A **globally embedded surface** $\mathcal{S} \subset \mathbb{R}^n$ is the image of a regular surface patch $\phi : U \to \mathbb{R}^n$ for which $\phi : U \to \mathcal{S}$ is bijective and

$$\phi^{-1} : \mathcal{S} \to U$$

is smooth either in the sense that there exists an open set $W \subset \mathbb{R}^n$ with $\mathcal{S} \subset W$ and (there exists) an extension

$$\overline{\phi^{-1}} \in C^\infty(W \to \mathbb{R}^2) \qquad \text{with} \qquad \overline{\phi^{-1}}\Big|_{\mathcal{S}} \equiv \phi^{-1}.$$

or in the (equivalent) sense that for each $\mathbf{p} \in \mathcal{S}$ and every injective regular surface patch $\tilde{\phi} : \tilde{U} \to \mathbb{R}^n$ with

$$\mathbf{p} \in \{\tilde{\phi}(\xi, \eta) : (\xi, \eta) \in \tilde{U}\} \subset \mathcal{S},$$

one has

$$\tilde{\phi}^{-1} \circ \phi\Big|_{\phi^{-1}(\tilde{U})} \in C^\infty(\phi^{-1}(\tilde{U}) \to \mathbb{R}^2) \qquad \text{and} \qquad \phi^{-1} \circ \tilde{\phi} \in C^\infty(\tilde{U} \to \mathbb{R}^2).$$

The **Besicovich area density** at a point $\mathbf{p}$ in a globally embedded surface as above is defined to be

$$\lim_{a \searrow 0} \frac{\mathcal{H}^2(\mathcal{S} \cap B_a(\mathbf{p}))}{\pi a^2}.$$

Problem 8 *(Besicovich area density)*

(a) *(example) Let* $\mathbf{e}_j$ *and* $\mathbf{e}_k$ *be distinct standard unit basis vectors in* $\mathbb{R}^n$*. Consider* $\phi : \mathbb{R}^2 \to \mathbb{R}^n$ *by* $\phi(\xi, \eta) = \xi \mathbf{e}_j + \eta \mathbf{e}_k$*. Show*

$$\mathcal{S} = \{\phi(\xi, \eta) : (\xi, \eta) \in \mathbb{R}^2\}$$

defines a globally embedded surface, and compute the Besicovich area density at each $\mathbf{p} \in \mathcal{S}$*.*

(b) *(Besicovich area density for linear injections* $L : \mathbb{R}^2 \to \mathbb{R}^3$*.) If* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are linearly independent (column) vectors in* $\mathbb{R}^3$ *and* $L(\mathbf{x}) = A\mathbf{x}$ *defines a linear function on column vectors* $\mathbf{x} \in \mathbb{R}^2$ *where* A *is the matrix with* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *in the columns, then the* **area scaling factor** *for* L *is given by*

$$\sigma = \sqrt{|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2} = \sqrt{\det(A^T A)}.$$

This means that if you want to calcuate the area of the image of any set $U \subset \mathbb{R}^2$ *under* L*, you should integrate* σ*:*

$$A = \int_U \sqrt{|\mathbf{v}_1|^2 |\mathbf{v}_2|^2 - (\mathbf{v}_1 \cdot \mathbf{v}_2)^2}.$$

Calculate the Besicovich area density of the image of L *at* $\mathbf{0} = (0, 0, 0)^T \in \mathbb{R}^3$*. Hint: What is the image* $\mathcal{S} = \{L\mathbf{x} : \mathbf{x} \in \mathbb{R}^2\}$ *of* L*, and what is the area of* $\mathcal{S} \cap B_a(\mathbf{0})$*?*

(c) *What is the Besicovich area density at each point* $\mathbf{p} \in \mathcal{S}$ *in a globally embedded surface* $\mathcal{S} \subset \mathbb{R}^3$*? Note: You may not be able to give all the rigorous details, but you should know the answer.*

