Cauchy's Theorem(s)

John McCuan

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We have now complex integration along a curve denoted by

$$\int_\alpha f. \tag{1}$$

There are two essential elements in such an integral: The curve and the function. The curve is assumed to be parameterized by a function $\alpha : [a, b] \to \mathbb{C}$ on a real interval, to be at least piecewise regular, and to have image $\Gamma \subset \mathbb{C}$. The function $f : \Gamma \to \mathbb{C}$ is assumed to be continuous. Recall then, that the value of the integral (1) is given by (the hybrid integral)

$$\int_a^b f \circ \alpha(t)\ \alpha'(t)\, dt.$$

We introduce here a natural broader context in which to consider complex integration. Namely, let us assume $f : \Omega \to \mathbb{C}$ is a complex valued function defined on an open set $\Omega \subset \mathbb{C}$. We will consider curves $\alpha : [a, b] \to \Gamma \subset \Omega$, but restrict attention to **closed curves**, that is, curves for which $\alpha(b) = \alpha(a)$.

Thus, we may think of three essential elements in relation to the complex integral (1) over a closed curve:

(i) The curve,

(ii) The function, and

(iii) The domain Ω.

Let us call a result a "Cauchy type theorem" if the conclusion is

$$\int_\alpha f = 0.$$

This conclusion may hold either for a specific closed curve or a class of closed curves, a specific domain or a class of domains, and/or a specific function or a class of functions. We know one such theorem already which is an important one:

Theorem 1 *(Existence of a primitive; Corollary 2 in my notes on integration or Corollary 3.3 in S&S) If* $f : \Omega \to \mathbb{C}$ *is holomorphic and there exists a function* $g : \Omega \to \mathbb{C}$ *holomorphic with*

$$g' = f \qquad \text{on} \qquad \Omega,$$

then

$$\int_\alpha f = 0$$

for any closed curve $\alpha : [a, b] \to \Omega$*.*

Note that in this theorem, the function may be viewed as quite special, but the domain and the curve can essentially be anything.

The next result of Cauchy type is quite different. Essentially the domain and the curve are very special, but the result applies to many functions.

Theorem 2 *(Goursat's theorem) If* $f : \Omega \to \mathbb{C}$ *is complex differentiable and* $\mathcal{U}$ *is a triangular domain with boundary a triangle*

$$T = \partial\mathcal{U} \qquad \text{with} \qquad T \cup \mathcal{U} = \overline{\mathcal{U}} \subset \Omega,$$

then

$$\int_\alpha f = 0$$

where $\alpha : [a, b] \to T$ *is a parameterization of the triangle* $T = \partial\mathcal{U}$*.*

We will combine these two results to get the next result:

Theorem 3 *(Cauchy's theorem in a triangle) If* $\Omega = \mathcal{U}$ *is a triangular domain and* $f : \Omega \to \mathbb{C}$ *is holomorphic, then*

$$\int_\alpha f = 0$$

for any closed curve $\alpha : [a, b] \to \Omega$*.*

In this result one sees many of the main features of what is considered Cauchy's theorem.[1]

[1]...rather than just a "Cauchy type theorem."

Theorem 4 *(Cauchy's theorem in a rectangle) If $\Omega = \mathcal{R}$ is a rectangular domain and $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_\alpha f = 0$$

for any closed curve $\alpha : [a, b] \to \Omega$.

Theorem 5 *(Cauchy's theorem in a disk) If $\Omega = D_r(z_0)$ is a disk domain and $f : \Omega \to \mathbb{C}$ is holomorphic, then*

$$\int_\alpha f = 0$$

for any closed curve $\alpha : [a, b] \to \Omega$.

There are more general versions, but these are a good start. Let us prove Goursat's theorem and Cauchy's theorem in a triangle.

Proof of Goursat's theorem: Every triangular domain $\mathcal{U}$ can be partitioned into four "half size" similar triangular subdomains

$$\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$$

with the vertices/corners of the subdomains either the midpoints of the sides of $T = \partial\mathcal{U}$ or the vertices of T. Note that all the linear dimensions of each of the triangular domains is half that of the original triangular domain. In particular, if $T_j = \partial\mathcal{U}_j$ for $j = 1, 2, 3, 4$, then

$$\text{length}(T_j) = \frac{1}{2}\ \text{length}(T).$$

Furthermore, if α_j is a (counterclockwise) parameterization of T_j for $j = 1, 2, 3, 4$, then

$$\int_\alpha f = \sum_{j=1}^{4} \int_{\alpha_j} f.$$

This gives

$$\left| \int_\alpha f \right| \le \sum_{j=1}^{4} \left| \int_{\alpha_j} f \right| \le 4 \max_j \left| \int_{\alpha_j} f \right|.$$

Let $\mathcal{U}^{(1)}$ with $T^{(1)} = \partial\mathcal{U}^{(1)}$ parameterized by $\alpha^{(1)}$ be a/the triangular domain among $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3$, and $\mathcal{U}_4$ satisfying

$$\left|\int_{\alpha^{(1)}} f\right| = \max_j \left|\int_{\alpha_j} f\right|.$$

Applying the same construction to $\mathcal{U}^{(1)}$, its successor $\mathcal{U}^{(2)}$ and so on, we obtain a sequence of nested triangular domains

$$\mathcal{U}^{(1)} \supset \mathcal{U}^{(2)} \subset \mathcal{U}^{(3)} \supset \cdots$$

with boundaries $T^{(m)} = \partial\mathcal{U}^{(m)}$ parameterized by $\alpha^{(m)}$ for $m = 1, 2, 3, \ldots$ and for which for which

$$\left|\int_{\alpha} f\right| \le 4^m \left|\int_{\alpha^{(m)}} f\right|.$$

and

$$\text{length}(T^{(m)}) \le \frac{1}{2^m}\,\text{length}(T).$$

Notice that the closures

$$\overline{\mathcal{U}^{(1)}} \supset \overline{\mathcal{U}^{(2)}} \supset \overline{\mathcal{U}^{(3)}} \supset \cdots$$

are a sequence of **nonempty nested compact sets**. Consequently,

$$\bigcap_{m=1}^{\infty} \overline{\mathcal{U}^{(m)}} = \{z_0\}$$

for some unique $z_0 \in \mathcal{U} \subset \Omega$.

Since f is complex differentiable at z_0 we have an approximation formula we can apply (or at least can try to apply) to estimate

$$\int_{\alpha^{(m)}} f.$$

This approximation formula can be written as

$$f(z) = f(z_0) + f'(z_0)\ (z - z_0) + \circ(|z - z_0|). \tag{2}$$

Let g_0 denote the global primitive for the constant function $f(z_0)$. That is, $g_0(z) = f(z_0)z$ and

$$g_0' = f(z_0).$$

Similarly, let $g_1(z) = f'(z_0)(z - z_0)^2/2$ so that

$$g_1' = f'(z_0)\ (z - z_0).$$

Using the existence of these primitives we can integrate the approximation formula (2) to get

$$\int_{\alpha^{(m)}} f = \int_{\alpha^{(m)}} \circ(|z - z_0|). \tag{3}$$

This may look a little unsettling and unfamiliar with an integral of a "little O" function like this; in principle there is no reason to believe a "little O" function is continuous or that this integral makes sense. Just bear with me a moment, and I'll come back and "fix it up" later. For now, let's estimate as follows:

$$\begin{aligned}
\left|\int_\alpha f\right| &\le 4^m \left|\int_{\alpha^{(m)}} f\right| \\
&\le 4^m\ \max \circ(|z - z_0|)\ \text{length}(T^{(m)}) \\
&= 4^m\ \frac{\max \circ(|z - z_0|)}{|z - z_0|}\ |z - z_0|\ \frac{1}{2^m}\,\text{length}(T) \\
&= 2^m\ \frac{\max \circ(|z - z_0|)}{|z - z_0|}\ |z - z_0|\ \text{length}(T).
\end{aligned}$$

