Assignment 3:
Power Series
Due Tuesday February 8, 2022

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Problem 1 *(Antiholomorphic Functions) Remember that given a function* $f : \Omega \to \mathbb{C}$ *with* $f = u + iv$ *and* $u, v \in C^1(U)$*, we say* f *is holomorphic on* Ω *if*

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] = 0.$$

This condition is equivalent to the existence of the limit(s)

$$\lim_{\mathbb{C} \ni h \to 0} \frac{f(z+h) - f(z)}{h} \in \mathbb{C}$$

for every $z \in \Omega$.

Given $g : \Omega \to \mathbb{C}$ *with* $g = \tilde{u} + i\tilde{v}$ *and* $\tilde{u}, \tilde{v} \in C^1(U)$*, we say* g *is* **antiholomorphic** *on* Ω *if*

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + i\left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}\right)\right] = 0 \qquad \text{on } \Omega.$$

(a) *Give a condition for* g *to be antiholomorphic in terms of the existence of the limit of a "difference quotient."*

(b) *Given an antiholomorphic function* $g = \tilde{u} + i\tilde{v}$ *as above, compute the value of the limit of the "difference quotient" you defined in the previous part.*

Solution:

(a) Recall that one obtains from the condition

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h} = L \in \mathbb{C}$$

on $f = u + iv$ that

$$L = \lim_{\mathbb{R}\ni h\to 0}\frac{f(z+h)-f(z)}{h} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and

$$L = \lim_{i\mathbb{R}\ni h\to 0}\frac{f(z+h)-f(z)}{h} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right).$$

From this follows the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

From the Cauchy-Riemann equations follows the condition

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] = 0.$$

And all this is reversible. Now, we want to mimic this starting with $\partial f/\partial z = 0$ and working backwards. Just plugging in starting with

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + i\left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}\right)\right] = 0,$$

we get the antiholomorphic Cauchy-Riemann equations which read

$$\frac{\partial \tilde{u}}{\partial x} = -\frac{\partial \tilde{v}}{\partial y} \qquad \text{and} \qquad \frac{\partial \tilde{v}}{\partial x} = \frac{\partial \tilde{u}}{\partial y}.$$

These suggest, again mimicking the next backward step for holomorphic functions, that we have some limit $\tilde{L} \in \mathbb{C}$ for which

$$\tilde{L} = \frac{\partial \tilde{u}}{\partial x} + i\frac{\partial \tilde{v}}{\partial x} \tag{1}$$

and

$$\begin{aligned}\tilde{L} &= i\frac{\partial \tilde{u}}{\partial y} - \frac{\partial \tilde{v}}{\partial y} \\ &= i\left(\frac{\partial \tilde{u}}{\partial y} + i\frac{\partial \tilde{v}}{\partial y}\right). \end{aligned} \tag{2}$$

Now, we can see from the holomorphic case how to get expressions like these (or at least sort of like these) from limits of difference quotients with pure real increment or pure imaginary increment. In particular, a purely real increment $h \in \mathbb{R}$ gives (or can give) as we know

$$\begin{aligned}\tilde{L} &= \lim_{\mathbb{R}\ni h\to 0} \frac{\tilde{u}(x+h,y) - \tilde{u}(x,y)}{h} + i \lim_{\mathbb{R}\ni h\to 0} \frac{\tilde{v}(x+h,y) - \tilde{v}(x,y)}{h} \qquad (3) \\ &= \lim_{\mathbb{R}\ni h\to 0} \frac{g(z+h) - g(z)}{h}\end{aligned}$$

which is the expression in (1). This is precisely like in the holomorphic case. But we have to do something different to get the expression in (2). In particular, if we write $i\mathbb{R} \ni h = i\epsilon$, then

$$\begin{aligned}\lim_{i\mathbb{R}\ni h\to 0} \frac{g(z+h) - g(z)}{h} &= -i \lim_{\mathbb{R}\ni \epsilon\to 0} \frac{g(z+i\epsilon) - g(z)}{\epsilon} \\ &= -i\left(\frac{\partial \tilde{u}}{\partial y} + i\frac{\partial \tilde{v}}{\partial y}\right)\end{aligned}$$

which is **not** (2). A little reflection suggests the following "difference quotient:"

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{\bar{h}} = \tilde{L} \in \mathbb{C} \tag{4}$$

with the complex conjugate $\bar{h} = h_1 - ih_2$ in the denominator. If the limit condition (4) holds, then according to (3) the limit in (1) is still obtained. Furthermore, taking the limit $i\mathbb{R} \ni h = i\epsilon \to 0$, we get

$$\begin{aligned}\tilde{L} &= \lim_{i\mathbb{R}\ni h\to 0} \frac{g(z+h) - g(z)}{\bar{h}} \\ &= i \lim_{\mathbb{R}\ni \epsilon\to 0} \frac{g(z+i\epsilon) - g(z)}{\epsilon} \\ &= i\left(\frac{\partial \tilde{u}}{\partial y} + i\frac{\partial \tilde{v}}{\partial y}\right) \\ &= i\frac{\partial \tilde{u}}{\partial y} - \frac{\partial \tilde{v}}{\partial y}\end{aligned}$$

and consequently the antiholomorphic Cauchy-Riemann equations hold. Finally then the definition given above of antiholomporphicity for g holds. This constitutes one solution of part **(a)** which I will state as a lemma:

Lemma 1 *If $g = \tilde{u} + i\tilde{v} : \Omega \to \mathbb{C}$ is a given complex valued function defined on an open set $\Omega \subset \mathbb{C}$ and for every $z \in \Omega$ the limit*

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{\bar{h}} = \tilde{L} \in \mathbb{C}$$

exists, then g is antiholomorphic in Ω in the sense (defined above) that

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + i\left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}\right)\right] = 0 \qquad \textit{on } \Omega.$$

We have not shown the two conditions are equivalent, but strictly speaking the problem did not ask that we do so. In order to do this it would be enough to assume the antiholomorphic Cauchy-Riemann equations hold throughout Ω for some $\tilde{u}, \tilde{v} \in C^1(\Omega)$ and then prove the modified "difference quotient"

$$\frac{g(z+h) - g(z)}{\bar{h}}$$

has a limit as $\mathbb{C} \ni h \to 0$ for every $z \in \Omega$. I will perhaps leave this to you.