Problem 9 *(Branches of the real square root) Here we are going to consider a subset $\mathcal{R}_0$ of the Riemann surface $\mathcal{R}$ for z^2 described above, but still everything will be "real." More precisely, we consider the interval $(0, \infty)$ in the first sheet $\mathcal{R}_1$ and the interval $(0, \infty)$ in the second sheet $\mathcal{R}_2$ of $\mathcal{R}$. There are various ways to make this rigorous or precise. One way is to consider the disjoint union*

$$\mathcal{R}_0 = (0, \infty) \sqcup (0, \infty)$$

which usually involves labels $\{(x, j) : x \in (0, \infty),\ j = 1, 2\}$. I like to think of the first copy with $\mathrm{Arg}(x) = 0$ as a particular color, say black and the second copy of $(0, \infty)$ with $\mathrm{Arg}(x) = 2\pi$ as red.. In any case, we know there is a globally defined **inverse square root** *$\rho : \mathcal{R} \to \mathbb{C}$, and we can restrict ρ to $\mathcal{R}_0$. Call this restriction $\rho_0 : \mathcal{R}_0 \to \mathbb{R}$ by*

$$\rho_0(x) = \rho(x) \qquad \text{for } x \in \mathcal{R}_0 = (0, \infty) \sqcup (0, \infty).$$

(a) *Let $\alpha : \mathcal{R}_0 \to \mathbb{R}^2$ by*

$$\alpha(x) = (\rho_0(x), x).$$

Draw the image of α in $\mathbb{R}^2$.

(b) *Append a point p to $\mathcal{R}_0$ corresponding to the branch point at $z = 0$. Define a metric on $\mathcal{R}_0^* = \mathcal{R}_0 \cup \{p\}$, and show*

$$\lim_{\mathcal{R}_0^* \ni x \to p} \alpha(x) = \lim_{\mathcal{R}_0^* \ni x \to p} (\rho(x), x) = (0, 0) \in \mathbb{R}^2.$$

(c) *Is the image of α a singular curve or a smooth curve?*

(d) *Consider the maps*

$$\begin{aligned}
&\phi_1 : (0, \infty) \to \mathbb{R}^2 \qquad \text{by } \phi_1(\xi) = (\sqrt{\xi}, \xi),\\
&\phi_2 : (0, \infty) \to \mathbb{R}^2 \qquad \text{by } \phi_2(\xi) = (-\sqrt{\xi}, \xi),\\
&\phi_0 : \mathbb{R} \to \mathbb{R}^2 \qquad \text{by } \phi_0(\xi) = (\xi, \xi^2),\\
&\psi_1 : (0, \infty) \to \mathbb{R}^2 \qquad \text{by } \psi_1(t) = (t, t^2),\\
&\psi_2 : (-\infty, 0) \to \mathbb{R}^2 \qquad \text{by } \psi_2(t) = (t, t^2),\\
&\Phi : (0, \infty) \sqcup (0, \infty) \to \mathcal{R}_0 \qquad \text{by } \Phi(t, j) = t \in \mathcal{R}_j,\ j = 1, 2,\\
&\Psi : (-\infty, 0) \cup (0, \infty) \to \mathcal{R}_0 \qquad \text{by } \Phi(t) = t \in \mathcal{R}_j,\ j = (3 - t/|t|)/2,\\
&I : (0, \infty) \to (0, \infty) \sqcup (0, \infty) \qquad \text{by } I(t) = (t, 1), \text{ and}\\
&J : (0, \infty) \to (-\infty, 0) \cup (0, \infty) \qquad \text{by } J(t) = -t.
\end{aligned}$$

Show

$$\psi_1 = \phi_0\Big|_{(0,\infty)} \qquad \text{and} \qquad \psi_2 = \phi_0\Big|_{(-\infty,0)}$$

and

$$\begin{aligned} \phi_0 &= \gamma(x^2), \\ \phi_1 &= \gamma \circ \Phi \circ I, \ \ \textit{and} \\ \psi_2 &= \gamma \circ \Psi \circ J. \end{aligned}$$

Note that ϕ_0 is a global patch for the image (curve) of α in the sense of surfaces described above.