Note that

$$|z - z_0| \le \text{diam}(\mathcal{U}^{(m)}) = \frac{1}{2^m}\,\text{diam}(\mathcal{U}).$$

Therefore,

$$\left|\int_\alpha f\right| \le \frac{\max \circ(|z - z_0|)}{|z - z_0|}\ \text{diam}(\mathcal{U})\ \text{length}(T) \to 0$$

as $m \nearrow \infty$ so that $|z - z_0| \to 0$. If this argument is correct, it shows

$$\int_\alpha f = 0$$

and we are done. Stein makes the argument look a little more palatable by writing our function $\circ(|z - z_0|)$ as

$$\circ(|z - z_0|) = \psi(z)(z - z_0) \tag{4}$$

for some function ψ with

$$\lim_{z \to z_0} \psi(z) = 0.$$

Notice that saying

$$\lim_{z \to z_0} \frac{\circ(|z - z_0|)}{|z - z_0|} = 0$$

is precisely the same as saying

$$\lim_{z \to z_0} \frac{\psi(z)|z - z_0|}{|z - z_0|} = 0$$

when (4) holds. But it makes the argument/estimates look rather better:

$$\begin{aligned}
\left| \int_\alpha f \right| &\leq 4^m \left| \int_{\alpha^{(m)}} f \right| \\
&\leq 4^m \ \max[|\psi(z)||z - z_0|] \ \text{length}(T^{(m)}) \\
&\leq 4^m \ \max|\psi(z)| \ \text{diam}(\mathcal{U}^{(m)}) \ \frac{1}{2^m} \text{length}(T) \\
&\leq 2^m \ \max|\psi(z)| \ \frac{1}{2^m} \ \text{diam}(\mathcal{U}) \ \text{length}(T) \\
&\leq \max|\psi(z)| \ \text{diam}(\mathcal{U}) \ \text{length}(T) \to 0 \quad \text{as } z \to z_0.
\end{aligned}$$

This looks better, but still we have skipped the objectionable step where we first estimate the integral of f using ψ. That is, before we start the string of estimates above we should write a version of (3) that looks like

$$\int_{\alpha^{(m)}} f = \int_{\alpha^{(m)}} \psi(z)|z - z_0|.$$

Again, it looks good (or at least better), but the fact of the matter is that generally the definition of "little O" says nothing about the function ψ being continuous and hence integrable (along a curve). In this case, however, we are okay. In fact, in the definition of differentiability the function $\circ(|z - z_0|) = \psi(z)(z - z_0)$ is **not just any** function. In this case, we have

$$\psi(z) = \frac{f(z) - f(z_0) - f'(z_0)(z - z_0)}{z - z_0},$$

and this specific function **is continuous** for $z \neq z_0$ and hence integrable. (And it has the property that $\psi(z) \to 0$ as $z \to z_0$.) So the proof above turns out to be just fine. $\square$

Corollary 1 *(Quadrilateral Goursat theorem) If $f : \Omega \to \mathbb{C}$ is complex differentiable and $\mathcal{V}$ is a connected quadrilateral domain with boundary a quadrilateral*

$$Q = \partial\mathcal{V} \qquad \text{with} \qquad Q \cup \mathcal{V} = \overline{\mathcal{V}} \subset \Omega,$$

then

$$\int_\alpha f = 0$$

where $\alpha : [a, b] \to Q$ is a parameterization of the quadrilateral $Q = \partial\mathcal{V}$.

Exercise 1 *Prove the quadrilateral Goursat theorem by noting that any quadrilateral domain is a union of triangular domains.*

Let's now use Goursat's theorem to prove Cauchy's theorem for a triangular domain.

Proof of Theorem 3: We start with a curve Γ, a closed curve within a triangular domain $\mathcal{U}$. Presumably, Γ is parameterized in some way, but this is not important for us because we are going to use Theorem 1. That is, we will show the existence of a primitive for the holomorphic function $f : \mathcal{U} \to \mathbb{C}$ on a triangular domain. Technically, we will not immediately obtain a primitive for f on all of $\mathcal{U}$, but this is only a technical detail. The point is that all integrals

$$\int_\alpha f$$

over closed curves like Γ must vanish if we can verify the existence of a primitive.

To this end, we note that Γ is a compact subset of the open triangular domain $\mathcal{U}$. Consequently, we can take a dilation $\mathcal{U}_1$ giving a similar triangular domain slightly smaller than $\mathcal{U}$ for which

$$\Gamma \subset \mathcal{U}_1 \subset \overline{\mathcal{U}_1} \subset \mathcal{U}.$$

Let the vertices of $T_1 = \partial\mathcal{U}_1$ be a, b, and c in counterclockwise order and let us denote the opposite sides of the triangle T_1 by A, B, and C with A opposite a and B opposite b. Notice that for each $z \in \mathcal{U}_1$, there is a unique segment parallel to the side A opposite a connecting z to a point $\zeta \in T_1$ on the side C between a and b. Thus, for each $z \in \mathcal{U}_1$, there is a unique path consisting of the segment γ_1 from a to the point $\zeta \in C$ and the segment α_1 from ζ to z. Thus, a unique complex valued function is defined on $\mathcal{U}_1$ by

$$g(z) = \int_{\gamma_1} f + \int_{\alpha_1} f.$$

We claim $g : \mathcal{U}_1 \to \mathbb{C}$ is holomorphic with $g' = f$. That is, g is a primitive for f on $\mathcal{U}_1$.

Let $h \in \mathbb{C}$ have $|h|$ small enough so that $z + h \in \mathcal{U}_1$. Then the value $g(z+h)$ is computed using segments γ_2 in C from a to a point ζ_2 and a segment α_2 from ζ_2 to z.

Letting γ_0 denote the segment from ζ to ζ_2 we can write

$$g(z+h) - g(z) = \int_{\gamma_2} f + \int_{\alpha_2} f - \int_{\gamma} f - \int_{\alpha} f = \int_{\gamma_0} f + \int_{\alpha_2} f - \int_{\alpha} f.$$

If we add the integral

$$\int_{\delta} f$$

where δ is the segment from $z + h$ to z we obtain

$$g(z+h) - g(z) + \int_{\delta} f = \int_{\gamma_0} f + \int_{\alpha_2} f + \int_{\delta} f - \int_{\alpha} f.$$

Notice the concatenation of γ_0, α_2, δ, and $-\alpha$ on the right is an oriented quadrilateral path around a connected domain from ζ to ζ_2 to $z + h$ to z and back to ζ. By the quadrilateral Goursat theorem we have

$$g(z+h) - g(z) = -\int_{\delta} f.$$

That is, $g(z+h) - g(z)$ is the integral of f over the segment from z to $z + h$. Using the differentiability of f at z to approximate this integral we have

$$f(w) = f(z) + f'(z)(w - z) + \psi(w)(w - z)$$

where ψ is a continuous function of w for which

$$\lim_{w \to z} \psi(w) = 0.$$

We have a primitive $g_0(w) = f(z)w$ for the first term so that

$$\begin{aligned}
-\int_{\delta} f &= \int_{-\delta} [g_0' + f'(z)(w - z) + \psi(w)(w - z)] \\
&= g_0(z+h) - g_0(z) + \int_{-\delta} f'(z)(w - z) + \int_{-\delta} \psi(w)(w - z) \\
&= f(z)h + \int_{-\delta} f'(z)(w - z) + \int_{-\delta} \psi(w)(w - z).
\end{aligned}$$