Another solution should be possible. An alterntive to (1) and (2) is that we have some limit $\tilde{L} \in \mathbb{C}$ for which

$$\tilde{L} = \frac{\partial \tilde{u}}{\partial x} - i\frac{\partial \tilde{v}}{\partial x} \tag{5}$$

and

$$\begin{aligned} \tilde{L} &= -i\frac{\partial \tilde{u}}{\partial y} - \frac{\partial \tilde{v}}{\partial y} \\ &= -i\left(\frac{\partial \tilde{u}}{\partial y} - i\frac{\partial \tilde{v}}{\partial y}\right). \end{aligned} \tag{6}$$

The value of (5) can be obtained as the limit

$$\tilde{L} = \lim_{\mathbb{R}\ni h\to 0} \frac{\bar{g}(z+h) - \bar{g}(z)}{h}$$

where we have taken the conjugate of the function g. With this in mind, let us compute

$$\begin{aligned}
\lim_{\mathbb{R}\ni\epsilon\to 0} \frac{\bar{g}(z+i\epsilon)-\bar{g}(z)}{i\epsilon} &= -i \lim_{\mathbb{R}\ni\epsilon\to 0} \frac{\bar{g}(z+i\epsilon)-\bar{g}(z)}{\epsilon} \\
&= -i\left(\frac{\partial \tilde{u}}{\partial y} - i\frac{\partial \tilde{v}}{\partial y}\right) \\
&= -i\frac{\partial \tilde{u}}{\partial y} - \frac{\partial \tilde{v}}{\partial y}.
\end{aligned}$$

We see that these match the expressions given in (6) and hence give an alternative solution to the problem:

Lemma 2 *If $g = \tilde{u} + i\tilde{v} : \Omega \to \mathbb{C}$ is a given complex valued function defined on an open set $\Omega \subset \mathbb{C}$ and for every $z \in \Omega$ the limit*

$$\lim_{h\to 0} \frac{\bar{g}(z+h)-\bar{g}(z)}{h} = \tilde{L} \in \mathbb{C}$$

exists, then g is antiholomorphic in Ω in the sense (defined above) that

$$\frac{\partial g}{\partial z} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} + i\left(\frac{\partial \tilde{v}}{\partial x} - \frac{\partial \tilde{u}}{\partial y}\right)\right] = 0 \qquad \text{on } \Omega.$$

(b) The answer to this part depends on the choice of modified difference quotient in the previous part. If one takes

$$\tilde{L} = \lim_{h\to 0} \frac{g(z+h)-g(z)}{\bar{h}},$$

then the limit is

$$\frac{\partial \tilde{u}}{\partial x} + i\frac{\partial \tilde{v}}{\partial x} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} - \frac{\partial \tilde{v}}{\partial y} + i\left(\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y}\right)\right] = \frac{\partial g}{\partial \bar{z}}.$$

Alternatively, if one takes

$$\tilde{L} = \lim_{h\to 0} \frac{\bar{g}(z+h)-\bar{g}(z)}{h},$$

then the limit is

$$\frac{\partial \tilde{u}}{\partial x} - i\frac{\partial \tilde{v}}{\partial x} = \frac{1}{2}\left[\frac{\partial \tilde{u}}{\partial x} - \frac{\partial \tilde{v}}{\partial y} - i\left(\frac{\partial \tilde{v}}{\partial x} + \frac{\partial \tilde{u}}{\partial y}\right)\right] = \overline{\frac{\partial g}{\partial \bar{z}}}.$$

One may ask at this point if one of these choices of modified difference quotient is more "natural" than the other and should more properly be called the "antiholomorphic" and "anticomplex derivative". Based on the observation that for holomorphic functions

$$\frac{\partial f}{\partial \bar{z}} = 0 \qquad \text{and} \qquad f'(z) = \frac{\partial f}{\partial z},$$

one notes that with the first choice of modified difference quotient one has

$$\frac{\partial f}{\partial z} = 0 \qquad \text{and} \qquad \lim_{h\to 0} \frac{g(z+h) - g(z)}{\bar{h}} = \frac{\partial f}{\partial \bar{z}}.$$

Problem 2 *(Daniel's Antiholomorphic Functions) Recall that the function $h : \mathbb{C}\backslash\{0\} \to \mathbb{C}$ by $h(z) = 1/z$ is holomorphic on the punctured plane and has values that can be written in the form*

$$h(z) = \frac{\bar{z}}{|z|^2}.$$

Let $g : \mathbb{C} \to \mathbb{C}$ be given by $g(z) = \bar{z}$.

(a) *Find another antiholomorphic function f for which $h = f \circ g = g \circ f$.*

(b) *Choose a point $z_0 \in \{z \in \mathbb{C} : \mathrm{Re}(z), \mathrm{Im}(z) > 0\}$ and consider the curves $\gamma_1(t) = z_0 + t$ and $\gamma_2(t) = z_0 + it$ for $0 \le t \le 1$ along with the square $C = \{z_0 + s + it : 0 \le s \le 1\}$. Draw the following:*

(i) *The images of γ_1 and γ_2 and the square C.*

(ii) *The images of $f \circ \gamma_1$, $f \circ \gamma_2$, and $f(C)$.*

(iii) *The images of $g \circ \gamma_1$, $g \circ \gamma_2$, and $g(C)$.*

(iv) *The images of $h \circ \gamma_1$, $h \circ \gamma_2$, and $h(C)$.*

You may wish to use mathematical software (Something like Matlab, Mathematica, Maple, Octave, ...) in order to get a very accurate picture.

Problem 3 *(Linear Mappings of $\mathbb{R}^2$) Let $L : U = \mathbb{R}^2 \to \mathbb{R}^2$ by $L(x, y) = (u(x, y), v(x, y))$ be linear. Consider $f : \Omega = \mathbb{C} \to \mathbb{C}$ with $f = u + iv$.*

(a) *Classify all linear maps L for which f is holomorphic.*

(b) *Classify all linear maps L for which f is antiholomorphic.*

Solution: If L is linear we can write

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{7}$$

That is, L is given by matrix multiplication where $A = (a_{ij})$ is some (constant) matrix. The Cauchy-Riemann equations then give $a_{11} = a_{22} = a$ and $a_{12} = -a_{21} = -b$ for some constants a and b. Thus, all holomorphic maps corresponding to linear maps are of the form

$$f(z) = f(x + iy) = ax - by + i(bx + ay) = (a + bi)(x + iy) = (a + bi)z \tag{8}$$

with corresponding linear map

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{9}$$

The holomorphic function is given by a complex scaling which, as we should know, corresponds to a rotation and dilation of the complex plane. The linear function is given by matrix multiplication with a matrix which can be written as

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \sqrt{a^2 + b^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

where the angle θ is determined up to an additive multiple of 2π (or uniquely with $0 \leq \theta < 2\pi$) by

$$\cos\theta = \frac{a}{\sqrt{a^2 + b^2}} \qquad \text{and} \qquad \sin\theta = \frac{b}{\sqrt{a^2 + b^2}}.$$

It should be noted that the map L in (9) is a homogeneous dilation of the plane followed by (or proceeded by) a rotation. (Dilation and rotation commute.)