Problem 10 *(A branch point in a Riemann surface) Let $\mathcal{R}$ denote the Riemann surface for z^2 with $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ as described above. Recall that the square root $\rho : \mathcal{R} \to \mathbb{C}$ is globally defined on $\mathcal{R}$.*

(a) *Extend the domain of $\mathcal{R}$ of the square root to $\mathcal{R}^*$ by adjoining a point p^* corresponding to the branch point at $z = 0$ and define a metric on $\mathcal{R}^*$. Consider*

$$X : \mathcal{R} \to \mathbb{C} \times \mathbb{C} = \mathbb{C}^2 \qquad \textit{by } X(z) = (\rho(z), z).$$

Show

$$\lim_{\mathcal{R} \ni z \to 0} X(z) = (0,0) \in \mathbb{C}^2$$

so that X extends to $\mathcal{R}^*$. Denote this extended map by

$$X^* : \mathcal{R}^* = \mathcal{R} \cup \{0\} \to \mathbb{C} \times \mathbb{C}.$$

Show X^* is an injection.

(b) *Show $\psi : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$ by $\psi(w) = X^*(w^2)$ is an injection with image*

$$\{\psi(w) : w \in \mathbb{C}\} = \{X^*(z) : z \in \mathcal{R}^*\}.$$

Here it is important to use $\mathcal{R}$ as the codomain of the mapping $w \mapsto w^2$.

(c) *Use the identification $z = \xi + i\eta \in \mathbb{C}$ with $(\xi,\eta) \in \mathbb{R}^2$ to write down a function $\phi : \mathbb{R}^2 \to \mathbb{R}^4$ corresponding to ψ. Show*

$$\mathcal{S} = \{\phi(\xi,\eta) : (\xi,\eta) \in \mathbb{R}^2\}$$

is a globally embedded surface in $\mathbb{R}^4$. This means the Riemann surface is smooth and embedded across the branch point when realized in $\mathbb{R}^4 \cong \mathbb{C} \times \mathbb{C}$. Of course, you can't do this in $\mathbb{R}^3$.

(d) *Determine the Besicovich area density of $\mathcal{S}$ at the point $\mathbf{0} = (0,0,0,0) \in \mathbb{R}^4$ corresponding to the branch point. Hints:*

(i) *For $a > 0$, determine*

$$U_a = \{(\xi, \eta) \in \mathbb{R}^2 : |\phi(\xi, \eta)| < a\}.$$

Hint hint: This set is a disk.

(ii) *The area of $\{\phi(\xi, \eta) : (\xi, \eta) \in U_a\}$ is given by*

$$\int_{U_a} \sigma \qquad \text{where } \sigma = \sqrt{\det(D\phi^T D\phi)}$$

and

$$D\phi = \begin{pmatrix} \dfrac{\partial \phi_1}{\partial \xi} & \dfrac{\partial \phi_1}{\partial \eta} \\[2ex] \dfrac{\partial \phi_2}{\partial \xi} & \dfrac{\partial \phi_2}{\partial \eta} \\[2ex] \dfrac{\partial \phi_3}{\partial \xi} & \dfrac{\partial \phi_4}{\partial \eta} \\[2ex] \dfrac{\partial \phi_4}{\partial \xi} & \dfrac{\partial \phi_4}{\partial \eta} \end{pmatrix}.$$

(e) bonus *Extend the domain $\mathcal{R}$ of the square root ρ by adding an extra point at ∞ and extending the codomain to the Riemann sphere. Determine the nature of the isolated singularity at $\infty \in \mathcal{R}$.*