Decay Comparison

John McCuan

April 23, 2022

In studying the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{2\pi i x\xi}\, dx$$

two important classes of decaying functions are often considered. The first consists of function $f \in C^0(\mathbb{R} \to \mathbb{C})$ for which there is a constant $A > 0$ with

$$|f(x)| \le \frac{A}{1+x^2} \qquad \text{for} \qquad x \in \mathbb{R}. \tag{1}$$

These are functions for which the Fourier transform is well defined as a uniformly continuous function $\hat{f} : \mathbb{R} \to \mathbb{C}$. Under certain additional conditions on f, it can also be shown that the Fourier transform $\hat{f}$ satisfies a decay condition: There is some $B > 0$

$$|\hat{f}(\xi)| \le B\, e^{-2\pi y|\xi|} \qquad \text{for} \qquad \xi \in \mathbb{R} \tag{2}$$

for y in some interval $0 < y < b$. Thus, these exponentially decaying functions constitute a second class of decaying functions which are of interest.

These two classes of decaying functions interact in the following way: Given a function $f \in C^0(\mathbb{R} \to \mathbb{C})$ satisfying the decay condition (1), we obtain a function $\hat{f} \in C^0(\mathbb{R} \to \mathbb{C})$ satisfying the decay condition (2). To this function $\hat{f}$ we would like to apply the **Fourier inversion formula**

$$g(x) = \int_{-\infty}^{\infty} \hat{f}(\xi)\, e^{2\pi i \xi x}\, d\xi.$$

In order for this formula to be well-defined and define a function $g \in C^0(\mathbb{R} \to \mathbb{C})$, which presumably under appropriate conditions should be the function f, we need $\hat{f}$

to satisfy a decay condition

$$|\hat{f}(\xi)| \leq \frac{C}{1+\xi^2} \qquad \text{for} \qquad \xi \in \mathbb{R}$$

where $C > 0$ is some constant. Since the exponential decays faster than the reciprocal of the quadratic, it may be observed that for large $|\xi|$ in particular we should have

$$e^{-2\pi y|\xi|} < \frac{1}{1+\xi^2}.$$

In fact for $0 < y < b$

$$\lim_{|\xi|\to\infty} \frac{e^{2\pi y|\xi|}}{1+\xi^2} = \lim_{|\xi|\to\infty} \frac{2\pi y e^{2\pi y|\xi|}}{2\xi} = \lim_{|\xi|\to\infty} 2\pi^2 y^2 e^{2\pi y|\xi|} = +\infty.$$

Thus, we may define $R : (0,\infty) \to [0,\infty)$ by

$$R(y) = \min\{t \in [0,\infty) : \phi(\xi, y) \leq \psi(\xi) \text{ for } \xi \geq t\}$$

where

$$\phi(\xi) = \phi(\xi, y) = e^{-2\pi y|\xi|} \qquad \text{and} \qquad \psi(\xi) = \frac{1}{1+\xi^2}.$$

It will be observed that the two functions ϕ and ψ are even, so only values corresponding to $\xi \geq 0$ need be considered, and we may also assume ϕ is differentiable at $\xi = 0$ with derivative

$$\phi'(0) = \phi'(0^+) = -2\pi y < 0.$$

Furthermore since,

$$\psi'(\xi) = -\frac{2\xi}{(1+\xi^2)^2} \leq 0$$

with equality only for $\xi = 0$, there is some $t > 0$ for which $\phi(\xi) < \psi(\xi)$ for $0 < \xi < t$, and we may define $r : (0,\infty) \to (0,\infty]$ by

$$r(y) = \sup\{t \in [0,\infty) : \phi(\xi, y) < \psi(\xi) \text{ for } 0 < \xi < t\} > 0.$$

There are several obvious questions to ask about the nature of the functions $r = r(y)$ and $R = R(y)$ as well as the relation between the functions ϕ and ψ. Two main assertions are the following:

Theorem 1 *For $y > 0$ large enough, $\phi(\xi, y) < \psi(\xi)$ for $\xi > 0$. Consequently, $R(y) = 0$ and $r(y) = +\infty$.*

Theorem 2 *For $y > 0$ small enough there exist points $\xi > 0$ with $\phi(\xi, y) > \psi(\xi)$. Consequently, $0 < r(y) < R(y)$.*

To see the first assertion, note that for $\xi \geq 0$

$$\phi(\xi) = \frac{1}{\sum_{n=0}^{\infty} \frac{(2\pi y\xi)^n}{n!}} < \frac{1}{1 + \frac{(2\pi y\xi)^2}{2}} \leq \frac{1}{1+\xi^2} = \psi(\xi) \quad \text{if} \quad y \geq \frac{1}{2\pi}.$$

