Isometric embedding of branch points (project)

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Riemann surfaces are usually described using a sheet model, so that a Riemann surface $\mathcal{R}$ is a union of copies of the complex plane with branch points excluded and appropriate transition/gluing relations between the copies of $\mathbb{C}$ (the sheets). For our purposes we wish to include the branch points. There will not be a branch point for each sheet, but only one point. Thus, at least schematically, the inclusion of a single branch point in a Riemann surface may be represented as

$$\mathcal{R} = \cup \mathcal{R}_j \cup \{z_0\}$$

where each set $\mathcal{R}_j$ is a copy of $\mathbb{C}$ with the point corresponding to the branch point excluded, say $(\mathbb{C}\backslash\{z_0\}, j)$ and an appropriate identification of branch cuts. The simplest example, and presumably the only example required for considerations of what I am going to suggest below is the Riemann surface $\mathcal{R}$ associated with $f(z) = z^2$. Thus,

$$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \{0\}$$

has two sheets and a single branch point at $z = 0$ with $\mathcal{R}_1$ corresponding to $\mathbb{C}\backslash\{0\}$ and $0 \le \arg(z) < 2\pi$, and $\mathcal{R}_2$ corresponding to $\mathbb{C}\backslash\{0\}$ and $2\pi \le \arg(z) < 4\pi$ and the identification

$$\arg(z) = 0 \qquad \text{when} \qquad \lim_{\zeta \to z} \arg(\zeta) = 4\pi$$

for some points $\zeta \in \mathcal{R}$.

There is, generally, given the sheet structure a projection $\pi : \mathcal{R} \to \mathbb{C}$.

Exercise 1 $\mathcal{R}$ *(with the branch point included) is a (natural) metric space.*

Osserman showed in his thesis that there exist isometric embeddings $\phi : \mathcal{R} \to \mathbb{R}^3$ (for certain Riemann surfaces including the Riemann surface for $f(z) = z^2$ but perhaps excluding the branch points). The following (slightly vague) observations follow:

1. There also exist isometric embeddings $\phi : \mathcal{R} \to \mathbb{R}^n$ for $n \geq 3$.

2. There also exist isometric embeddings $\phi : \mathcal{R} \to \mathbb{C}^n$ for $n \geq 2$.

Let V denote either $\mathbb{R}^n$ for $n \geq 3$ or $\mathbb{C}^n$ for $n \geq 2$. Here are some questions:

1. Can you prove there is no isometric embedding $\phi : \mathcal{R} \to \mathbb{C}$?

2. Given an isometric embedding of $\mathcal{R}\backslash\{z_0\}$, does it extend to the branch point?

3. Is there a way to define/measure the "singularity" at the branch point?

4. Do there exist isometric embeddings $\phi : \mathcal{R} \to V$ with ϕ smooth away from the branch point, i.e., on $\mathcal{R}\backslash\{z_0\}$? (I think Osserman showed the answer is "yes," but I haven't read the thesis in detail.

5. Can you prove there is no smooth embedding $\phi : \mathcal{R} \to V$ (globally on all of $\mathcal{R}$)?

In regard to question 3: Let $\mathcal{S}$ be the image of an isometric embedding $\phi : \mathcal{R} \to V$, smooth away from the branch point z_0. Given $p = \phi(z)$ with $z \in \mathcal{R}\backslash\{z_0\}$ there is an isometry (a natural isometry)

$$\psi_z : \mathbb{C} \to T_p\mathcal{S}$$

considering the tangent plane $T_p\mathcal{S}$ as an affine subspace of V, the codomain of ϕ. A little more precisely, it is possible to (uniquely) orient ψ_z so that taking the bijective map $\phi : \mathcal{R} \to \mathcal{S}$ with modified/restricted codomain and $\phi^{-1} : \mathcal{S} \to \mathcal{R}$ and $d\phi^{-1} = d(\phi^{-1}) : T_p\mathcal{S} \to T_z\mathcal{R} \sim \mathbb{C}$ satisfy

$$d\psi_z^{-1} = d(\psi_z)^{-1} = d\phi^{-1}.$$

Claim: This makes ψ_z unique.

With the identification of $\mathbb{C}$ with $\{(x_1, x_2, 0, \ldots, 0) \in \mathbb{R}^n : x_1 + ix_2 \in \mathbb{C}\}$ or $\{z, 0, \ldots, 0) \in \mathbb{C}^n\}$, the natural isometry $\psi_z : \mathbb{C} \to T_p\mathcal{S}$ extends to a rigid motion

$$\psi_z : V \to V.$$

One can then consider

$$A(z) = \|\psi_z = \mathrm{id}\| \qquad \text{for some norm}$$

and/or

$$\lim_{z \to z_0} \psi_z$$

as measures of the "niceness" or "regularity" of the embedding ϕ.

Guess: Perhaps there is more to be said about the isometric embedding of branch points.

It would be nice of Osserman or Ahlfors were around to ask. But they are not, so it's (perhaps) up to us.