The antiholomorphic Cauchy-Riemann equations applied to the coordinate functions of the linear map (7) give $a_{11} = -a_{22} = a$ and $a_{12} = a_{21} = b$. Thus, the antiholomorphic functions corresponding to linear maps on $\mathbb{R}^2$ take the form

$$f(z) = f(x+iy) = ax + by + i(bx - ay) = (a+bi)(x-iy) = (a+bi)\bar{z} \tag{10}$$

with corresponding linear map

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ b & -a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}. \tag{11}$$

The mapping of the complex plane is clearly conjugation followed by a complex scaling, i.e., rotation and dilation. Similarly, the matrix for the linear map L can be written as

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix} = \sqrt{a^2+b^2}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, we have reflection in the $x - axis$ followed by a rotation and dilation of the plane. Naturally, the rotation and the reflection do not commute here, but their product

$$\begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

is the reflection in the line

$$\{(v_1, v_2) \in \mathbb{R}^2 : (1-\cos\theta)v_1 = \sin\theta\ v_2\}.$$

Thus, the mapping of the plane is a reflection followed by a dilation, and these two commute.

Problem 4 *(S&S Exercise 1.12) Consider* $f : \mathbb{C} \to \mathbb{C}$ *by* $f(x+iy) = \sqrt{|xy|}$. *Show that* f *satisfies the Cauchy-Riemann equations at* $z = x + iy = 0$, *but* f *is not (complex) differentiable at* $z = 0$.

Problem 5 *(S&S Exercise 1.13) Show that the following are equivalent for a holomorphic function $f : \Omega \to \mathbb{C}$ with $f = u + iv$ and $u, v \in C^1(U)$:*

(i) *f is constant on the open set Ω.*

(ii) *$f(\Omega) = \{f(z) : z \in \Omega\}$ is a subset of a line.*

(iii) *$f(\Omega) = \{f(z) : z \in \Omega\}$ is a subset of a circle.*

Hint: Consider the case where $f(\Omega)$ lies in a vertical line.

Solution: It is clear that **(i)** implies **(ii)** and **(iii)**. It is almost as clear that neither **(ii)** nor **(iii)** implies **(i)**. If Ω is an open set with three connected components $\Omega = C_1 \sqcup C_2 \sqcup C_3$, then we can take a holomorphic function $f : \Omega \to \mathbb{C}$ by $f(z) = c_j$ for $j = 1, 2, 3$ where c_1, c_2 and c_3 are three complex constants not lying on a line. This function is perfectly holomrophic, but not constant. Of course, it is piecewise constant on the components. Similarly, if Ω has four components, we can find a holomorphic function $f : \Omega \to \mathbb{C}$ taking four distinct values that do not lie on a circle.

What we can show is that if Ω is **connected** and one of **(ii)** or **(iii)** holds, then f is constant.

If **(ii)** holds, then there is some $w \in \mathbb{C}$ and some $\theta \in [0, 2\pi)$ such that

$$f(\Omega) \subset \{w + te^{i\theta} : t \in \mathbb{R}\}.$$

This means $g = e^{-i\theta}(f - w) = a$ is holomorphic for some real function $a = a(x, y)$. By the Cauchy-Riemann equations $Da \equiv \mathbf{0}$ on U which is also connected and it follows that for any $(x_0, y_0) \in U$ the sets

$$\{(x, y) \in U : a(x, y) = a(x_0, y_0)\} \qquad \text{and} \qquad \{(x, y) \in U : a(x, y) = a(x_0, y_0)\}$$

are disjoint, open, and have union U. Since the first set is nonempty and U is connected, the second open set must be empty, that is $a \equiv a(x_0, y_0)$ is constant. Therefore,

$$f \equiv w + ae^{i\theta} \quad \text{is constant as well.}$$

Now, consider the possibility that Ω is connected and f has image in a circle. Notice that the topological argument above (involving connectedness) applies whenever we can show each point in the connnected set U has a neighborhood, say a disk, on which f (restricted to the neighborhood) is constant. Now let's say there is some

$w \in \mathbb{C}$ and some $r > 0$ such that $f(z) \in \{w + re^{it} : t \in \mathbb{R}\}$ for all $z \in \Omega$. Given $z_0 \in \Omega$, there is some (unique) $t_0 \in [0, 2\pi)$ such that $f(z_0) = w + re^{it_0}$. Let $w_0 = w + re^{i(t_0+\pi)}$. By continuity, there is a disk $D_r(z_0) \subset \Omega$ such that the restriction $f : D_r(z_0) \to \mathbb{C}$ has image in $\{w + re^{it} : t \in \mathbb{R}\} \backslash \{w_0\}$. It follows that $g : D_r(z_0) \to \mathbb{C}$ by

$$g(z) = \frac{1}{f(z) - w_0}$$

is a holomorphic function with

$$g(z) = \frac{1}{w + re^{it} - w_0} = \frac{1}{r} \frac{1}{e^{it} - e^{i(t_0+\pi)}} = \frac{1}{re^{it_0}} \frac{1}{e^{i(t-t_0)} + 1} = \frac{1}{re^{it_0}} \frac{e^{-i(t-t_0)} + 1}{2[1 + \cos(t - t_0)]}.$$

Notice that

$$\zeta = \frac{e^{-i(t-t_0)} + 1}{1 + \cos(t - t_0)} = 1 + i\frac{\sin(t_0 - t)}{1 + \cos(t - t_0)}.$$

That is, all values of ζ lie in a vertical line $\operatorname{Re}\zeta = 1$. Then $\zeta/(2r)$ is also in a vertical line with $\operatorname{Re}(\zeta/(2r)) = 1/(2r)$. Thus, the rotation e^{it_0} gives an image also in a line. That is, $g : D_r(z_0) \to \mathbb{C}$ is a holomorphic function on a connected open set with image in a line. Thus, by the implication **(ii)** implies **(i)** we conclude that g is constant. This constant must also be nonzero, and it follows that $f(z) = w_0 + 1/g(z)$ is also constant for $z \in D_r(z_0)$. This means f satisfies the condition for the toplogical argument, and f is constant on Ω.

