Assignment 8: Meremorphic Functions and residue calculus Due Tuesday April 5, 2022

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Problem 1 *(S&S Chapter 1 Exercise 18; Assignment 3 Problem 9—second chance) Here is the original statement of the problem: Consider a power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

with radius of convergence $R > 0$. *Show that* f *has a (convergent) power series expansion*

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

with center z_0 *for any* $z_0 \in D_R(0)$.

This is a nice problem, and I don't think anyone got it quite correct. A good number of you gave some[1] *"hand-waving" assertion to the effect that because the series converges absolutely, you can "rearrange" terms freely. I've written up my solution with an explanation of why I don't think the rearrangement of terms applicable to an absolutely convergent series is applicable here. In any case, I offer the following as an opportunity for you to nail down some/the details. In particular, I'm adding the following:*

[1] This is my interpretation. Maybe you know exactly what you are talking about, but this is your chance to explain it to me.

(a) *A* **rearrangement** *of a sequence* $\{\alpha_n\}_{n=1}^{\infty}$ *of complex numbers is a sequence* $\{\alpha_{j(n)}\}_{n=1}^{\infty}$ *where* $j : \mathbb{N} \to \mathbb{N}$ *is a bijection. Show that if*

$$\sum_{n=1}^{\infty} \alpha_n \qquad \text{is absolutely convergent,}$$

then

$$\sum_{n=1}^{\infty} \alpha_{j(n)} \qquad \text{is convergent for every bijection } j : \mathbb{N} \to \mathbb{N},$$

with value

$$\sum_{n=1}^{\infty} \alpha_{j(n)} = \sum_{n=1}^{\infty} \alpha_n \in \mathbb{C}.$$

(b) *(conjecture) Let*

$$\sum_{n=1}^{\infty} \alpha_n$$

be an absolutely convergent series for which each α_n *satisfies*

$$\alpha_n = \sum_{m=1}^{\infty} \beta_{nm}$$

for some absolutely convergent series

$$\sum_{m=1}^{\infty} \beta_{nm}.$$

Then

$$\sum_{n=1}^{\infty} \alpha_n = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \beta_{nm}.$$

(c) *Give a correct solution of the original problem and show, moreoever, that the series expansion of* f *with center* z_0 *has radius of convergence (at least)* $R - |z_0|$. *Hint: Go back through my solution of the original problem and improve it.*

Problem 2 *(S&S Chapter 3 Exercise 13; Ahlfors' theorem on removable singularities) Let* $f : \Omega \backslash \{z_0\} \to \mathbb{C}$ *be holomorphic with an isolated singularity at* $z_0 \in \Omega$. *Complete the steps below to prove the following result:*

Theorem 1 *(Ahlfors' result on removeable singularities) There exists a holomorphic function* $f_1 : \Omega \to \mathbb{C}$ *with the restriction to* $\Omega\backslash\{z_0\}$ *satisfying*

$$f_1\Big|_{\Omega\backslash\{z_0\}} \equiv f$$

if and only if

$$\lim_{z\to z_0} (z - z_0)f(z) = 0. \tag{1}$$

(a) *Show that if (1) holds, then*

$$\lim_{\epsilon\searrow 0} \int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta - z} = 0$$

where $\alpha(t) = z_0 + \epsilon e^{it}$ *for* $0 \le t \le 2\pi$ *parameterizes a circle around* z_0.

(b) *Note that if* $D_r(z_0) \subset \Omega$, *then* $f_1 : D_{r/2}(z_0) \to \mathbb{C}$ *by*

$$f_1(z) = \frac{1}{2\pi i} \int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta - z}$$

where $\alpha(z) = z_0 + 3re^{it}/4$ *for* $0 \le t \le 2\pi$ *parameterizes a circle in* $D_r(z_0)\backslash D_{r/2}(z_0)$ *is a well-defined continuous function on* $D_{r/2}(z_0)$. *Use a difference quotient to show the function* f_1 *is holomorphic on all of* $D_{r/2}(z_0)$. *Hint: Keep the integral as a complex integral; do not write it as a hybrid integral on an interval.*

(c) *Show that* $f_1(z) = f(z)$ *for* $z \in D_{r/2}(z_0)\backslash\{z_0\}$. *Hint(s): Use Cauchy's theorem and a keyhole contour.*

(d) *Finish the details of proving Ahlfors' theorem.*

Problem 3 *(Laurent series) Let $\Omega = D_R(0)\backslash\overline{D_r(0)}$ be an annular region for fixed r and R with $0 < r < R$. Let $f : \Omega \to \mathbb{C}$ be holomorphic.*

(a) *Consider the function $f_1 : D_R(0) \to \mathbb{C}$ defined by*

$$f_1(z) = \frac{1}{2\pi i}\int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta - z}$$

where $\alpha = \rho e^{it}$ for $0 \le t \le 2\pi$ for some ρ with $|z| < \rho < R$. Show that f_1 is a well-defined holomorphic function on $D_R(0)$.