Therefore,

$$\frac{g(z+h)-g(z)}{h} = f(z) + \frac{1}{h}\int_{-\delta} f'(z)(w-z) + \frac{1}{h}\int_{-\delta} \psi(w)(w-z).$$

Estimating the last two terms we find

$$\left|\frac{1}{h}\int_{-\delta} f'(z)(w-z)\right| \le |f'(z)||h| \to 0$$

and

$$\left|\frac{1}{h}\int_{-\delta} \psi(w)(w-z)\right| \le \max|\psi||h| \to 0$$

as $h \to 0$. That is, $g' = f$. $\square$

1 Under the Rug Part I

We have given proofs of Goursat's theorem (for triangular subdomains) and Cauchy's theorem in a triangle. Hopefully these proofs have been tolerably convincing. We have followed the exposition of Stein for the most part with a couple exceptions. One exception is that we stated (and proved) Cauchy's theorem in a triangle rather than a disk. I think there is a tangible advantage to the approach I have taken in this regard. The proof of Cauchy's theorem in this case is basically about creating a primitive g by integrating along particular simple paths that can be "resolved" in terms of triangular paths. Stein's proof, similar to Ahlfors' in a disk depends on paths starting at the center z_0 and consisting of two segments, the first horizontal and the second vertical connecting to a point z. There is a minor ambiguity in that for some points the required two-segment path must go right (or left) and then up, while for others the path must go right (or left) then down, and for yet others the path is degenerate consisting only of a horizontal segment or only of a vertical segment. If one wants to be (extremely) careful with this there are several cases to check. There is a similar ambiguity in the proof I've given above in a triangular domain, but at least all paths proceed from one vertex of the triangle along a specified side and then "up." The cases I've swept under the rug involve the position/value of the increment h, and the same ambiguity (or appearance of various irritating cases to check) is present in the proof on the disk, but their[2] number of cases is multiplied by the first choice of path.

[2] Stein's and Ahlfors'.

For example, if you start with a horizontal path from the center of a disk z_0 to a point $z = z_0 + a$ with $a > 0$, then for h small, you may need a two-segment (horizontal plus up) or a two-segment (horizontal plus down) path to write down the desired integral from z_0 to $z + h$ defining $g(z + h)$. This multiplication of cases does not happen in the case of the triangular domain, as I have argued, starting from the corner. In fact, I will make some effort below to consider (extremely) carefully the cases in a triangular domain associated with the choice of h. I will do this in connection with some alternative constructions which I think are somewhat interesting and instructive.

A kind of second exception may be thought of in the opposite direction: Rather than state and prove a "simpler" version of the result as I've done with Goursat's theorem, I've used, without formal statement a corollary of Goursat's theorem involving a rather quite general quadrilateral, leaving the proof as an exercise, while Stein states and proves Corollary 1.2 which is essentially a version of Goursat's theorem for rectangular subdomains. Again, I will try to come back and "clean up" my result on quadrilateral subdomains below on connection with some auxilliary constructions.

The real issue, it seems to me, however is the question of the Jordan curve theorem. Stein says briefly that he is going to come back to it, but for a serious graduate text in complex analysis I think what he has done here may be fairly viewed as a significant deficiency. I say this with all due respect as I think the overall exposition is really quite impressive. Two things can alert the careful reader to what is happening here. The first is the appearance of "toy domains," which I have not so far mentioned. These are not even defined precisely **and** the cursory definition of them involves the word "obvious." What is supposed to be "obvious" about them is that "they ae so simple that the notion of their interior will be obvious." First of all, what he is trying to say is that the conclusion of the Jordan curve theorem is "obvious" for these domains, which may be more or less true, but without any explanation whatsoever this presentation does a real disservice to the serious graduate student. At least in my opinion in the composition of a serious exposition one should set a good example of critical thinking for the student and at least state precisely what is being swept under the rug. Second, and the second "red flag" for the reader, is the nonstandard use of the term "interior." The interior of a set is a well-defined topological term at this point. Here the term is being used to refer to **the unique bounded component of the complement of a simple closed curve**; the existence of this set is basically the assertion of the Jordan curve theorem and is anything but obvious.

Having made my complaint(s), let me see if I can do a bit better.

2 Goursat Subdomains

Given an open set $\Omega \subset \mathbb{C}$ a **simple Goursat subdomain** is an open set $\mathcal{U}$ with $\overline{\mathcal{U}} \subset \Omega$ having the following properties: First $\partial\mathcal{U}$ is a closed curve/contour.[3] Second, there exists a fixed natural number ν and there exists a fixed scale $\mu \in (0,1)$ such that

(i)
$$\overline{\mathcal{U}} = \bigcup_{j=1}^{\nu} \overline{\mathcal{U}_j}$$

for some subdomains $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_\nu \subset \mathcal{U}$ satisfying

(ii) $\mathcal{U}_k \cap \mathcal{U}_j = \phi$ for $j \neq k$,

(iii)
$$\mathcal{H}^1\left(\overline{\mathcal{U}_j} \cap \overline{\mathcal{U}_k}\right) = 0 \qquad \text{for } j \neq k,$$

and

(iv) each $\mathcal{U}_j$ is geometrically similar to $\mathcal{U}$ with

$$\mathcal{U}_j = \{\mu z + w_j : z \in \mathcal{U}\}$$

for some $w_j \in \mathbb{C}$, $j = 1, 2, \ldots, \nu$.

The triangular domain is an example of a Goursat subdomain with $\nu = 4$ and $\mu = 1/2$. The rectangular domain is also an example of a Goursat subdomain with $\nu = 4$ and $\mu = 1/2$. Ahlfors proves Goursat's theorem for a rectangular subdomain.

Exercise 2 *Give an example of a Goursat subdomain with $\nu \neq 4$ and/or $\mu \neq 1/2$. Give an example of a Goursat subdomain which is not a triangular domain or a rectangular domain.*

If α is a counterclockwise parameterization of the boundary of a simple Goursat subdomain $\mathcal{U}$, then

$$\int_\alpha = \sum_{j=1}^{\nu} \int_{\alpha_j} f \tag{5}$$

[3] By "contour" we mean a curve admitting a piecewise regular parameterization—a curve Γ that can be used to construct a complex integral of a continuous function $f : \Gamma \to \mathbb{C}$.

where α_j is a counterclockwise parameterization of the boundary of $\mathcal{U}_j$ for $j = 1, 2, \ldots, \nu$. I'm not sure if it's easy to prove (5) or not. Perhaps this assertion is not true according to the definition I've given. If not, perhaps (5) should simply be added to the definition (or be essentially taken as the definition).

Exercise 3 *If (5) holds for a simple Goursat subdomain $\mathcal{U}$, then each set $\mathcal{U}_j$ for $j = 1, 2, \ldots, \nu$ is a Goursat subdomain.*

Theorem 6 *(Goursat's theorem) If $f : \Omega \to \mathbb{C}$ is complex differentiable and $\mathcal{U}$ is a simple Goursat subdomain with respect to Ω, then*

$$\int_\alpha f = 0$$

where $\alpha : [a, b] \to \Omega$ is a parameterization of $\partial\mathcal{U}$.

Proof: See below.

Definition 1 *A* **general Goursat subdomain** *with respect to an open set $\Omega \subset \mathbb{C}$ is a domain $\mathcal{U} \subset \mathbb{C}$ for which $\partial\mathcal{U}$ is a closed contour and $\overline{\mathcal{U}} \subset \Omega$ with*

$$f : \Omega \to \mathbb{C} \text{ holomorphic} \qquad \Longrightarrow \qquad \int_\alpha f = 0$$

where α parameterizes $\Gamma = \partial\mathcal{U}$ and $f : \Omega \to \mathbb{C}$ is any holomorphic function.