Problem 6 *(S&S Exercise 1.16) Find the radius of convergence of the following series:*

(a)

$$\sum_{n=0}^{\infty} (\log n)^2 \ z^n.$$

(b)

$$\sum_{n=0}^{\infty} n! \ z^n.$$

(c)

$$\sum_{n=0}^{\infty} \frac{n^2}{4^n + 3n} \ z^n.$$

Solution: Some of these may be a bit easier with the ratio test/limit from the next problem, but I am assuming from the ordering Stein intended them to be done without that. They can be done without the next exercise, and the middle one needs something a little more than the next exercise. Most of you got these correct, but some of you had slick solutions I may want to remember, so I'm going to write/type them up while I remember them. Especially the slick estimate for $n!$ in part **(b)** which two of you used.

(a)

$$\sum_{n=0}^{\infty} (\log n)^2 \ z^n.$$

Taking $a_n = (\log n)^2$ and $n > 1$, we can write

$$\log |a_n|^{1/n} = \log(\log n)^{2/n} = (2/n)\log(\log n) = 2\frac{\log(\log n)}{n}.$$

This is an indeterminate form ∞/∞, so we can apply L'Hopital's rule:

$$\frac{\log(\log n)}{n} \sim \frac{\frac{1}{\log n}\frac{1}{n}}{1} = \frac{1}{n\log n} \to 0$$

as $n \nearrow \infty$. Thus,

$$L = \lim_{n\to\infty} |a_n|^{1/n} = e^0 = 1,$$

and $R = 1$.

An alternative is to estimate using $\log n < n$ so that

$$\log |a_n|^{1/n} = \log(\log n)^{2/n} < (2/n)\log n = 2\frac{\log n}{n}.$$

An easy application of L'Hopitals rule gets

$$L = \lim_{n\to\infty} |a_n|^{1/n} \le e^0 = 1,$$

so $R \ge 1$. but then you're stuck getting the reverse inequality. This is not so difficult however since for any z with $|z| \ge 1$,

$$\lim_{n\to\infty} |a_n z^n| \ge \lim_{n\to\infty} (\log n)^2 = +\infty,$$

and the pointwise series fails the basic test for convergence at all such points. This means, of course, that the Hadamard radius satisfies $R \le 1$.

(b)

$$\sum_{n=0}^{\infty} n!\ z^n.$$

If I remember correctly, I applied the logarithm in this case as well, and it was complicated. Here are two slick approaches I (more or less) read in your papers. One person did something like this: Let $a_n = n!$. Then taking $n = 2k > 2$ we see

$$a_n = (2k)(2k-1)\cdots k \cdots 2 \cdot 1 > k^k.$$

Therefore,

$$L = \limsup |a_n|^{1/n} \geq \limsup (k^k)^{1/(2k)} = \limsup \sqrt{k} = +\infty.$$

Therefore, $R = 0$.

An alternative is to use the series (real or complex) for the exponential function, and note that

$$e^n = \sum_{k=0}^{\infty} \frac{k^k}{k!} > \frac{n^n}{n!}.$$

This means

$$n! > \left(\frac{n}{e}\right)^n$$

which is a really nice estimate. From this we get

$$L = \lim \frac{n}{e} = +\infty,$$

and $R = 0$ follows as before.

(c)

$$\sum_{n=0}^{\infty} \frac{n^2}{4^n + 3n}\ z^n.$$

Let

$$a_n = \frac{n^2}{4^n + 3n}.$$

Obviously the basic idea is that the polynomial/rational parts make no difference and the radius is the same as if the coefficient were $a_n \sim 1/4^n$, that is $L = 1/4$ and $R = 4$. To see this one can use a logarithm:

$$\log a_n^{1/n} = \frac{1}{n}\left[\log n^2 - \log(4^n + 3n)\right].$$

Note that

$$\lim_{n\to\infty} \frac{\log n^2}{n} = 0,$$

so we can consider

$$b_n = \frac{\log(4^n + 3n)}{n}.$$

Clearly,

$$b_n < \frac{\log 4^n}{n} = \log 4.$$

But also (at least for n large)

$$b_n > \frac{\log(4^n/2)}{n} = \log 4 - \frac{\log 2}{n} \to \log 4.$$

We conclude, as expected

$$L = \lim_{n\to\infty} e^{-b_n} = \frac{1}{4},$$

so $R = 4$.

Problem 7 *(S&S Exercise 1.17) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of coefficients in $\mathbb{C}$. If*

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

exists as a real number in $(0, \infty)$, then (show) this is the Hadamard radius. Note: the quantity R here is the reciprocal of the quantity one would get from the ratio test. Hint: Stackexchange.

Solution: Under the assumption

$$0 < R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} < \infty,$$

let

$$M = \frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Then of course, $0 < M < \infty$, and for any $\epsilon > 0$ there is some $N > 0$ with

$$M - \epsilon < \frac{|a_{n+1}|}{|a_n|} < M + \epsilon \qquad \text{for all } n \geq N.$$

We may also take $\epsilon > 0$ small enough so that

$$0 < M - \epsilon < \frac{|a_{n+1}|}{|a_n|} < M + \epsilon \qquad \text{for all } n \geq N.$$

Consider $n > N$. We have then

$$|a_n| = |a_N| \, \frac{|a_{N+1}|}{|a_N|} \, \frac{|a_{N+2}|}{|a_{N+1}|} \cdots \frac{|a_n|}{|a_{n-1}|}$$

where the last factor can be written as $|a_{N+(n-N)}|/|a_{n-1}|$ so we see there are $n - N$ quotient factors. It follows that

$$|a_N|(M - \epsilon)^{n-N} < |a_n| < |a_N|(M + \epsilon)^{n-N},$$

and

$$|a_N|^{1/n}(M-\epsilon)^{1-N/n} = (M-\epsilon)\left(\frac{|a_N|}{(M-\epsilon)^N}\right)^{1/n} < |a_n|^{1/n} < (M+\epsilon)\left(\frac{|a_N|}{(M+\epsilon)^N}\right)^{1/n}. \tag{12}$$

Noting that $|a_N|/(M \pm \epsilon)^N$ is a fixed positive number we have

$$\log\left(\frac{|a_N|}{(M \pm \epsilon)^N}\right)^{1/n} = \frac{1}{n}\log\left(\frac{|a_N|}{(M \pm \epsilon)^N}\right) \to 0 \quad \text{as } n \to 0,$$

and

$$\lim_{n\to\infty}\left(\frac{|a_N|}{(M \pm \epsilon)^N}\right)^{1/n} = 1.$$

Therefore, taking the limit as n tends to infinity on both sides of (12) we have

$$M - \epsilon \leq \liminf_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq M + \epsilon$$

for $\epsilon > 0$ arbitrarily small. That is,

$$L = \lim_{n\to\infty} |a_n|^{1/n} = M.$$

Hence the Hadamard radius is $1/L = 1/M = R$.

