Assignment 6: Cauchy's Theorem(s)
Due Tuesday March 22, 2022

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Problem 1 *(Dan Romik's notes: The Fundamental Theorem of Algebra) Let us return to part* **(j)** *of Problem 1 of Assignment 5. Recall that this concerns a monic quadratic polynomial* $p(z) = a_0 + a_1 z + z^2$ *under the assumptions*

(i) $a_0 \neq 0$,

(ii) $a_1 \neq 0$, *and*

(iii) $a_1^2 \neq 4a_0$.

Notice that $p(0) = a_0$. *In particular, for* $r = 0$

$$\{p(re^{it}) : 0 \leq t \leq 2\pi\} = \{a_0\} \in \mathbb{C}\backslash\{0\}.$$

The objective here is to use a "blow-up" argument to understand

$$\Gamma_0 = \{p(\epsilon e^{it}) : 0 \leq t \leq 2\pi\}$$

for $\epsilon > 0$ *small. If you need to review the overall elements of Romik's approach and the context of these problems, you can look back at Problem 1 of Assignment 5.*

Consider for $\epsilon > 0$ *the curve*

$$C_\epsilon = \left\{ \left(\frac{1}{\epsilon}\right) \left[p(\epsilon e^{it}) - a_0\right] : 0 \leq t \leq 2\pi \right\},$$

parameterized by

$$\beta_\epsilon(t) = \frac{1}{\epsilon} \left[p(\epsilon e^{it}) - a_0\right] \qquad \text{for } 0 \leq t \leq 2\pi$$

and the parameterized curve/circle $\beta_0 : [0, 2\pi] \to \mathbb{C}$ *by*

$$\beta_0(t) = a_1 e^{it}.$$

The curve C_ϵ *is called a (normalized) "blow-up" of the image curve* Γ_0*. The idea is that as* $\epsilon \searrow 0$*, the curve* $\Gamma_0 = \Gamma_0(\epsilon)$ *is getting "very small" and "very close" to the point* $\{a_0\}$*. The blow-up allows us to see what* Γ_0 *looks like on a fixed visible scale in order to see the geometry of the limit.*

(a) *Show that*

$$\lim_{\epsilon \searrow 0} p(\epsilon e^{it}) = a_0$$

uniformly in $t \in [0, 2\pi]$*.*

(b) *For* $k = 0, 1, 2, \ldots$*, let us define the* C^k *norm on the (real) differentiability space* $C^k([0, 2\pi] \to \mathbb{C})$ *by*

$$\|\beta - \beta_0\|_{C^k} = \sum_{j=0}^{k} \max\{|\beta^{(j)}(t) - \beta_0^{(j)}(t)| : 0 \le t \le 2\pi\}.$$

Show

$$\lim_{\epsilon \searrow 0} \|\beta_\epsilon - \beta_0\|_{C^k} = 0$$

uniformly in $t \in [0, 2\pi]$*.*

(c) *What does part* **(b)** *above tell you about the image* Γ_0 *for* $\epsilon > 0$ *small?*

(d) *Prove that for* $\epsilon > 0$ *small enough,*

$$\int_\alpha \frac{1}{z} = 0$$

where $\alpha(t) = p(\epsilon e^{it})$ *for* $0 \le t \le 2\pi$*. Hint: If you have trouble with part* **(d)** *here, go on to Problem 2 below, and come back to this one.*

Problem 2 *(Dan Romik's notes: The Fundamental Theorem of Algebra) Here we revisit the case* $k = 1$ *of Problem 1 of Assignment 5, and part* **(k)** *in particular. Remember we wish to calculate*

$$\int_\alpha \frac{1}{z}$$

where $\alpha(t) = a_0 + a_1 r e^{it}$ *for* $0 \leq t \leq 2\pi$ *and where* a_0 *and* a_1 *are nonzero complex numbers. Let us assume* $\mathrm{Re}(a_0) \geq 0$ *and* $0 < r < r_0 = |a_0|/|a_1|$.