The following result allows one to treat domains, like triangular domains or rectangular domains in (a) "standard position."

Theorem 7 *The conformal image of a general Goursat subdomain is a general Goursat subdomain in the following sense: If $\mathcal{U}$ is a general Goursat subdomain in Ω and*

(i) *$\phi : \Omega \to W$ is a surjective holomorphic function onto an open set $W \subset \mathbb{C}$,*

(ii)] $\phi(\overline{\mathcal{U}}) = \overline{\mathcal{V}}$ for some open set $\mathcal{V} \subset W$ with $\partial\mathcal{V}$ a closed contour with parameterization $\beta = \phi \circ \alpha$ where α is a parameterization of $\partial\mathcal{U}$ as in the definition,

then

$$\int_\beta f = \int_\alpha f \circ \phi \; \phi' = 0$$

for any holomorphic function $f : W \to \mathbb{C}$.

3 Cauchy's Theorem for (various) Domains

We have given a proof of Cauchy's theorem for triangular domains above and stated Cauchy's theorem for a rectangular domain and for a disk. We rework and generalize this discussion below.

3.1 Cauchy Domains

Let us define a **Cauchy domain** to be an open set $\Omega \subset \mathbb{C}$ with the property that

$$\int_\alpha f = 0$$

for every closed (piecewise regular) contour $\Gamma \subset \Omega$ (parameterized by α) and every holomorphic function $f : \Omega \to \mathbb{C}$. One might be tempted to think of a Cauchy domain as (simply) a simply connected domain: An open set $\Omega \subset \mathbb{C}$ is **simply connected** if Ω if every closed curve $\Gamma \subset \Omega$ is **homotopic** in Ω to a point (any particular point) $z_0 \in \Omega$. Two "curves," i.e., continuous functions $\alpha : [a, b] \to \mathbb{C}$ and $\beta : [a, b] \to \mathbb{C}$ are **homotopic** if there is a continuous function $h : [a, b] \times [0, 1] \to \mathbb{C}$ with

$$\begin{aligned} \alpha(t) &= h(t, 0) \qquad \text{for } t \in [a, b], \text{ and} \\ \beta(t) &= h(t, 1) \qquad \text{for } t \in [a, b]. \end{aligned}$$

Given homotopic curves α and β, the continuous function $h : [a, b] \times [0, 1] \to \mathbb{C}$ deforming one to the other is called a **homotopy**. Two homotopic curves are **homotopic in an open set** $\Omega \subset \mathbb{C}$ if the homotopy satisfies (or more properly if there exists a homotopy satisfying)

$$h : [a, b] \times [0, 1] \to \Omega,$$

i.e., the codomain of h is Ω. A homotopy deforming one curve α to another β is said to be a **fixed endpoint homotopy** and the curves are said to be **fixed endpoint homotopic** if

$$h(a, s) = \alpha(a) \quad \text{and} \quad h(b, s) = \alpha(b) \qquad \text{for all } s \in [0, 1].$$

Two (homotopic) closed curves (or loops) are said to be **homotopic** or **homotopic as loops** if the/a homotopy satisfies

$$h(a, s) = h(b, s) \qquad \text{for all } s \in [0, 1].$$

In the definition of **simply connected** above we are saying that given any closed contour, i.e., a continuous, piecewise regular, parameterization $\alpha : [a, b] \to \Omega$ and any point $z_0 \in \Omega$, there is a homotopy

$$h : [a, b] \times [0, 1] \to \Omega$$

with

$$h(a, s) = h(b, s) \qquad \text{for all } s \in [0, 1]$$

and $h(t, 1) \equiv z_0$. Notice that the function $\beta : [a, b] \to \Omega$ by $\beta(t) = h(t, 1) \equiv z_0$ in this case is definitely not regular, and there is no requirement that $\gamma(t) = h(t, s)$ for s fixed parameterizes a regular curve for $s > 0$. Nevertheless, thinking of a Cauchy domain as a simply connected domain is not entirely wrong.

Theorem 8 *Every simply connected open set $\Omega \subset \mathbb{C}$ is a Cauchy domain.*

This theorem is proved in Appendix B (Theorem 1.1) of Stein and Shakarchi. Perhaps some clarification is in order: First of all, Stein and Shakarchi have given a definition of **simply connected** which differs, at least superficially, from mine. Their definition is given on the top of page 96 in Chapter 3. It is crucial for this definition that one look back at the definition of "region" given on page 7 of Chapter 1. The term "region" does not have a standard meaning in mathematics, so one would think Stein and Shakarchi would put some emphasis on the particular meaning they have attached to this word. In any case, Stein and Shakarchi intend that a "region" is an open subset of $\mathbb{C}$ which is apriori connected. Some authors require also that a "region" is bounded. Some use the term synonomously with "open set" or even just "set." The word "domain" is a little bit similar. I tend to use the term "domain" to mean simply an open subset of $\mathbb{C}$. More generally, I might be a little sloppy sometimes and mean simply a "set" on which a function is defined, i.e., the domain of a function. In any case, one should note that Stein and Shakarchi have an underlying apriori assumption of connectedness when they define a domain Ω to be simply connected if every pair of curves with the same endpoints is fixed endpoint homotopic within Ω to one another.

Exercise 4 *Show that the definition of* **simply connected** *given by Stein and Shakarchi is equivalent to the one I've given above in which each loop is homotopic (as a loop) to any single point in the domain.*

Stein and Shakarchi also introduce a notion of **holomorphically simply connected** which looks superficially like my definition of a Cauchy domain above. The difference is that, again, Stein and Shakarchi require/assume a holomorphically simply connected domain is a "region," that is, it is connected, but my definition of a Cauchy domain does not assume the domain Ω is connected. It is nice that Stein and Shakarchi get an equivalence (Theorem 1.1 of Appendix B). On the other hand, it is reasonable to point out that the union Ω of two disjoint open disks is a domain with the property that

$$\int_\alpha f = 0$$

for any holomorphic function $f : \Omega \to \mathbb{C}$ and any closed curve (parameterized by α) in Ω. The union of two disks is a Cauchy domain, but it is not holomorphically simply connected because it is not connected; each component is simply connected.

In a certain sense, I think the focus on the Jordan curve theorem (for piecewise smooth curves) in Stein and Shakarchi's Appendix B is a little unusual and unwaranted, but it's a nice result to consider. On the other hand, the phrasing of Theorem 2.3 in Appendix B of Stein and Shakarchi, especially for domains with boundary a piecewise smooth Jordan curve, is almost a reduction to the consideration of (general) Goursat (sub) domains as introduced above.

In Appendix B Stein and Shakarchi cover what I consider relatively important material including

1. The properties of **simply connected** domains,
2. The **winding number**, and
3. Theorem 2.9 which they call a **general form of Cauchy's theorem**.

As mentioned above Stein and Shakarchi discuss homotopies and simply connected domains briefly in Chapter 3. They also hold off on mentioning the **Riemann mapping theorem** until Chapter 8, and I think it is nice to mention that theorem a bit earlier—even if one does not prove it. I am going to try to cover (at least some of) these topics below including a more general form of Cauchy's theorem which does not require reference to a Jordan (simple closed) curve or the Jordan curve theorem. I'm also going to try to give an extended/alternative treatment of Goursat's theorem and the special cases of Cauchy's theorem. In short I'm going to try to tie together some topics which are separated (probably for the sake of clarity) in Stein and Shakarchi.