This result may be applied in parts **(a)** and **(c)** of Problem 6 above. For part **(a)** we have $a_n = (\log n)^2$, so

$$\frac{|a_{n+1}|}{|a_n|} = \left(\frac{\log(n+1)}{\log n}\right)^2.$$

This is an ∞/∞ indeterminate form to which L'Hopital's rule applies (inside):

$$\frac{|a_{n+1}|}{|a_n|} = \left(\frac{\log(n+1)}{\log n}\right)^2 \sim \left(\frac{n}{n+1}\right)^2 \to 1 \quad \text{as } n \to \infty.$$

Therefore, we get $R = 1$ in agreement with our work in Problem 6.

For part **(c)** of Problem 6, we have

$$a_n = \frac{n^2}{4^n + 3n},$$

so

$$M = \frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^2}{4^{n+1} + 3(n+1)} \frac{4^n + 3n}{n^2} = \frac{1 + 2/n + 1/n^2}{4 + 3(n+1)/4^n} (1+3n/4^n) \to \frac{1}{4} \quad \text{as } n \to \infty.$$

Therefore, $R = 1/M = 4$.

So that worked reasonably well for parts **(a)** and **(c)**. It will be noted that what we have cannot be applied directly to part **(b)** of Problem 6. In that case, $a_n = n!$. The quotient of consecutive terms is easy enough to evaluate:

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{n!} = n + 1.$$

But this ratio does not have a finite nonzero limit. The limit is $+\infty$. An easy modification of the argument above gives the result we want however:

If

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \infty \quad \text{or equivalently} \quad \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = 0,$$

then for any $M > 0$, we can take N large enough (fixed) and $n > N$ so that

$$|a_n| > |a_N| M^{n-N}.$$

Then

$$|a_n|^{1/n} > |a_N|^{1/n} M^{1-N/n} = M \left(\frac{|a_N|}{M^N}\right)^{1/n}.$$

It follows then that

$$L = \lim\sup n \to \infty |a_n|^{1/n} \geq M,$$

and since M is arbitrarily large, $L = \infty$ and the Hadamard radius is $R = 1/L = 0$.

This result applies to part **(b)** of Problem 6 where the series converges only for $z = 0$.

While we're at it, we can also treat the complementary case

$$\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = 0 \quad \text{or equivalently} \quad \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|} = \infty.$$

In this case, we get for any $\epsilon > 0$

$$|a_n| < |a_N|\epsilon^{n-N}.$$

Therefore,

$$|a_n|^{1/n} < \epsilon \left(\frac{|a_N|}{\epsilon^N}\right)^{1/n} \to \epsilon \quad \text{as } n \to \infty.$$

Thus,

$$\lim_{n\to\infty} |a_n|^{1/n} = 0,$$

and

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}} = +\infty.$$

Note: I adapted the argument(s) above from a discussion of this question I found on Stackexchange including a rather elegant solution, and the hint was intended to suggest looking up this online discussion. However, when I went to type up my solution, I searched for that discussion on Stackexchange, and I don't think I could find it. I did find a different one with a similar solution, so I used that. One of the reasons I decided to type up the solution, in fact, was because I couldn't find the Stackexchange page I had in mind with the hint. Based on a remark Katherine Booth included in her solution, probably the basic idea of this argument goes back to Rudin's *Principles of Mathematical Analysis.* At least a lot of people have read about it there...presumably including me, though I don't remember it. Who knows where Rudin got it!

Second Note:

There is an essentially different approach to this problem. Instead of showing

$$L = \limsup_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} |a_{n+1}|/|a_n|$$

directly, one can show directly that the series converges absolutely for

$$|z| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

and diverges for

$$|z| > \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}.$$

Then the main theorem on the convergence of complex power series, which the first approach uses in a rather different way, identifies this number

$$\lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

as the unique Hadamard radius. I won't go through the details of this approach. It's rather similar to the argument I've given above, but then one must incorporate z and compare to a geometric series.

Problem 8 *(S&S Exercise 1.16) Find the radius of convergence of the hypergeometric series:*

$$F(\alpha, \beta, \gamma, z) = 1 + \sum_{n=1}^{\infty} \frac{c_n}{n!}\ z^n$$

where

$$c_n = \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\cdots(\beta+n-1)}{\gamma(\gamma+1)\cdots(\gamma+n-1)}.$$

Hint: Use the ratio test.

Solution: We've seen that the series in Problem 6 can be treated either by computing the Hadamard radius directly or by using some variation of the limit ratio test. This hypergeometric series can also be treated using the Hadamard radius definition directly, using also a logarithm. At least I think I did it, but it was rather complicated (to say the least). This one is a natural for the limit ratio test of Problem 7.

$$\frac{|c_{n+1}|/(n+1)!}{|c_n|/n!} = \frac{1}{n+1}\frac{(\alpha+n)(\beta+n)}{\gamma+n} = \frac{1}{1+1/n}\frac{(\alpha/n+1)(\beta/n+1)}{\gamma/n+1} \to 1 \quad \text{as } n \to \infty.$$

Thus, $R = 1$ using the basic result of Problem 7.

Problem 9 *(S&S Exercise 1.18) Consider a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence $R > 0$. Show that f has a (convergent) power series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

with center z_0 for any $z_0 \in D_R(0)$.

Solution: Let me start with a formal calculation (which most of you also made):

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n z^n &= \sum_{n=0}^{\infty} a_n [(z - z_0) + z_0]^n \\
&= \sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k \qquad (13) \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \; a_n \binom{n}{k} z_0^{n-k} (z - z_0)^k \qquad (14) \\
&= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} \; a_n \binom{n}{k} z_0^{n-k} (z - z_0)^k \qquad (15) \\
&= \sum_{k=0}^{\infty} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k. \qquad (16)
\end{aligned}$$

The expression (13) is obtained from the binomial expansion formula, and there is no problem with that. The expression in (15) however is obtained by a kind of formal rearrangement. The usual rearrangement of a series, however, assumes a given ordering $\{\alpha_n\}_{n=1}^{\infty}$ of the terms in a series

$$\sum_{n=1}^{\infty} \alpha_n$$

and a bijection $j : \mathbb{N} \to \mathbb{N}$ according to which those terms are rearranged. One can indeed start with the expression in (14) and set

$$\beta_{nk} = a_n \binom{n}{k} z_0^{n-k} (z - z_0)^k$$

to obtain a double indexed sequence which can be enumerated (or reindexed) in preparation for rearrangement. The rearrangement represented in (15) however does not seem to correspond to a simple bijection. One is required to "run out to infinity" along a subsequence of the terms, and then do this infinitely many times. Thus, it is difficult to say—or more properly impossible to say—what is the $j(n)$-th term. What is happening here is not rearrangement according to, say, Rudin (*Principles of Mathematical Analysis*). Even if one takes a nominally more general notion of the sum of a series

$$\sum_{\gamma\in\Gamma} x_\gamma$$

of non-negative terms x_γ indexed by a set Γ of arbitrary cardinality, for example that of Terrence Tao in his book *An Introduction to Measure Theory,* where one takes the supremum of finite sums over subsets of the indexing set Γ, I still don't see that one can get a simple "rearrangement" result of the kind needed here. Maybe you can, but I'm not seeing it. As I recall Tao emphasizes that his construction is pretty specific to non-negative terms.