(a) *Note there is a branch of the logarithm* $\log : \Omega \to \mathbb{C}$ *given by*

$$\log z = \log|z| + i\arg(z)$$

on an appropriate domain containing $\Gamma = \{\alpha(t) : 0 \leq t \leq 2\pi\}$.

(b) *Conclude*

$$\int_\alpha \frac{1}{z} = 0.$$

Hint: You have a primitive.

(c) *If you attempt to calculate*

$$\int_\alpha \frac{1}{z} = \int_0^{2\pi} \frac{1}{a_0 + re^{it}}\ ire^{it}\,dt$$

directly, then in principle, you should be able to write down a formula for a function $g : [0, 2\pi] \to \mathbb{C}$ *for which* $g(0) = g(2\pi)$ *and*

$$g'(t) = \frac{1}{a_0 + re^{it}}\ ire^{it}$$

though this may not be so easy. Determine conditions on r *and* a_0 *for which*

$$\arg(z) = \tan^{-1}\frac{\mathrm{Im}(z)}{\mathrm{Re}(z)} \qquad \text{for } z = a_0 + re^{it},\ 0 \leq t \leq 2\pi,$$

and find an expression for the function g *under these conditions.*

Problem 3 *(removable singularities and Taylor's formula) Let Ω be an open subset of $\mathbb{C}$ with $z_0 \in \Omega$. A holomorphic function $f : \Omega\backslash\{z_0\} \to \mathbb{C}$ is said to have an* **isolated singularity** *at z_0.*

We will later prove the following result:

Theorem 1 *If f has an isolated singularity at $z_0 \in \Omega$ and*

$$\lim_{z \to z_0} (z - z_0) f(z) = 0, \tag{1}$$

then there exists an extension $g : \Omega \to \mathbb{C}$ with g holomorphic and

$$g\Big|_{z \in \Omega\backslash\{z_0\}} = f. \tag{2}$$

Conversely, if f has an isolated singularity at $z_0 \in \Omega$ and there exists an extension $g : \Omega \to \mathbb{C}$ such that g is holomorphic and (2) holds, then (1) holds as well.

An isolated singularity satisfying one of the equivalent conditions (1) or (2) is called a **removable singularity**.

For this problem, let $h : \Omega \to \mathbb{C}$ be holomorphic and consider a point $z_0 \in \Omega$. The point is to obtain a Taylor expansion formula for h.

(a) *Apply Theorem 1 to the function $f : \Omega\backslash\{z_0\} \to \mathbb{C}$ by*

$$f(z) = \frac{h(z) - h(z_0)}{z - z_0}$$

to obtain a holomorphic function $g_1 : \Omega \to \mathbb{C}$ for which

$$h(z) = h(z_0) + g_1(z)(z - z_0).$$

(b) *What is $g_1(z_0)$?*

(c) *Apply Theorem 1 to appropriate functions to obtain holomorphic functions $g_j : \Omega \to \mathbb{C}$ for $j = 2, 3, 4, \ldots$ such that*

$$g_j(z) = g_j(z_0) + g_{j+1}(z)(z - z_0).$$

(d) *Conclude*

$$h(z) = \sum_{n=0}^{N} \frac{h^{(n)}(z_0)}{n!}(z - z_0)^n + g_{N+1}(z)(z - z_0)^{N+1}.$$

Solution:

(a) The function $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ defined by

$$f(z) = \frac{h(z) - h(z_0)}{z - z_0}$$

satisfies (1) because

$$\lim_{z \to z_0} (z - z_0) f(z) = \lim_{z \to z_0} [h(z) - h(z_0)] = 0,$$

and h is continuous. Therefore Theorem 1 applies, and we can take $g_1 = g$ from the theorem so that for $z \neq z_0$

$$\frac{h(z) - h(z_0)}{z - z_0} = g_1(z).$$

Thus,

$$h(z) = h(z_0) + g_1(z)(z - z_0),$$

and this equality clearly also holds for $z = z_0$.