Here are four general abstract results concerning Cauchy domains and/or holomorphically simply connected domains:

Theorem 9 *If Ω is a connected Cauchy domain (holomorphically simply connected in the terminology of Stein), then given any holomorphic function $f : \Omega \to \mathbb{C}$ and any $z_0 \in \Omega$ the function $g : \Omega \to \mathbb{C}$ by*

$$g(z) = \int_\alpha f \tag{6}$$

where α is any countour connecting z_0 to z is a well-defined complex valued function. More generally, if $\mathcal{C}$ is any connected component of a Cauchy domain with $z_0 \in \mathcal{C}$, then (6) gives a well-defined function $g : \mathcal{C} \to \mathbb{C}$.

Proof: If α and β are two contours connecting z_0 to z, then $\alpha - \beta$ is a closed contour, so

$$\int_{\alpha-\beta} f = 0,$$

and consequently

$$g(z) = \int_\alpha f = \int_\beta f$$

is well-defined. $\square$

This general abstract result, to a large extent at least, reduces the proof of theorems about the existence of primitives to the question of calculating the well-defined function g given by (6) without worrying about which particular path one is using to find the value of g.

Here is a second general abstract result about Cauchy domains:

Theorem 10 *The conformal image of a Cauchy domain is a Cauchy domain in the following sense: If Ω is a Cauchy domain and $\phi : \Omega \to W$ is a bijective holomorphic function (conformal map) onto an open set $W \subset \mathbb{C}$ so that any parameterization β of a contour in W can be written as $\beta = \phi \circ \alpha$ for some parameterization α of a contour in Ω, then*

$$\int_\beta f = \int_\alpha f \circ \phi \; \phi' = 0$$

for any holomorphic function $f : W \to \mathbb{C}$.

Proof: This is just the change of variables formula for complex integrals. $\square$

This result allows us to put domains in standard position. See the section on triangular domains below.

A third general abstract result generalizes the first one (Theorem 9):

Theorem 11 *Let $\mathcal{C}$ be a connected subdomain of a Cauchy domain Ω and let $z_0 \in \mathcal{C}$. Given a holomorphic function $f : \Omega \to \mathbb{C}$, the following hold:*

(i) *The function $g : \mathcal{C} \to \mathbb{C}$ by*

$$g(z) = \int_\alpha f \tag{7}$$

where $\alpha : [a, b] \to \mathcal{C}$ is any contour connecting z_0 to z in $\mathcal{C}$ is well-defined and satisfies

(ii) *The increment*

$$g(z+h) - g(z) = h \int_0^1 f(z+ht)\, dt$$

for any h small enough so that $z + th \in \mathcal{C}$ for $0 \le t \le 1$, and

(iii) *$g' = f$ on $\mathcal{C}$ so that f has a primitive given by (7) on $\mathcal{C}$.*

Proof: The fact that g is well-defined follows as in Theorem 9.

Let β parameterize a path in $\mathcal{C}$ connecting z_0 to $z + h$, and let α parameterize a path in $\mathcal{C}$ connecting z_0 to z. Then

$$g(z+h) - g(z) = \int_\beta f - \int_\alpha f = \int_\beta f - \int_\alpha f + \int_{-\gamma} f + \int_\gamma f$$

where $\gamma(t) = z + th$ for $0 \le t \le 1$ is a path connecting z to $z+h$. Note that $\beta - \alpha - \gamma$ is an oriented closed contour starting at z_0 and going to $z + h$ (along the image of α) and then from $z + h$ to z (along the image of $-\gamma$) and then from z to z_0 along the image of $-\beta$. Thus, the integral along this closed countour vanishes because Ω is a Caucny domain, and

$$g(z+h) - g(z) = \int_\gamma f = \int_0^1 f(z_0 + ht)\, h\, dt.$$

It follows also that

$$\lim_{h\to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h\to 0} \int_0^1 f(z_0 + ht)\, dt = f(z). \qquad \square$$

Finally, we have this:

Theorem 12 *If Ω_1, Ω_2 and $\Omega_1 \cap \Omega_2$ are nonempty connected Cauchy domains, then $\Omega_1 \cup \Omega_2$ is a (connected) Cauchy domain.*

Proof: It is enough to construct a primitive for any holomorphic function $f : \Omega_1 \cup \Omega_2 \to \mathbb{C}$. Let $z_0 \in \Omega_1 \cap \Omega_2$ and consider $g : \Omega_1 \cup \Omega_2 \to \mathbb{C}$ by

$$g(z) = \begin{cases} \displaystyle\int_{\alpha_1} f, & z \in \Omega_1 \\ \\ \displaystyle\int_{\alpha_2} f, & z \in \Omega_2 \end{cases}$$

where α_1 parametrizes a contour in Ω_1 connecting z_0 to z and α_2 parameterizes a contour in Ω_2 connecting z_0 to z. We claim the values agree when $z \in \Omega_1 \cap \Omega_2$ so that g is well-defined. To see this, take a parameterization β of a contour in $\Omega_1 \cap \Omega_2$ connecting z_0 to z. Then

$$\int_{\alpha_1} f = \int_{\alpha_1} f + \int_{-\beta} f + \int_{\beta} f = \int_{\beta} f$$

since $\alpha - \beta$ is a closed contour in Ω_1 which is a Cauchy domain. Similarly,

$$\int_{\alpha_2} f = \int_{\beta} f.$$

Therefore, the function is well-defined. By the properties of Theorem 11 we see $g' = f$.
□

3.2 Method 1 for Cauchy's Theorem(s)

The above results can be used to show various domains are Cauchy domains. We will first show several kinds of domains Ω are Cauchy domains using the following approach:

Step 1 Consider a closed curve $\Gamma \subset \Omega$. Since Γ is compact and is contained in the open set Ω, it is possible to **find a scaling**, i.e., a domain W that is of the form

$$W = \{\lambda z + z_1\}$$

where $\lambda > 0$ and $z_1 \in \mathbb{C}$, i.e., a domain that is similar to Ω, for which

$$\Gamma \subset W \subset \overline{W} \subset \Omega.$$

Step 2 Consider **unique paths** connecting $z_0 \in \partial W \cap \Omega$ to $z \in W$.

Step 3 Given $f : \Omega \to \mathbb{C}$, use the approach of Theorem 11 to **define** $g : W \to \mathbb{C}$ by

$$g(z) = \int_\beta f$$

where β is **some unique contour** connecting z_0 to $z \in W$.

Step 4 **Use Goursat's theorem** to show (in all cases) condition **(ii)**

$$g(z+h) - g(z) = h \int_0^1 f(z+ht)\, dt$$

of Theorem 11 holds for $z \in W$ and $h \in \mathbb{C}$ with $|h|$ small enough.

Step 5 Conclude

$$\lim_{h \to 0} \frac{g(z+h) - g(z)}{h} = \lim_{h \to 0} \int_0^1 f(z_0 + ht)\, dt = f(z)$$

as in Theorem 11.

Step 6 **Use Corollary 3.3** of Stein and Shakarchi to conclude that since f has a primitive on W,

$$\int_\alpha f = 0$$

where α is a parameterization of Γ.

We now carry out these steps in detail for some simple domains.