Instead, I would use the formal calculation in the following way: We can take the coefficients

$$b_k = \sum_{n=k}^{\infty} a_n \left(\begin{array}{c} n \\ k \end{array} \right) z_0^{n-k} \tag{17}$$

appearing in (16) just at face value as "candidate coefficients" for our expansion at z_0. This means, we will still need to justify the following:

(i) The numbers b_k are well-defined, and

(ii) The resulting series converges to f in some disk with center z_0.

At this point, furthermore, I seem to be forced to break my considerations into two cases, which none of you seemed to consider. The first case is when $z_0 = 0$. Of course, this one is easy because we have the original series expansion, and that gives the desired assertion. Nevertheless, it seems to me significant that henceforth I can restict to $z_0 \neq 0$.

Some of you observed that **(i)** holds. I'm not sure anyone gave the details. The details can be provided as follows:

$$\sum_{n=k}^{\infty} a_n \left(\begin{array}{c} n \\ k \end{array} \right) z_0^{n-k} = \frac{1}{z_0} \sum_{n=k}^{\infty} a_n \left(\begin{array}{c} n \\ k \end{array} \right) z_0^{n}$$

involves a formal series with Hadamard limit

$$L_k = \limsup_{n\to\infty} \left| a_n \binom{n}{k} \right|^{1/n} = \limsup_{n\to\infty} |a_n|^{1/n} \le 1/R$$

because

$$1 \le \binom{n}{k}^{1/n} \le \left(\frac{n^k}{k!}\right)^{1/n} \to 1 \qquad \text{as } n \to \infty \text{ for fixed } k. \tag{18}$$

Therefore, the candidate coefficients b_k given in (17) are at least well-defined complex numbers for $k = 0, 1, 2, \ldots$, and we can consider the formal power series

$$\sum_{k=0}^{\infty} b_k (z - z_0)^k. \tag{19}$$

Note that a perfectly good alternative to my reasoning above is to re-index the series

$$b_k = \sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} = \sum_{n=0}^{\infty} a_{n+k} \binom{n+k}{k} z_0^n$$

and show this series converges absolutely in a similar manner. In particular, though I have used the assumption $0 < |z_0| < R$, one can (with a little more work) show that for each $k = 0, 1, 2, 3, \ldots$

$$\limsup_{n\to\infty} \left| a_n \binom{n}{k} \right|^{1/(n-k)} = \limsup_{n\to\infty} |a_n|^{1/n} \le 1/R.$$

Incidentally, an easy induction shows the interesting (if not unexpected) identity

$$b_k = \frac{f^{(k)}(z_0)}{k!}.$$

I'm not really sure, without Cauchy's theorem/integral formula and Taylor's theorem, how to actually use this at this point. The problem is one has (at the moment) no good estimates on the sequence of derivatives $f^{(k)}(z_0)$ without the Cauchy integral formula.

It remains to show **(ii)** that the formal series (19) converges to f in some disk around z_0. Here is a proof of this fact which seems to be correct:

The overall strategy is to break a partial sum into three terms

$$\sum_{k=0}^{K} b_k (z - z_0)^k = \sum_{k=0}^{K} \left[\sum_{n=k}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k = I \ + \ II \ + \ III$$

with

$$I = \sum_{k=0}^{K} \left[\sum_{n=k}^{K} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k,$$

$$II = \sum_{k=0}^{K} \left[\sum_{n=K+1}^{M} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k,$$

and

$$III = \sum_{k=0}^{K} \left[\sum_{n=M+1}^{\infty} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k.$$

Given $\epsilon > 0$, we will show that for z in some suitable disk $D_r(z_0)$ we can take K large enough so that the terms satisfy $|I - f(z)| < \epsilon/3$, $|II| < \epsilon/3$ and $|III| < \epsilon/3$. Consequently, we will have

$$|I \;+\; II \;+\; III - f(z)| \leq |I \;- f(z)| + |II| + |III| < \epsilon,$$

and we will be done. To this end, let us fix $\epsilon > 0$ and note the following: Given any disk $D_{(1-\delta)R}(0)$ with $0 < \delta < 1$ fixed, the series

$$\sum_{n=0}^{\infty} a_n z^n$$

converges uniformly and absolutely for $z \in \overline{D_{(1-\delta)R}}(0)$ to f. This means, first of all, we can take $K_1 = K_1(\delta) > 0$ so that for $K > K_1$ we have

$$\left| \sum_{n=0}^{K} a_n z^n - f(z) \right| < \frac{\epsilon}{3} \qquad \text{for} \qquad z \in \overline{D_{(1-\delta)R}(0)}. \tag{20}$$

Furthermore, there is a real valued function $g : D_R(0) \to \mathbb{R}$ given by

$$g(z) = \sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{\infty} |a_n| \; |z|^n \qquad \text{for} \qquad z \in D_R(0),$$

and there is some $K_2 = K_2(\delta) > 0$ so that for $K > K_2$ we have

$$\left| \sum_{n=0}^{K} |a_n||z|^n - g(z) \right| < \frac{\epsilon}{3} \qquad \text{for} \qquad z \in \overline{D_{(1-\delta)R}(0)}.$$

We can, and will, also use this last inequality in the form

$$\sum_{n=K}^{\infty} |a_n||z|^n < \frac{\epsilon}{6} \qquad \text{for} \qquad z \in \overline{D_{(1-\delta)R}(0)}. \tag{21}$$

Notice that (20) already gives us the desired condition $|I - f(z)| < \epsilon/3$ when $K > K_1$. More precisely, for finitely many terms we can change the order of summation so that

$$\begin{aligned}
|I - f(z)| &= \left| \sum_{k=0}^{K} \left[\sum_{n=k}^{K} a_n \binom{n}{k} z_0^{n-k} \right] (z - z_0)^k - f(z) \right| \\
&= \left| \sum_{k=0}^{K} \sum_{n=k}^{K} a_n \binom{n}{k} z_0^{n-k} (z - z_0)^k - f(z) \right| \\
&= \left| \sum_{n=0}^{K} a_n \sum_{k=0}^{n} \binom{n}{k} z_0^{n-k} (z - z_0)^k - f(z) \right| \\
&= \left| \sum_{n=0}^{K} a_n z^n - f(z) \right| \\
&\leq \frac{\epsilon}{3}.
\end{aligned}$$

