Assignment 2 Problem 7
Second Partial Solution

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Here is an alternative proof of the chain rule for a composition of holomorphic functions. It is based on my previous solution of this problem but the argument is streamlined and the estimates are tightened. For clarity with regard to the two arguments/solutions, let me point out that the invoking of differentiability of the functions g and f (and the roles of δ_1 and δ_2) has been reversed.

Again I write down what I want to prove. That is,

$$(g \circ f)(z+h) - (g \circ f)(z) - (g' \circ f)(z)\ f'(z)\ h = \circ(h). \tag{1}$$

This means precisely that

$$\lim_{h \to 0} \frac{(g \circ f)(z+h) - (g \circ f)(z) - (g' \circ f)(z)\ f'(z)\ h}{h} = 0,$$

or for any $\epsilon > 0$, there is some $\delta > 0$ for which $0 < |h| < \delta$ implies

$$\left| \frac{(g \circ f)(z+h) - (g \circ f)(z) - (g' \circ f)(z)\ f'(z)\ h}{h} \right| < \epsilon. \tag{2}$$

To this end, let $\epsilon > 0$. Since f is differentiable at z, there is some $\delta_1 > 0$ for which $0 < |h| < \delta_1$ implies

$$\left| \frac{f(z+h) - f(z) - f'(z)\ h}{h} \right| < \min\left\{1,\ \frac{\epsilon}{2}\ \ \frac{1}{|g'(f(z))| + 1}\right\}. \tag{3}$$

As motivation for the choice of the positive tolerance appearing on the right in (3), let us consider the estimation of the quantity on the left of (2) for any complex number h with $|h| > 0$. By the triangle inequality

$$\left| \frac{(g \circ f)(z+h) - (g \circ f)(z) - (g' \circ f)(z)\ f'(z)\ h}{h} \right| \le Q_2 + Q_1$$

where

$$Q_2 = \left| \frac{(g \circ f)(z+h) - (g \circ f)(z) - g'(f(z))\ k}{h} \right|,$$

$$Q_1 = \left| \frac{g'(f(z))\ k - g'(f(z))\ f'(z)\ h}{h} \right|,$$

and

$$k = f(x+h) - f(x). \tag{4}$$

Notice that

$$Q_1 = |g'(f(z))| \left| \frac{f(z+h) - f(z) - f'(z)\ h}{h} \right|,$$

and the factor on the right vanishes in the limit as $h \to 0$. This is made quantitatively precise by our estimate (3) so that for $0 < |h| < \delta_1$ we know

$$Q_1 < \frac{\epsilon}{2}.$$

We wish now to obtain a similar estimate for Q_2. In fact with our definition of k we see that the argument $f(z+h) = f(z) + k$, so we can write

$$Q_2 = \left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{h} \right|.$$

This looks rather like the definition of differentiability of g at $f(z)$. In fact, for $k = f(z+h) - f(z) = 0$, the quantity Q_2 vanishes, and for $k \neq 0$, we can write

$$Q_2 = \left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{k} \right| \left| \frac{f(z+h) - f(z)}{h} \right|.$$

In order to estimate the factor

$$\left| \frac{f(z+h) - f(z)}{h} \right|$$

we consider the condition (3) holding for $0 < |h| < \delta_1$ and obtain

$$\left| \frac{f(z+h) - f(z)}{h} \right| \leq \left| \frac{f(z+h) - f(z)}{h} - f'(z) \right| + |f'(z)| < 1 + |f'(z)|. \tag{5}$$

Consequently, we invoke the definition of differentiability of g at $f(z)$ in the following form: There exists some δ_2 for which the condition $0 < |\mu| < \delta_2$ on a complex number μ implies

$$\left| \frac{g(f(z)+\mu) - g(f(z)) - g'(f(z))\ \mu}{\mu} \right| < \frac{\epsilon}{2}\ \frac{1}{1 + |f'(z)|}. \tag{6}$$

In order for this to apply with $\mu = k = f(z+h) - f(z)$ we need $|f(z+h) - f(z)| < \delta_2$. Thus, we consider once again (3) or alternatively (5). From these we see that if $0 < |h| < \delta_1$, then

$$|k| = |f(z+h) - f(z)| < (1 + |f'(z)|)|h|.$$

Therefore, we make the final specification

$$0 < |h| < \delta = \min\left\{ \delta_1, \frac{\delta_2}{1 + |f'(z)|} \right\}.$$

If this condition holds $|k| = |f(z+h) - f(z)| < \delta_2$. In particular, if $k \neq 0$, then (6) and (5) together imply

$$Q_2 = \left| \frac{g(f(z)+k) - g(f(z)) - g'(f(z))\ k}{k} \right| \left| \frac{f(z+h) - f(z)}{h} \right| < \frac{\epsilon}{2}.$$

As mentioned above when $k = 0$, the inequality $Q_2 < \epsilon/2$ holds trivially. In all cases then we have established that $0 < |h| < \delta$ implies

$$\left| \frac{(g \circ f)(z+h) - (g \circ f)(z) - (g' \circ f)(z)\ f'(z)\ h}{h} \right| \leq Q_1 + Q_2 < \epsilon.$$

That is,

$$(g \circ f)(z+h) - (g \circ f)(z) - (g' \circ f)(z)\ f'(z)\ h = \circ(h). \qquad \square$$