(b) *Consider the function $f_2 : \mathbb{C}\backslash\overline{D_r(0)} \to \mathbb{C}$ by*

$$f_2(z) = -\frac{1}{2\pi i}\int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta - z}$$

where $\alpha = \rho e^{it}$ for $0 \le t \le 2\pi$ for some ρ with $r < \rho < |z|$. Show that f_2 is a well-defined holomorphic function on $\mathbb{C}\backslash\overline{D_r(0)}$.

(c) *Prove that $f(z) = f_1(z) + f_2(z)$ for $z \in \Omega$. Hint: Cauchy's theorem.*

(d) *Consider $g : D_{1/r}(0)\backslash\{0\}$ by*

$$g(w) = f_2(1/w).$$

Show g has a removable singularity at $w = 0$.

(e) *Conclude that*

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

with

$$a_n = \frac{1}{2\pi i}\int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta^{n+1}}.$$

Note: If one expands on an annulus with a different center z_0, then the series becomes

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

and the coefficients become

$$a_n = \frac{1}{2\pi i}\int_{\zeta=\alpha} \frac{f(\zeta)}{(\zeta - z_0)^{n+1}}.$$

Problem 4 *(winding number) Given a closed path* Γ *in* $\mathbb{C}\backslash\{z_0\}$ *parameterized by* α *the* **winding number** *of* Γ *with respect to* z_0 *is defined by*

$$n(\Gamma, z_0) = \frac{1}{2\pi i}\int_\alpha \frac{1}{z - z_0}.$$

(a) *Prove* $g : \mathbb{C}\backslash\{z_0\} \to \mathbb{C}$ *by*

$$g(z) = \frac{1}{z - z_0}$$

does **not** *have a primitive on any domain* Ω *with* $z_0 \in \Omega$.

(b) *Given any integer* $k \in \mathbb{Z}$, *find a path with*

$$n(\Gamma, z_0) = k.$$

(c) *Do you think you can find a closed loop in* $\mathbb{C}\backslash\{z_0\}$ *with winding number*

$$n(\Gamma, z_0) \notin \mathbb{Z}?$$

Problem 5 *(S&S Chapter 3 Exercise 1) Recall that*

$$\sin(\pi z) = \frac{e^{i\pi z} - e^{-i\pi z}}{2i}.$$

(a) *Identify all the zeros of* $\sin(\pi z)$.

(b) *Find the power series expansion of* $\sin(\pi z)$ *with center at each zero.*

(c) *Find the singular expansion of*

$$f(z) = \frac{1}{\sin(\pi z)}$$

at each pole z_0 *and find* $\operatorname{res}(f, z_0)$ *at that pole.*

Problem 6 *(S&S Chapter 3 Exercise 2) Use residue calculus to compute*

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^4}\, dx.$$

Problem 7 *(S&S Chapter 3 Exercise 6) Use residue calculus to show*

$$\int_{-\infty}^{\infty} \frac{1}{(1+x^2)^{n+1}}\, dx = \frac{(2n)!}{4^n (n!)^2}\ \pi.$$

Problem 8 *(S&S Chapter 3 Exercise 9) Use residue calculus to show*

$$\int_0^1 \log(\sin \pi x)\, dx = -\log 2.$$

Problem 9 *(S&S Chapter 3 Exercise 12) Let $u \in \mathbb{R}\backslash\mathbb{Z}$.*

(a) *Compute*

$$\lim_{k\to\infty} \int_\alpha \frac{\pi \cot \pi z}{(u+z)^2}$$

where $\alpha(t) = (k+1/2)e^{it}$ for $0 \leq t \leq 2\pi$ and $k \geq |u|$.

(b) *Conclude that*

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

(c) *What changes/happens if $u \in \mathbb{C}\backslash\mathbb{Z}$?*

Problem 10 *(S&S Chapter 3 Exercise 14) Assume $f : \mathbb{C} \to \mathbb{C}$ is entire. If f is one-to-one, show there exist $a_0, a_1 \in \mathbb{C}$ such that*

$$f(z) = a_1 z + a_0.$$