3.3 Triangular domains

Perhaps the simplest, most natural, form in which to consider a triangular domain in a kind of standard position is

$$\mathcal{U} = \left\{ z \in \mathbb{C} : \cot\theta \operatorname{Im} z < \operatorname{Re} z < a - \left[\frac{a}{r} \csc\theta - \cot\theta \right] \operatorname{Im} z,\ 0 < \operatorname{Im} z < r \sin\theta \right\} \quad (8)$$

where θ is an angle satisfying $0 < \theta < \pi$ and a and r are positive (real) numbers; see Figure 1. Notice that the upper bound for $\operatorname{Re} z$ in (8) can also be written as

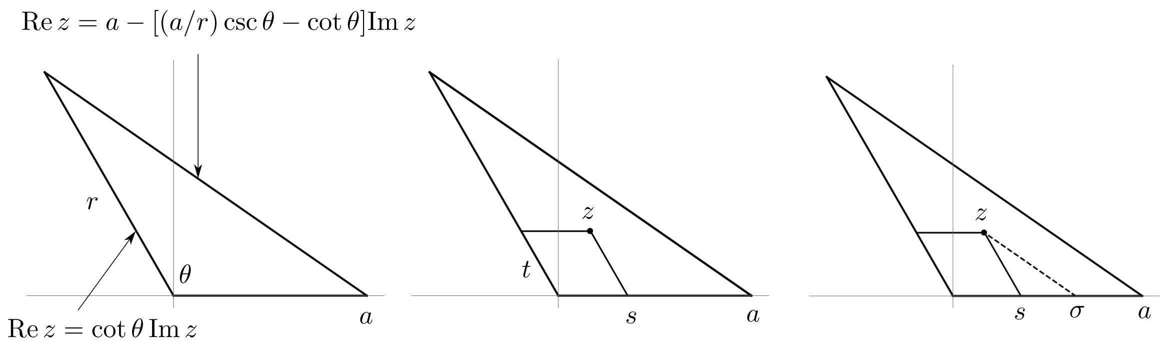

Figure 1: Trianglular domain in standard position with various coordinates.

$$a - \frac{1}{r}\csc\theta\,(a - r\cos\theta)\operatorname{Im} z.$$

It is useful to consider this triangular domain (carefully) in at least two alternative forms. The first

$$\mathcal{U} = \{s + te^{i\theta} : 0 < t < (1 - s/a)r,\ 0 < s < a\} \tag{9}$$

allows us to locate points within $\mathcal{U}$ along lines parallel to the primary side opposite $a \in \mathbb{R}$. Notice that for each $z \in \mathcal{U}$, writing

$$z = \operatorname{Re} z + i \operatorname{Im} z = s + t\cos\theta + it\sin\theta, \tag{10}$$

there are unique values $s = s_0(z)$ with $0 < s < a$ given by

$$s = \operatorname{Re} z - \cot\theta\ \operatorname{Im} z \tag{11}$$

and $t = t_0(z)$ with $0 < t < (1 - s/a)r$ given by

$$t = \frac{\operatorname{Im} z}{\sin\theta} = \csc\theta\ \operatorname{Im} z \tag{12}$$

for which (10) holds. Conversely, if $0 < s < a$ and $0 < t < (1-s/a)r$, then $z = s+te^{i\theta}$ determines a unique point in the set defined in (8) since on the one hand

$$0 < t\sin\theta = \operatorname{Im} z < r\sin\theta - (s/a)r\sin\theta < r\sin\theta,$$

and on the other hand, the inequality $0 < t < (1 - s/a)r$ implies $s < a - (a/r)t$, so

$$\begin{aligned}\cot\theta\ \operatorname{Im} z = t\cos\theta < s + t\cos\theta = \operatorname{Re} z \\ < a - (a/r)t + t\cos\theta = a - \frac{1}{r}\csc\theta(a - r\cos\theta)\operatorname{Im} z.\end{aligned}$$

An alternative to (9) is to locate points in $\mathcal{U}$ along lines parallel to the side of the triangular boundary opposite the origin:

$$\mathcal{U} = \{(1-\tau)\sigma + \tau\sigma(r/a)e^{i\theta} : 0 < \tau < 1,\ 0 < \sigma < a\} \tag{13}$$

Again, the relation

$$\sigma - \tau\sigma + \tau\sigma(r/a)\cos\theta + i\tau\sigma(r/a)\sin\theta = \operatorname{Re} z + i\operatorname{Im} z$$

determines unique values $\sigma = \sigma_0(z)$ and $\tau = \tau_0(z)$ given by

$$\sigma = \operatorname{Re} z + \left[\frac{a}{r}\csc\theta - \cot\theta\right]\operatorname{Im} z = \operatorname{Re} z + \frac{1}{r}\csc\theta(a - r\cos\theta)\operatorname{Im} z \tag{14}$$

and

$$\tau = \frac{a\operatorname{Im} z}{r\sin\theta\ \operatorname{Re} z + (a - r\cos\theta)\operatorname{Im} z} \tag{15}$$

satisfying the appropriate inequalities in (13). Conversely, a given point

$$z = (1-\tau)\sigma + \tau\sigma(r/a)e^{i\theta}$$

determined as in (13) satisfies

$$0 < \operatorname{Im} z = \tau\sigma(r/a)\sin\theta < r\sin\theta,$$

since $\tau\sigma/a < 1$, and

$$\begin{aligned}\cot\theta\operatorname{Im} z = \tau\sigma(r/a)\cos\theta < \operatorname{Re} z = \sigma - \frac{\tau\sigma}{a}(a - r\cos\theta) \\ < a - \frac{1}{r}\csc\theta(a - r\cos\theta)\operatorname{Im} z\end{aligned}$$

since $\operatorname{Im} z = \tau\sigma r\sin\theta/a$.

Finally, it may be of use to record the relations between points

$$s + te^{i\theta} = (1-\tau)\sigma + \tau\sigma(r/a)e^{i\theta} \in \mathcal{U}.$$

Namely,

$$s = \sigma(1-\tau) \qquad \text{and} \qquad t = \tau\sigma(r/a), \tag{16}$$

and

$$\sigma = s + \frac{a}{r}t = \frac{at+rs}{r} \qquad \text{and} \qquad \tau = \frac{at}{at+rs}. \tag{17}$$

As a consequence of our considerations so far, we have essentially completed **Step 2** in Method 1 to prove Cauchy's theorem, at least for triangle domains in standard position.

Lemma 2 *Given $z = s_0 + t_0 e^{i\theta} \in \mathcal{U}$ where $\mathcal{U}$ is a triangular domain in standard position given by (9), there is a unique path/contour $\alpha_0 + \alpha_1$ given by*

$$\begin{array}{rcll} \alpha_0(s) & = & s & \textit{for } 0 \le s \le s_0, \\ \alpha_1(t) & = & s_0 + te^{i\theta} & \textit{for } 0 \le t \le t_0 \end{array}$$

connecting 0 *to* z.

Exercise 5 *State and prove a corresponding result for*

$$z = (1-\tau_0)\sigma_0 + \tau_0\sigma_0(r/a)e^{i\theta} \in \mathcal{U}$$

where $\mathcal{U}$ is a triangular domain in standard position given by (13) giving a unique path connecting 0 *to z along a segment parallel to the side of $\partial\mathcal{U}$ opposite* 0.