(b) Since g_1 is continuous at z_0 and equal to the difference quotient away from z_0, we have

$$g_1(z_0) = \lim_{z \to z_0} g_1(z) = h'(z_0).$$

(c) Since g_1 is holomorphic, just like h, we can take $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ by

$$f(z) = f_2(z) = \frac{g_1(z) - g_1(z_0)}{z - z_0}$$

and apply the same argument as in part (a) to get

$$g_1(z) = g_1(z_0) + g_2(z)(z - z_0).$$

Note that from part (b) we also get $g_2(z_0) = g_1'(z_0)$. This is interesting of course because it **does not mean** $g_2(z_0) = h''(z_0)$. Plugging back in to our definition of f_2 we get something like

$$g_2(z) = f_2(z) = \frac{h(z) - h(z_0) - h'(z_0)(z - z_0)}{(z - z_0)^2}.$$

Obviously, if we knew the Taylor expansion formula already (or even the next step of the calculation we are presently making) we could get

$$g_2(z_0) = \frac{h''(z_0)}{2}.$$

Question: Whence cometh the factor of 2!?

In any case, I think it's clear now that

$$g_j(z) = g_j(z_0) + g_{j+1}(z)(z - z_0)$$

follows by induction if we set

$$f_{j+1}(z) = \frac{g_j(z) - g_j(z_0)}{z - z_0}$$

to get a function $f_{j+1} : \Omega\backslash\{z_0\} \to \mathbb{C}$ with an isolated singularity at z_0 to which Theorem 1 applies giving a holomorphic function $g_{j+1} = g$. We'll also always get $g_{j+1}(z_0) = g_j'(z_0)$.

(d) We've got so far that

$$h(z) = h(z_0) + g_1(z)(z - z_0)$$

with $g_1(z_0) = h'(z_0)$ and

$$h(z) = h(z_0) + h'(z_0)(z - z_0) + g_2(z)(z - z_0)^2. \tag{3}$$

This is a good start for the induction, and while we're at it, let's note that we can differentiate (3) to get

$$h'(z) = h'(z_0) + g_2'(z)(z - z_0)^2 + 2g_2(z)(z - z_0);$$

there's the factor of 2! (!), and

$$h''(z) = g_2''(z)(z - z_0)^2 + 2g_2'(z)(z - z_0) + 2g_2(z). \tag{4}$$

Therefore,

$$g_2(z_0) = \frac{h''(z_0)}{2}$$

as expected. It's looking like it would be nice to have some kind of recursive formula generalizing (4). Let's see what we can do:

$$h^{(N+1)}(z) = \sum_{n=0}^{N+1} \begin{pmatrix} N+1 \\ n \end{pmatrix} \frac{(N+1)!}{(N+1-n)!} \, g_{N+1}^{(N+1-n)}(z)(z - z_0)^{N+1-n} \tag{5}$$

with

$$h^{(k)}(z) = \sum_{n=k}^{N} \frac{h^{(n)}(z_0)}{(n-k)!}(z-z_0)^{n-k}$$
$$+ \sum_{n=0}^{k} \left(\begin{array}{c} k \\ n \end{array} \right) \frac{(N+1)!}{(N+1-n)!}\, g_{N+1}^{(k-n)}(z)(z-z_0)^{N+1-n} \qquad (6)$$

for $0 \le k \le N$. I'll admit that I needed to write this down on scratch paper to get a reasonable inductive hypothesis. (It may not be quite correct yet, but it should be in the right direction, and if I (or you) read through it once it should be easilly corrected.)