The only problem is that for this to hold we need $K > K_1(\delta)$ and we have not yet chosen δ. Let's get that out of the way and choose the radius r for the disk $D_r(z_0)$ while we're at it. We know $|z_0| < R$. Therefore, there is some $\delta > 0$ for which

$$|z_0| < (1 - \delta)R.$$

Recall that we have also restricted to the case $0 < |z_0|$. Let

$$r = \frac{1}{2} \min \{(1 - \delta)R - |z_0|, |z_0|\}. \tag{22}$$

This is where I use $|z_0| > 0$. Note that for $z \in D_r(z_0)$ we have

$$|z| \leq |z - z_0| + |z_0| < \frac{1}{2}[(1 - \delta)R - |z_0|] + |z_0| = (1 - \delta)R.$$

Thus, $z \in \overline{D_{(1-\delta)R}(0)}$ and in fact $D_r(z_0) \subset \overline{D_{(1-\delta)R}(0)}$. With these choices of δ and r fixed, we get the desired inequality $|I - f(z)| < \epsilon/3$ for $z \in D_r(z_0)$ as long as $K > K_1$.

We can also reverse the order of summation in the second term:

$$
\begin{aligned}
II &= \sum_{k=0}^{K}\left[\sum_{n=K+1}^{M} a_n \binom{n}{k} z_0^{n-k}\right](z-z_0)^k \\
&= \sum_{k=0}^{K}\sum_{n=K+1}^{M} a_n \binom{n}{k} z_0^{n-k}(z-z_0)^k \\
&= \sum_{n=K+1}^{M} a_n \sum_{k=0}^{K} \binom{n}{k} z_0^{n-k}(z-z_0)^k.
\end{aligned}
$$

Notice the upper index of summation on the inner sum is not n, so we are not in a position to reverse the binomial expansion. Nevertheless, for $|z-z_0|<r$

$$
\begin{aligned}
|II| &\le \sum_{n=K+1}^{M} |a_n| \sum_{k=0}^{K} \binom{n}{k} |z_0|^{n-k}|z-z_0|^k \\
&\le \sum_{n=K+1}^{M} |a_n| \sum_{k=0}^{n} \binom{n}{k} |z_0|^{n-k}|z-z_0|^k \\
&\le \sum_{n=K+1}^{M} |a_n|[|z_0|+|z-z_0|]^n \\
&\le \sum_{n=K+1}^{M} |a_n|[(1-\delta)R]^n \\
&\le \sum_{n=K+1}^{\infty} |a_n|[(1-\delta)R]^n \\
&\le \frac{\epsilon}{3}
\end{aligned}
$$

as long as $K > K_2$ according to (21). Notice that this conclusion holds for any $M > K+1$. Thus, if we can obtain the desired inequality on $|III|$ for some large enough M we are done. In particular, at this point we may consider K fixed with $K > \max\{K_1, K_2\}$.

In order to choose M, let us return to the limit assertion (18) above. Recall that $|z_0| < (1+\delta)R$. Therefore, there is some $\mu > 0$ for which

$$(1+\mu)|z_0| < (1+\delta)R.$$

For each fixed k there is some $m = m(k) > k$ for which

$$1 \leq \begin{pmatrix} n \\ k \end{pmatrix}^{1/n} \leq 1 + \mu \qquad \text{if } n > m(k). \tag{23}$$

Let us take then $M > \max\{m(0), m(1), m(2), \ldots, m(K)\}$. Then

$$\begin{aligned}
|III| &= \left| \sum_{k=0}^{K} \left[\sum_{n=M}^{\infty} a_n \begin{pmatrix} n \\ k \end{pmatrix} z_0^{n-k} \right] (z - z_0)^k \right| \\
&= \left| \sum_{k=0}^{K} \sum_{n=M}^{\infty} a_n \begin{pmatrix} n \\ k \end{pmatrix} z_0^{n-k} (z - z_0)^k \right| \\
&\leq \sum_{k=0}^{K} \sum_{n=M}^{\infty} |a_n| \begin{pmatrix} n \\ k \end{pmatrix} |z_0|^{n-k} |z - z_0|^k \\
&\leq \sum_{k=0}^{K} \sum_{n=M}^{\infty} |a_n| \left[\begin{pmatrix} n \\ k \end{pmatrix}^{1/n} \right]^n |z_0|^{n-k} \left(\frac{|z_0|}{2} \right)^k \\
&\leq \sum_{k=0}^{K} \sum_{n=M}^{\infty} |a_n| (1 + \mu)^n |z_0|^n \, \frac{1}{2^k} \\
&\leq \sum_{k=0}^{K} \frac{1}{2^k} \sum_{n=M}^{\infty} |a_n| [(1 + \mu)|z_0|]^n \\
&\leq \sum_{k=0}^{K} \frac{1}{2^k} \sum_{n=M}^{\infty} |a_n| [(1 + \delta)R]^n \\
&\leq 2 \, \frac{\epsilon}{6} \\
&= \frac{\epsilon}{3}.
\end{aligned}$$

Thus we have obtained the result, but we have used the condition $|z - z_0| < r \leq |z_0|/2$ in a seemingly essential way. In particular, it does not follow from this argument that one has convergence on $D_r(z_0)$ with $r = R - |z_0|$ as one would expect.

Problem 10 *(S&S Exercise 1.20, Ahlfors Exercise 2.4.1) For each positive integer $m = 1, 2, 3, \ldots$, find the power series expansion for the function*

$$f(z) = \frac{1}{(1-z)^m}$$

with center at the origin, and find the radius of convergence.