We have skipped **Step 1**, but we are also now ready to return to it. If Γ is a closed curve in $\mathcal{U}$ given by (9), then consider

$$\mathcal{S} = \{(1-\epsilon_1)z : z \in \mathcal{U}\}$$

for $0 < \epsilon_1 < 1$ and

$$\mathcal{W} = \{(1-\epsilon_1)z + \epsilon_2 e^{i\theta/2} : z \in \mathcal{U}\}$$

for $0 < \epsilon_1 < 1$ and $\epsilon_2 > 0$ as indicated in Figure 2. The domain $\mathcal{S}$ is geometrically similar to $\mathcal{U}$. In fact the domain $\mathcal{S}$ is a triangular domain in standard position with sides determined by $(1-\epsilon_1)a \in \mathbb{R}$ and $(1-\epsilon_1)re^{i\theta}$. In particular, Lemma 2 applies to $\mathcal{S}$ giving unique paths. Clearly $\mathcal{W}$ is a translation of $\mathcal{S}$ and each point $z \in \mathcal{W}$ determines a unique path $\alpha_0 + \alpha_1$ by

$$\begin{array}{rcll} \alpha_0(s) & = & \epsilon_2 e^{i\theta/2} + s & \text{for } 0 \le s \le s_0, \\ \alpha_1(t) & = & \epsilon_2 e^{i\theta/2} + s_0 + te^{i\theta} & \text{for } 0 \le t \le t_0 \end{array} \tag{18}$$

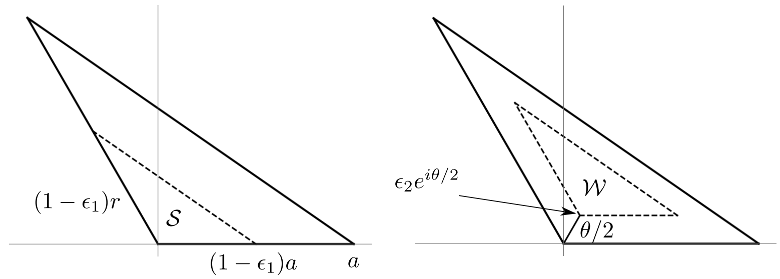

Figure 2: Trianglular subdomain $\mathcal{S}$ in standard position and translated triangular domain $\mathcal{W}$.

connecting $e^{i\theta/2}$ to z where, in this instance,

$$s_0 = \operatorname{Re}(z - \epsilon_2 e^{i\theta/2}) - \cot\theta \operatorname{Im}(z - \epsilon_2 e^{i\theta/2}) = \operatorname{Re} z - \cot\theta \operatorname{Im} z - \frac{\epsilon_2}{2}\sec(\theta/2)$$

and

$$t_0 = \csc\theta \operatorname{Im}(z - \epsilon_2 e^{i\theta/2}) = \csc\theta \operatorname{Im} z - \frac{\epsilon_2}{2}\sec(\theta/2).$$

Furthermore $\mathcal{S}$ is clearly a subset of $\mathcal{U}$. The translation $\mathcal{W}$, however, will only be a subset of $\mathcal{U}$ for ϵ_2 small enough; see Figure 3. Let's try to make this precise. Note first that each z in the sector

$$\{\rho e^{i\phi} : \rho > 0,\, 0 < \phi < \theta\}$$

determines unique real numbers $\tilde{a} = ca$ and $\tilde{r} = cr$ for which z is on the boundary of

$$\tilde{\mathcal{U}} = \{s + te^{i\theta} : 0 < t < (1 - s/\tilde{a})\tilde{r},\, 0 < s < \tilde{a}\}.$$

In fact, the relation (14) applies even for $z \notin \mathcal{U}$ to give

$$\tilde{a} = \sigma = \operatorname{Re} z + \frac{1}{r}\csc\theta(a - r\cos\theta)\operatorname{Im} z > 0. \tag{19}$$

The corresponding $\tilde{r}$ is given by similarity of the triangular boundaries of $\partial\mathcal{U}$ and $\partial\tilde{\mathcal{U}}$:

$$\tilde{r} = \frac{r}{a}\tilde{a} = \frac{r}{a}\operatorname{Re} z + \frac{1}{a}\csc\theta(a - r\cos\theta)\operatorname{Im} z > 0.$$

The expressions for $\tilde{a}$ and $\tilde{r}$ here are determined so that z is on the portion of $\partial\tilde{\mathcal{U}}$ opposite 0. If we take a typical point

$$z = s + te^{i\theta} + \epsilon_2 e^{i\theta/2} \in \mathcal{W}$$

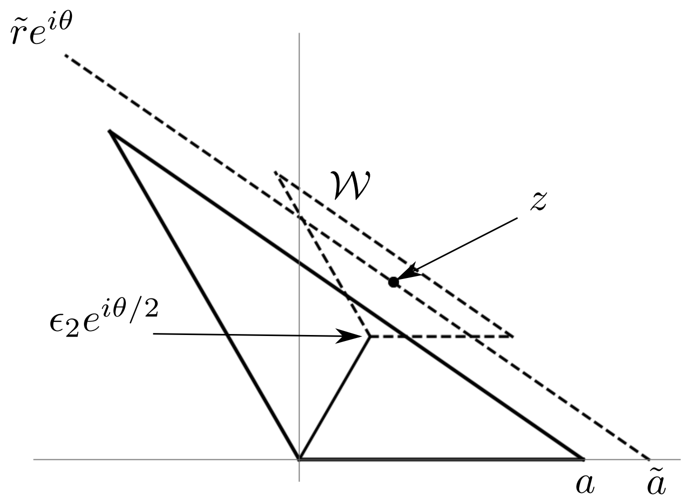

Figure 3: Translated triangular domain $\mathcal{W}$ and the triangular domain $\tilde{\mathcal{U}}$ determined by a point z in the sector determined by θ.

with $0 < s < (1-\epsilon_1)a$ and $0 < t < (1-\epsilon_1 - s/a)r$, then the corresponding $\tilde{a}$ determined by (19) is

$$\begin{aligned}
\tilde{a} &= s + t\cos\theta + \epsilon_2\cos(\theta/2) + \frac{1}{r\sin\theta}(a - r\cos\theta)[t\sin\theta + \epsilon_2\sin(\theta/2)] \\
&= s + t\cos\theta + \epsilon_2\cos(\theta/2) + \frac{t}{r}(a - r\cos\theta) + \frac{\epsilon_2}{2}(a - r\cos\theta)\frac{1}{r\cos(\theta/2)} \\
&= s + t\left(\frac{a}{r}\right) + \frac{\epsilon_2}{2}\left(2\cos(\theta/2) + \frac{a}{r\cos(\theta/2)} - \frac{\cos\theta}{\cos(\theta/2)}\right) \\
&= s + t\left(\frac{a}{r}\right) + \frac{\epsilon_2}{2}\left(2\cos(\theta/2) + \frac{a}{r\cos(\theta/2)} - \frac{2\cos^2(\theta/2) - 1}{\cos(\theta/2)}\right) \\
&= s + t\left(\frac{a}{r}\right) + \frac{\epsilon_2}{2}(a + r)\frac{1}{r\cos(\theta/2)} \qquad (20) \\
&< s + (1 - \epsilon_1 - s/a)a + \frac{\epsilon_2}{2}(1 + a/r)\sec(\theta/2) \\
&= (1-\epsilon_1)a + \frac{\epsilon_2}{2}(1 + a/r)\sec(\theta/2).
\end{aligned}$$

From this we see $\mathcal{W} \subset \mathcal{U}$ for

$$(1-\epsilon_1)a + \frac{\epsilon_2}{2}(1 + a/r)\sec(\theta/2) \le a,$$

that is

$$\epsilon_2 \leq 2\epsilon_1 \frac{a}{1+a/r} \cos(\theta/2) = \frac{2ar\cos(\theta/2)}{a+r}\, \epsilon_1. \tag{21}$$

The open set $\mathcal{W}$ is **compactly contained** in $\mathcal{U}$, which means

$$\overline{\mathcal{W}} \subset \mathcal{U}$$

and is denoted by $\mathcal{W} \subset\subset \mathcal{U}$, if strict inequality holds in (21).