Here's the induction: Given

$$h(z) = \sum_{n=0}^{N} \frac{h^{(n)}(z_0)}{n!}(z-z_0)^n + g_{N+1}(z)(z-z_0)^{N+1} \qquad (7)$$

satisfying also (5) and (6) so that in particular,

$$g_{N+1}(z_0) = \frac{h^{(N+1)}(z_0)}{(N+1)!},$$

we get from part (c) above

$$g_{N+1}(z) = g_{N+1}(z_0) + g_{N+2}(z)(z-z_0).$$

Substituting this into (7) we find

$$\begin{aligned} h(z) &= \sum_{n=0}^{N} \frac{h^{(n)}(z_0)}{n!}(z-z_0)^n + g_{N+1}(z_0)(z-z_0)^{N+1} + g_{N+2}(z)(z-z_0)^{N+2} \\ &= \sum_{n=0}^{N+1} \frac{h^{(n)}(z_0)}{n!}(z-z_0)^n + g_{N+2}(z)(z-z_0)^{N+2}. \end{aligned}$$

That was easy. Of course, now we need to verify the derivative values inductively. That's a little unpleasant, but modulo typos it does seem to work out. Differentiating we get

$$h'(z) = \sum_{n=1}^{N} \frac{h^{(n)}(z_0)}{(n-1)!}(z-z_0)^{n-1}$$
$$+ g'_{N+2}(z)(z-z_0)^{N+2} + (N+2)g_{N+2}(z)(z-z_0)^{N+1}$$

which is the case $k = 1$ of (6) with N replaced by $N + 1$. We also have the case $k = 0$ of course. For $k \leq N$, we may rely on a secondary induction on k to obtain

$$
\begin{aligned}
h^{k+1}(z) = & \sum_{n=k+1}^{N+1} \frac{h^{(n)}(z_0)}{(n-k-1)!}(z-z_0)^{n-k-1} \\
& + \sum_{n=0}^{k} \binom{k}{n} \frac{(N+2)!}{(N+2-n)!} \, g_{N+2}^{(k+1-n)}(z)(z-z_0)^{N+2-n} \\
& + \sum_{n=0}^{k} \binom{k}{n} \frac{(N+2)!}{(N+1-n)!} \, g_{N+2}^{(k-n)}(z)(z-z_0)^{N+1-n} \\
= & \sum_{n=k+1}^{N+1} \frac{h^{(n)}(z_0)}{[n-(k+1)]!}(z-z_0)^{n-(k+1)} + g_{N+2}^{(k+1)}(z)(z-z_0)^{N+2} \\
& + \sum_{n=1}^{k} \binom{k}{n} \frac{(N+2)!}{(N+2-n)!} \, g_{N+2}^{(k+1-n)}(z)(z-z_0)^{N+2-n} \\
& + \sum_{n=1}^{k} \binom{k}{n-1} \frac{(N+2)!}{(N+2-n)!} \, g_{N+2}^{(k+1-n)}(z)(z-z_0)^{N+2-n} \\
& + \frac{(N+2)!}{(N+1-k)!} \, g_{N+2}(z)(z-z_0)^{N+1-k} \\
= & \sum_{n=k+1}^{N+1} \frac{h^{(n)}(z_0)}{[n-(k+1)]!}(z-z_0)^{n-(k+1)} + g_{N+2}^{(k+1)}(z)(z-z_0)^{N+2} \\
& + \sum_{n=1}^{k} \binom{k+1}{n} \frac{(N+2)!}{(N+2-n)!} \, g_{N+2}^{(k+1-n)}(z)(z-z_0)^{N+2-n} \\
& + \frac{(N+2)!}{(N+1-k)!} \, g_{N+2}(z)(z-z_0)^{N+1-k} \\
= & \sum_{n=k+1}^{N+1} \frac{h^{(n)}(z_0)}{[n-(k+1)]!}(z-z_0)^{n-(k+1)} \\
& + \sum_{n=0}^{k+1} \binom{k+1}{n} \frac{(N+2)!}{(N+2-n)!} \, g_{N+2}^{(k+1-n)}(z)(z-z_0)^{N+2-n}.
\end{aligned}
$$

