Elementary Convergence Theorems For Complex Power Series

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Given a formal power series

$$\sum_{n=0}^{\infty} a_n z^n$$

determined by complex numbers/coefficients $a_0, a_1, a_2, a_3, \ldots$, the basic theorem about power series asserts that the series converges and determines a complex differentiable function $f : D_R(0) \to \mathbb{C}$ on a disk $D_R(0)$ determined by the **Hadamard radius**

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}.$$

Moreover the series **diverges** for $|z| > R$. Daniel Savio asked a question in class concerning exactly what it means that the series diverges here. In particular, Daniel asked if it means

$$\sum_{n=0}^{\infty} a_n z^n = \infty \tag{1}$$

when $|z| > R$? This question does make sense, but I'm pretty sure the answer is "no." That is also the answer I gave in class, but I also asserted that this is "never" the case...or at least I implied such an assertion. I am not sure that is correct either, but let's try to consider the question a bit more carefully.

First of all let's recall/clarify exactly the meaning of the divergence suggested by (1). What this means is that for any $M > 0$, there is some $N \in \mathbb{N}$ for which $k > N$ implies

$$\left| \sum_{n=0}^{k} a_n z^n \right| > M.$$

A good place to start with a consideration like this is with the geometric series

$$f(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{on } D_1(0).$$

In fact, the Hadamard radius in this case is clearly $R = 1$, so the question becomes what happens with the partial sums

$$S_k = \sum_{n=0}^{k} z^n$$

when $|z| > 1$? Using the polar form we can write

$$S_k = \sum_{n=0}^{k} |z|^n e^{in\theta}$$

where θ is the argument of z. It seems pretty clear that if S_k is going to converge to ∞ (in the Riemann sphere) for $|z| > 1$, then the top order term is going to have to dominate all the other terms, so (perhaps) we should write

$$S_k = |z|^k e^{in\theta} + \sum_{n=0}^{k-1} |z|^n e^{in\theta}.$$

Thus,

$$|S_k| \geq |z|^k - \sum_{n=0}^{k-1} |z|^n = |z|^k \left(1 - \sum_{n=0}^{k-1} \frac{1}{|z|^{k-n}}\right).$$

Notice the sum

$$\sum_{n=0}^{k-1} \frac{1}{|z|^{k-n}} = \sum_{n=0}^{k-1} \left(\frac{1}{|z|}\right)^{k-n}$$

appearing here is geometric with

$$\sum_{n=0}^{k-1} \left(\frac{1}{|z|}\right)^{k-n} = \sum_{m=1}^{k} \left(\frac{1}{|z|}\right)^{m} = \frac{1-(1/|z|)^{k+1}}{1-1/|z|} - 1.$$

Thus we have shown

$$|S_k| \geq |z|^k \left(2 - \frac{1-(1/|z|)^{k+1}}{1-1/|z|}\right) \geq |z|^k \left(2 - \frac{1}{1-1/|z|}\right).$$

Now, looking at this, we note first of all that if $|z| > 2$, then $1/|z| < 1/2$ so that $1 - 1/|z| > 1/2$ and

$$\frac{1}{1 - 1/|z|} < 2 \qquad \text{fixed.}$$

This means that, given any $\epsilon > 0$, taking k large enough we can conclude $1 - 1/|z|^{k+1} < 1 + \epsilon$ and consequently for some fixed $\epsilon > 0$

$$\frac{1 + \epsilon}{1 - 1/|z|} < 2 \qquad \text{fixed,}$$

and

$$|S_k| \geq |z|^k \left(2 - \frac{1 + \epsilon}{1 - 1/|z|}\right) \qquad \text{for all } k \text{ sufficiently large.}$$

In particular, this implies the divergent series

$$\sum_{n=0}^{\infty} z^n$$

does indeed diverge to ∞ in the Riemann sphere when $|z| > 2$. We have certainly not shown the same holds for $1 < |z| \leq 2$, but this is very suggestive that that may be the case. That is to say, from what we have above (if my estimates are correct) then Daniel's suggestion may very well hold at least in some cases and my assertion may very well be wrong.

I think I dont want to work harder on this question at the moment, which as far as I can see for me would take the immediate form of two or three obvious things:

1. Examine more closely what happens with the geometric series in the annulus $1 < |z| \leq 2$.

2. Assuming the geometric series satisfies Daniel's suggestion, attempt to find other (counter) examples by adjusting the coefficients a_j in $\sum a_j z^j$ with $|a_j| = 1$ so that cancellation occurs for some values of z with $|z| > 1$ leading to "bounded divergence" or "oscillatory divergence" of the associated partial sums.