It may be recalled, after this lengthy discussion of the translated domain $\mathcal{W}$ that we started our discussion of **Step 1** with mention of a closed curve Γ contained in $\mathcal{U}$. In order to finally relate $\mathcal{W}$ to Γ, we note first that since Γ is a compact subset of the open set $\mathcal{U}$ we have

$$\operatorname{dist}(\Gamma, \partial\mathcal{U}) = \inf\{|z-\zeta| : z \in \Gamma,\ \zeta \in \partial\mathcal{U}\} = \delta > 0$$

is a positive number. Generally, given a point $z \in \mathcal{U}$,

$$\operatorname{dist}(z, \partial\mathcal{U}) = \min\{\operatorname{dist}(z, L_0), \operatorname{dist}(z, L_1), \operatorname{dist}(z, L_2)\} \tag{22}$$

where L_0, L_1, and L_2 are the three lines containing the sides of $\partial\mathcal{U}$. Taking L_0 to be the side opposite 0 and L_1 the bottom side (on the real axis), we can rewrite (22) as

$$\operatorname{dist}(z, \partial\mathcal{U}) = \min\left\{ \frac{r\sin\theta}{a\sqrt{a^2+r^2-2ar\cos\theta}}\left[a^2 - a\operatorname{Re} z - \csc\theta(a - r\cos\theta)\operatorname{Im} z\right],\right.$$
$$\left. \operatorname{Im} z, \qquad \sin\theta \operatorname{Re} z - \cos\theta \operatorname{Im} z \right\}.$$

The expression for $\operatorname{dist}(z, L_0)$ is obtained by taking the value $\tilde{a}$ given in (19) associated with z and then using similar triangles. Note that the quantity

$$a^2 - a\operatorname{Re} z - \csc\theta(a - r\cos\theta)\operatorname{Im} z > 0$$

whenever $z \in \mathcal{U}$. The expression for $\operatorname{dist}(z, L_2)$ is obtained by taking the value s given in (11) associated with z and projecting onto the line perpendicular to L_2.

Now let us assume

$$\epsilon_2 < \frac{2ar\cos(\theta/2)}{a+r}\, \epsilon_1,$$

that is strict inequality holds in (21) so that $\mathcal{U} \subset\subset \mathcal{U}$, and also cosider the three lines M_0, M_1, and M_2 containing the three corresponding sides of $\partial\mathcal{W}$. If M_0 is the line containing the side opposite $\epsilon_2 e^{i\theta/2}$, then

$$\operatorname{dist}(M_0, L_0) = \frac{r\sin\theta}{\sqrt{a^2+r^2-2ar\cos\theta}}\left[\epsilon_1 a - \frac{\epsilon_2}{2}(1+a/r)\sec(\theta/2)\right]$$

and $\mathrm{dist}(M_1, L_1) = d(M_2, L_2) = \epsilon_2 \sin(\theta/2)$. Given a point $z \in \mathcal{U}$ with $\mathrm{dist}(z, L_j) > \mathrm{dist}(z, M_j)$ for $j = 0, 1, 2$ one must have $z \in \mathcal{W}$. In particular, if $z \in \Gamma$, then $\mathrm{dist}(z, L_j) \geq \delta > 0$ for $j = 0, 1, 2$, so if

$$\epsilon_1 < \frac{a\delta}{r \sin\theta} \sqrt{a^2 + r^2 - 2ar\cos\theta}$$

and

$$\epsilon_2 < \min\left\{ \frac{\delta}{\sin(\theta/2)}, \frac{2ar\cos(\theta/2)}{a+r}\ \epsilon_1 \right\}$$

we have $\Gamma \subset \mathcal{W} \subset \overline{\mathcal{W}} \subset \mathcal{U}$ as desired.

Exercise 6 *Show that given a curve $\Gamma \subset \mathcal{U}$, where $\mathcal{U}$ is simply some open subset of $\mathbb{C}$, it is not always possible to find a subdomain $\mathcal{W}$ geometrically similar to $\mathcal{U}$ with*

$$\Gamma \subset \mathcal{W} \subset \overline{\mathcal{W}} \subset \mathcal{U}. \tag{23}$$

Show that it is possible to obtain (23) when $\mathcal{U}$ is convex or star shaped, i.e., there is a point z_0 so that the segment

$$\{(1-t)z_0 + t\zeta : 0 \leq t < 1\} \subset \mathcal{U}$$

for every $\zeta \in \partial\mathcal{U}$.

We have completed **Step 1** and **Step 2**. Turning to **Step 3**, we take the path $\alpha = \alpha_0 + \alpha_1 : [0, s_0] \sqcup [0, t_0] \to \overline{\mathcal{W}}$ defined[4] in (18) above and derived from Lemma 2 and define $g : \overline{\mathcal{W}} \to \mathbb{C}$ by

$$g(z) = \int_\beta f.$$

The function g is well-defined because the path β connecting $\epsilon_2 e^{i\theta/2}$ to z is unique.

[4]The symbol "$\sqcup$" is used here to denote the **formally disjoint union** of two sets. Notice that the two intervals $[0, s_0]$ and $[0, t_0]$ are not disjoint, but a path is typically defined on a single interval $[a, b]$. There are various ways to deal with the concatenation. One possibility is to define $\beta(t) = \alpha_0(t)$ for $0 \leq t \leq s_0$ and $\beta(t) = \alpha_1(t - s_0)$ for $s_0 \leq t \leq s_0 + t_0$ so that $[a, b] = [0, s_0 + t_0]$. Here we have considered $[a, b] \approx [0, s_0] \sqcup [0, t_0]$ instead. Formally this means we consider the set of pairs $I_0 = \{(t, 0) : 0 \leq t \leq s_0\} \cup \{(t, 1) : 0 \leq t \leq t_0\}$. Making a topological identification of the points $(s_0, 0)$ and $(0, 1)$ in I_0 we obtain a set I which can be given a topology (and even a metric) so that it is isometric to an interval $[a, b]$. This is the somewhat cumbersome meaning of $[a, b] \approx [0, s_0] \sqcup [0, t_0]$.

We next consider **Step 4** which in some sense is the reason for the various steps of Method 1. We recall the form of the path β in this case:

$$\begin{array}{rcll} \alpha_0(s) & = & \epsilon_2 e^{i\theta/2} + s & \text{for } 0 \leq s \leq s_0, \\ \alpha_1(t) & = & \epsilon_2 e^{i\theta/2} + s_0 + te^{i\theta} & \text{for } 0 \leq t \leq t_0 \end{array}$$

where

$$s_0 = s_0(z) = \operatorname{Re} z - \cot\theta \operatorname{Im} z \qquad \text{and} \qquad t_0 = t_0(z) = \csc\theta \operatorname{Im} z.$$

Notice then that the unique path connecting $z_0 = \epsilon_2 e^{i\theta/2}$ to $z + h$ has

$$s_0(z+h) = \operatorname{Re} z - \cot\theta \operatorname{Im} z + \operatorname{Re} h - \cot\theta \operatorname{Im} h \qquad \text{and} \qquad t_0(z+h) = \csc\theta \operatorname{Im} z + \csc\theta \operatorname{Im} h.$$

This leads to four cases

(i) $\operatorname{Re} h - \cot\theta \operatorname{Im} h < 0$,

(ii) $\operatorname{Re} h - \cot\theta \operatorname{Im} h = 0$ and $\operatorname{Im} h < 0$,

(iii) $\operatorname{Re} h - \cot\theta \operatorname{Im} h = 0$ and $\operatorname{Im} h > 0$, and

(iv) $\operatorname{Re} h - \cot\theta \operatorname{Im} h < 0$

as indicated in Figure 4.

Let us consider a triangular domain:

Theorem 13 *(Cauchy's theorem on a triangular domain in standard position) The domain*

$$\mathcal{U} = \{s + te^{i\theta} : 0 < t < (1 - s/a)r,\ 0 < s < a\}$$

is a Cauchy domain.

Figure 4: Different cases for the path to the increment $z + h$.