This is (6) with N replaced with $N + 1$ and k replaced with $k + 1$. Evaluating this with $k = N$ we see

$$h^{N+1}(z) = h^{(N+1)}(z_0) + \sum_{n=0}^{N+1} \begin{pmatrix} N+1 \\ n \end{pmatrix} \frac{(N+2)!}{(N+2-n)!}\; g_{N+2}^{(N+1-n)}(z)(z-z_0)^{N+2-n}.$$

Differentiating one last time:

$$\begin{aligned} h^{N+2}(z) = & \sum_{n=0}^{N+1} \begin{pmatrix} N+1 \\ n \end{pmatrix} \frac{(N+2)!}{(N+2-n)!}\; g_{N+2}^{(N+2-n)}(z)(z-z_0)^{N+2-n} \\ & + \sum_{n=0}^{N+1} \begin{pmatrix} N+1 \\ n \end{pmatrix} \frac{(N+2)!}{(N+1-n)!}\; g_{N+2}^{(N+1-n)}(z)(z-z_0)^{N+1-n} \\ = & \, g_{N+2}^{(N+2)}(z)(z-z_0)^{N+2} \\ & + \sum_{n=1}^{N+1} \begin{pmatrix} N+1 \\ n \end{pmatrix} \frac{(N+2)!}{(N+2-n)!}\; g_{N+2}^{(N+2-n)}(z)(z-z_0)^{N+2-n} \\ & + \sum_{n=1}^{N+1} \begin{pmatrix} N+1 \\ n-1 \end{pmatrix} \frac{(N+2)!}{(N+2-n)!}\; g_{N+2}^{(N+2-n)}(z)(z-z_0)^{N+2-n} \\ & + (N+2)!\; g_{N+2}(z) \\ = & \sum_{n=0}^{N+2} \begin{pmatrix} N+2 \\ n \end{pmatrix} \frac{(N+2)!}{(N+2-n)!}\; g_{N+2}^{(N+2-n)}(z)(z-z_0)^{N+2-n}. \end{aligned}$$

This is (5) with N replaced with $N+1$, and this completes the induction. □

Problem 4 *(Liouville's theorem) Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. Assume f satisfies a* **sequential growth estimate** *as follows: There is a sequence of radii R_j $j = 1, 2, 3, \ldots$ with $R_j \nearrow \infty$ such that*

$$|f(z)| < \sqrt[3]{|z|} \qquad \text{for} \qquad |z| = R_j.$$

Assume also an **integral conformal factor estimate**

$$|f'(z)| \leq \left| \frac{1}{2\pi} \int_{\zeta=\alpha} \frac{f(\zeta)}{|\zeta - z|^{3/2}} \right|$$

where α parameterizes $\partial D_r(z)$ for any $z \in \mathbb{C}$.

Prove f is constant.

Problem 5 *(Complex power series) Let* Ω *be an open subset of* $\mathbb{C}$ *with* $z_0 \in \mathbb{C}$*, and let* $f : \Omega \to \mathbb{C}$ *be a function. Show that if* f *is* **complex analytic** *at* $z_0 \in \Omega$*, i.e., there is some* $r > 0$ *for which* $D_r(z_0) \subset \Omega$ *and*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad \text{for } z \in D_r(z_0),$$

then the coefficients a_n *for* $n = 0, 1, 2, 3, \ldots$ *are uniquely determined.*

Problem 6 *(algebra) Use the fundamental theorem of algebra, as in the proof of Corollary 4.7 of Chapter 2 in Stein and Shakarchi, to prove that every polynomial*

$$p(z) = \sum_{n=0}^{k} a_n z^n$$

with $a_k \neq 0$ *can be written as*

$$p(z) = a_k \prod_{n=1}^{k} (z - w_j)$$

for some complex numbers $w_1, w_2, \ldots, w_k$*.*

Problem 7 *(pointwise limits—real functions) This problem is provided as contrast with the assertion that a pointwise limit of holomorphic functions converging uniformly on compact subsets is holomorphic (Theorem 5.2 of Chapter 2 of Stein & Shakarchi).*

(a) *Consider the function* $\eta_0 : \mathbb{R} \to [0, \infty)$ *by*

$$\eta_0(x) = \begin{cases} e^{-\frac{1}{1-x^2}}, & |x| < 1 \\ 0, & |x| \geq 1. \end{cases}$$

Plot η_0 *and show* $\eta_0 \in C^\infty(\mathbb{R})$.