Solution: Remember, one does not yet have the complex Taylor expansion theorem at this point. A couple of you used clever inductions which I record here:

We can start with the geometric series

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

which we know has radius of convergence $R = 1$. We also have the term-by-term differentiation theorem, so

$$\frac{d}{dz}\left(\frac{1}{1-z}\right) = \frac{1}{(1-z)^2} = \sum_{n=1}^{\infty} nz^{n-1}$$

has the same radius of convergence. In general, we will accumulate coefficient powers on both the left within the sum on the right:

$$\frac{d^{m-1}}{dz^{m-1}}\left(\frac{1}{1-z}\right) = \frac{(m-1)!}{(1-z)^m} = \sum_{n=m-1}^{\infty} n(n-1)\cdots[n-(m-2)]z^{n-(m-1)}.$$

(This is an easy induction.) That is, for $m \geq 2$,

$$\begin{aligned}
\frac{1}{(1-z)^m} &= \sum_{n=m-1}^{\infty} \frac{n(n-1)\cdots(n-m+2)}{(m-1)!}\ z^{n-m+1} \\
&= \sum_{n=0}^{\infty} \frac{(n+m-1)(n+m-2)\cdots(n+1)}{(m-1)!}\ z^n \\
&= \sum_{n=0}^{\infty} \frac{(n+m-1)!}{(m-1)!n!}\ z^n \\
&= \sum_{n=0}^{\infty} \binom{n+m-1}{m-1}\ z^n.
\end{aligned}$$

Here is another one rather closer to the approach I had in mind:

$$\left(\frac{1}{1-z}\right)\left(\frac{1}{1-z}\right) = (1+z+z^2+\cdots)(1+z+z^2+\cdots) = 1+2z+3z^2+4z^3+\cdots.$$

We've checked that $n^{1/n}$ tends to 1 and n tends to ∞ which means this series has the same radius of convergence as the original geometric series. More generally, let us assume by induction that

$$\frac{1}{(1-z)^m} = \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n \tag{24}$$

with radius of convergence $R = 1$. Then we can write

$$\frac{1}{(1-z)^{m+1}} = \left(\sum_{\ell=0}^{\infty} z^\ell\right) \sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n.$$

This means

$$\begin{aligned}
\frac{1}{(1-z)^{m+1}} &= \sum_{k=0}^{\infty} \sum_{\ell+n=k} \binom{n+m-1}{m-1} z^k \\
&= \sum_{k=0}^{\infty} \left[\sum_{\ell+n=k} \binom{n+m-1}{m-1}\right] z^k \\
&= \sum_{k=0}^{\infty} \binom{k+m}{m} z^k.
\end{aligned}$$

The combinatorial assertion

$$\sum_{\ell+n=k} \binom{n+m-1}{m-1} = \binom{k+m}{m}$$

is a relatively easy induction on k since

$$\begin{aligned}
\sum_{\ell+n=k} \binom{n+m-1}{m-1} &= \sum_{n=0}^{k} \binom{n+m-1}{m-1} \\
&= \sum_{n=0}^{k-1} \binom{n+m-1}{m-1} + \binom{k+m-1}{m-1},
\end{aligned}$$

and
$$\binom{k-1+m}{m} + \binom{k+m-1}{m-1} = \binom{k+m}{m}.$$
In this instance, since we are only rearranging finitely many terms in the sequence for each k, we can get away with quoting a result about rearrangement of terms in an absolutely convergent series.

We can also take (24) formally writing
$$\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} z^n = \sum_{n=0}^{\infty} b_n z^n. \tag{25}$$
It is clear that the coefficients $b_n = b_n^m$ are positive. Furthermore, since $b_n^1 \equiv 1$ for the geometric series and
$$b_n^{m+1} = \sum_{k=0}^{n} b_k^m$$
it follows inductively that the sequence of coefficients $b_0, b_1, b_2, \ldots$ is increasing, more precisely nondecreasing for the geometric series ($m = 1$) and strictly increasing for $m > 1$, with
$$nb_1^m < b_n^{m+1} < nb_n^m \qquad \text{for} \qquad n > 1.$$
Since
$$\lim_{n\to\infty} (nb_1)^n = 1 \qquad \text{and} \qquad \lim_{n\to\infty} (nb_n)^n = 1$$
we see that the radius of convergence for each formal series (25) is $R = 1$. Consequently, for $|z| < 1$ we have
$$\lim_{N\to\infty} \sum_{n=N}^{\infty} b_n |z|^n = 0. \tag{26}$$

With this in mind, we consider a partial sum

$$\begin{aligned}
\sum_{n=0}^{N} \left(\begin{array}{c} n+m-1 \\ m-1 \end{array} \right) z^n &= \sum_{k=0}^{N} \sum_{\ell+k=n} \left(\begin{array}{c} k+m-1 \\ m-1 \end{array} \right) z^k \\
&= \left(\sum_{\ell=0}^{N} z^\ell \right) \sum_{k=0}^{N} \left(\begin{array}{c} k+m-1 \\ m-1 \end{array} \right) z^k \\
&\qquad - \sum_{k=N+1}^{2N} \left[\sum_{\ell=k-N}^{k} \left(\begin{array}{c} \ell+m-1 \\ m-1 \end{array} \right) \right] z^k \\
&= \left(\sum_{\ell=0}^{N} z^\ell \right) \left(\sum_{k=0}^{N} b_k z^k \right) - \sum_{k=N+1}^{2N} \left[\sum_{\ell=k-N}^{k} b_\ell \right] z^k. \qquad (27)
\end{aligned}$$

By inductive hypothesis we have

$$\lim_{N\to\infty} \sum_{k=0}^{N} b_k z^k = \frac{1}{(1-z)^m}.$$

We also know

$$\lim_{N\to\infty} \sum_{\ell=0}^{N} z^\ell = \frac{1}{1-z}.$$

Therefore, the leading product in (27) satisfies

$$\lim_{N\to\infty} \left(\sum_{\ell=0}^{N} z^\ell \right) \left(\sum_{k=0}^{N} b_k z^k \right) = \frac{1}{(1-z)^m}.$$

The remaining term can be estimated as follows:

$$\begin{aligned}
\left| \sum_{k=N+1}^{2N} \left[\sum_{\ell=k-N}^{k} b_\ell \right] \ z^k \right| &= \left| \sum_{k=N+1}^{2N} \left[\sum_{\ell=k-N}^{k} b_\ell^m \right] \ z^k \right| \\
&\leq \sum_{k=N+1}^{2N} \left[\sum_{\ell=k-N}^{k} b_\ell^m \right] \ |z|^k \\
&\leq \sum_{k=N+1}^{2N} \left[\sum_{\ell=0}^{k} b_\ell^m \right] \ |z|^k \\
&= \sum_{k=N+1}^{2N} b_k^{m+1} \ |z|^k \\
&\leq \sum_{k=N+1}^{\infty} b_k^{m+1} \ |z|^k.
\end{aligned}$$

In view of (26) we have shown this quantity tends to zero as $N \to \infty$. Thus, taking the limit in (27) we have

$$\sum_{n=0}^{\infty} \binom{n+m-1}{m-1} \ z^n = \sum_{n=0}^{\infty} b_n \ z^n = \frac{1}{(1-z)^m}.$$