3. In the direction of trying to prove what we'll call Savio's conjecture for the moment, assume one has a divergent series

$$\sum_{n=0}^{\infty} a_n z^n$$

with $|z| > R$ but for which the partial sums

$$S_k = \sum_{n=0}^{k} a_n z^n$$

have a subsequence which remains bounded (and then try to get a contradiction of the **basic necessary condition** for the convergence of a series considered below).

Instead, what I think I want to do is demonstrate some of the more elementary properties and results about complex series and power series. Before we do that, let me record here precisely Daniel's suggestion as I understand it and as it stands:

Conjecture 1 (Savio's Conjecture) *If $R \in (0, \infty)$ is the Hadamard radius for a formal complex power series*

$$\sum_{n=0}^{\infty} a_n z^n,$$

then

$$\sum_{n=0}^{\infty} a_n z^n = \infty \quad \textit{in the Riemann sphere when } |z| > R,$$

in the sense that for any $M > 0$, there is some $N > 0$ such that $k > n$ implies

$$\left| \sum_{n=0}^{k} a_n z^n \right| > M.$$

1 Basic Necessary Condition

Theorem 1 *If the formal series $\sum a_n$ determined by complex numbers $a_0, a_1, a_2, a_3, \ldots$ converges to a complex number $w \in \mathbb{C}$, then*

$$\lim_{n \to \infty} a_n = 0.$$

Proof: Let $\epsilon > 0$. We know that there is some $N > 0$ such that $k > N$ implies

$$\left| \sum_{n=0}^{k} a_n - w \right| < \frac{\epsilon}{2}.$$

This means that for $k > N+1$ we have also

$$\begin{aligned} |a_k| &= \left| \sum_{n=0}^{k} a_n - w - \sum_{n=0}^{k-1} a_n + w \right| \\ &\le \left| \sum_{n=0}^{k} a_n - w \right| + \left| - \sum_{n=0}^{k-1} a_n + w \right| \\ &< \epsilon. \qquad \square \end{aligned}$$

An essentially equivalent proof is the following:

$$a_n = S_n - S_{n-1}$$

where

$$S_k = \sum_{n=0}^{k} a_n$$

is the n-th partial sum of the series. Thus, if the series converges

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = w - w = 0. \qquad \square$$

2 Absolute Convergence

Recall that a complex series $\sum a_n$ determined by the complex numbers $a_0, a_1, a_2, a_3, \ldots$ is **absolutely convergent** if the real series

$$\sum_{n=0}^{\infty} |a_n| \qquad \text{converges to a real number,} \tag{2}$$

i.e., the sequence of partials sums

$$M_k = \sum_{n=0}^{k} |a_n|$$

of the series of nonnegative terms in (2) is bounded above by some real number.

We can also consider the following condition: The formal series $\sum a_n$ is **absolutely Cauchy** if given any $\epsilon > 0$, there is some N for which $N < k < \ell$ implies

$$\sum_{n=k+1}^{\ell} |a_n| < \epsilon. \tag{3}$$

Theorem 2 *If $\sum a_n$ is absolutely Cauchy, then the series is absolutely convergent.*

Proof: In order to show a series is absolutely convergent, it is enough to show the sequence of partial sums

$$M_k = \sum_{n=0}^{k} |a_n|$$

is a Cauchy sequence (since $\mathbb{R}$ is metrically complete). To this end, let $\epsilon > 0$ and take N as in the definition above so that (3) holds. Then

$$|M_\ell - M_k| = M_\ell - M_k = \sum_{n=k+1}^{\ell} |a_n| < \epsilon. \qquad \square$$

Theorem 3 *If $\sum a_n$ is absolutely convergent, then the series is convergent in $\mathbb{C}$.*

Proof: Since $\mathbb{C}$ is also complete, it is enough to to show the sequence of partial sums

$$S_k = \sum_{n=0}^{k} a_n$$

is Cauchy in $\mathbb{C}$. Let $\epsilon > 0$. Since every convergent sequence is Cauchy and we know the sequence determined by

$$M_k = \sum_{n=0}^{k} |a_n|$$

is convergent, we can take N so that $N < k < \ell$ implies $|M_\ell - M_k| < \epsilon$. Therefore, if $N < k < \ell$ we have

$$|S_\ell - S_k| = \left| \sum_{n=k+1}^{\ell} a_n \right| \le \sum_{n=k+1}^{\ell} |a_n| = M_\ell - M_k < \epsilon. \qquad \square$$