(b) *For* $k = 1, 2, 3, \ldots$, *let* $\eta_k : \mathbb{R} \to [0, \infty)$ *by*

$$\eta_k(x) = k\ \eta_0(kx).$$

Plot η_k *and compute*

$$\int_{\mathbb{R}} \eta_k.$$

Hint: Your answer will be in terms of the positive constant

$$\int_{\mathbb{R}} \eta_0.$$

(c) *Consider* $u_k : \mathbb{R} \to [0, \infty)$ *by*

$$u_k(x) = \int_{\xi \in \mathbb{R}} \eta_k(\xi)|x - \xi|.$$

Plot u_k *and show that as* k *tends to infinity,* u_k *converges uniformly on all of* $\mathbb{R}$ *to a well-known function* $u : \mathbb{R} \to [0, \infty)$. *Show that* $u \in C^0(\mathbb{R}) \backslash C^1(\mathbb{R})$.

Problem 8 *(S&S Chapter 2 Exercise 6) Let* Ω *be an open subset of* $\mathbb{C}$ *with* $z_0 \in \Omega$ *and* $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ *holomorphic. Show that if* $\mathcal{U}$ *is an open triangular domain with*

$$z_0 \in \mathcal{U} \subset \overline{\mathcal{U}} \subset \Omega$$

and there exists some M *such that*

$$|f(z)| \leq M \qquad \text{for } z \in \mathcal{U}, \tag{8}$$

then

$$\int_\alpha f = 0$$

where α *is a parameterization of the triangular contour* $\partial\mathcal{U}$.

Note that this problem is related to Problem 3 on removable singularities above. How does condition (8) relate to (1)?

Problem 9 *(S&S Chapter 2 Exercise 8) Consider the strip* $\Omega = \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$. *If* $f : \Omega \to \mathbb{C}$ *is holomorphic and there are positive real numbers* M *and* μ *so that*

$$|f(z)| \le M(1 + |z|)^\mu \qquad \textit{for } z \in \Omega,$$

then show that for each $n = 1, 2, 3, \ldots$, *there is a positive real number* M_n *for which*

$$|f^{(n)}(x)| \le M_n(1 + |x|)^\mu \qquad \textit{for } x \in \mathbb{R}.$$

Problem 10 *(S&S Chapter 2 Exercise 9) Let* Ω *be an open bounded subset of* $\mathbb{C}$ *and consider a holomorphic function* $\phi : \Omega \to \Omega$. *Prove the following: If there exists some* $z_0 \in \Omega$ *for which*

$$\phi(z_0) = z_0 \qquad \text{and} \qquad \phi'(z_0) = 1,$$

then there exist constants $a, b \in \mathbb{C}$ *such that* $\phi(z) = az + b$. *Hint(s):*

(a) *Reduce to the case* $z_0 = 0$.

(b) *Use analyticity to write*

$$\phi(z) = z + a_N z^N + O(z^{N+1}) \qquad \textit{as } z \to 0$$

and some $N > 1$.

(c) *Consider the* k*-fold composition* $f_k(z) = \phi \circ \phi \circ \cdots \circ \phi$ *and show*

$$f_k(z) = z + k a_N z^N + O(z^{N+1}).$$

(d) *Take the limit as* k *tends to* ∞ *to conclude* $a_N = 0$. *Notice* $|f_k^{(N)}(0)|$ *tends to* ∞ *if* $a_N \neq 0$.