For the second assertion, note that ψ is independent of y while $\phi = \phi(x, y)$ is decreasing in y with

$$\frac{\partial \phi}{\partial y} = -2\pi|\xi|e^{-2\pi y|\xi|} \tag{3}$$

with $\phi(\xi, y)$ converging uniformly to 1 on sets $K \subset\subset [0, \infty)$ as $y \searrow 0$. Specifically, we can say that for any $\epsilon > 0$ and any $t > 0$, there is some $\delta > 0$ such that $0 < y < \delta$ implies

$$0 < 1 - \phi(\xi, y) < \epsilon \qquad \text{for} \qquad 0 < \xi < t.$$

Since $\psi(\xi) < 1$ for every fixed $\xi > 0$, there we will clearly have

$$\phi(\xi, y) > 1 - \epsilon > \psi(\xi) \qquad \text{for } \epsilon > 0 \text{ small enough and } \delta = \delta(\epsilon) > 0 \text{ small enough.}$$

Having established Theorem 1 and Theorem 2, we can conclude that the set

$$U = U(y) = \{\xi \in (0, \infty) : \phi(\xi, y) > \psi(\xi)\}$$

satisfies for some $y_* > 0$ and some $y_{**} \geq y_*$

(i) $U(y) = \phi$ for $y > y_{**}$, and

(ii) $U(y) \neq \phi$ for $0 < y < y_*$.

Furthermore, it follows from (3) that

$$U(y_2) \subset U(y_1) \qquad \text{for} \qquad 0 < y_1 \leq y_2. \tag{4}$$

We conclude that for some unique[1]

$$y_0 = \min\{y \in (0, \infty) : U(y) = \phi\} \doteq 0.12808 > 0$$

the following hold:

[1]The transcendental equation leading to the numerical approximation of this value will be addressed below. See Lemma 1 and its proof.

(i) $U(y) = \phi$ for $y \geq y_0$,

(ii) $U(y) \neq \phi$ for $y < y_0$,

(iii) The set inclusion in (4) is strict unless $y_1 \geq y_0$, and

(iv) When $y = y_0$ we have $U(y_0) = 0$ so that

$$\phi(\xi, y_0) \leq \psi(\xi) \qquad \text{for} \qquad \xi > 0,$$

but there exists at least one point $\xi = \xi_0 > 0$ for which

$$\phi(\xi_0, y_0) = \psi(\xi_0).$$

At any such point $\xi_0 > 0$ there holds

$$\phi'(\xi_0) = \frac{\partial \phi}{\partial \xi}(\xi_0, y_0) = \psi'(\xi_0).$$

Lemma 1 *There is precisely one positive* $\xi_0 = \xi_0(y_0) \doteq 1.98029 > 0$ *satisfying the conditions described above with*

$$\begin{aligned} \phi(\xi_0) &= \psi(\xi_0) \text{ and} \\ \phi'(\xi_0) &= \psi'(\xi_0). \end{aligned}$$

Proof: The prescribed system of equations, which we know corresponds to at least one pair $(\xi_0, y_0) \in (0, \infty) \times (0, \infty)$ requires

$$e^{-2\pi y_0 \xi_0} = \frac{1}{1+\xi_0^2} \qquad \text{and} \qquad -2\pi y_0 \, e^{-2\pi y_0 \xi_0} = -\frac{2\xi_0}{(1+\xi_0^2)^2}. \tag{5}$$

Substituting the value of the exponential from the first equation into the second equation, we obtain a relation

$$-2\pi y_0 \, \frac{1}{1+\xi_0^2} = -\frac{2\xi_0}{(1+\xi_0^2)^2} \qquad \text{or} \qquad \pi y_0 \, \xi_0^2 - \xi_0 + \pi y_0 = 0. \tag{6}$$

Thus, we have an equation which is quadratic in ξ_0 and may be solved in the form

$$\xi_0 = \frac{1 \pm \sqrt{1 - 4\pi^2 y_0^2}}{2\pi y_0}.$$

Returning to the first equation in (5) we compute

$$\xi_0^2 = \frac{1 - 2\pi^2 y_0^2 \pm \sqrt{1 - 4\pi^2 y_0^2}}{2\pi^2 y_0^2} \qquad \text{and} \qquad 1 + \xi_0^2 = \frac{1 \pm \sqrt{1 - 4\pi^2 y_0^2}}{2\pi^2 y_0^2}$$

so that the first equation may be written as

$$e^{-1 \mp \sqrt{1-4\pi^2 y_0^2}} = \frac{2\pi^2 y_0^2}{1 \pm \sqrt{1 - 4\pi^2 y_0^2}}. \tag{7}$$

Thus, ξ_0 is eliminated from this equation. To simplify notation, let us write

$$\alpha = \sqrt{1 - 4\pi^2 y_0^2}.$$

Then

$$2\pi^2 y_0^2 = \frac{1 - \alpha^2}{2}$$

and (7) becomes

$$e^{-1 \mp \alpha} = \frac{1 - \alpha^2}{2} \, \frac{1}{1 \pm \alpha}. \tag{8}$$

Notice that the choice of sign is coordinated, so this becomes two equations

$$e^{-1-\alpha} = \frac{1 - \alpha}{2} \qquad \text{and} \qquad e^{-1+\alpha} = \frac{1 + \alpha}{2}. \tag{9}$$

The first of these (transcendental) equations,

$$\frac{1}{e} \, e^{-\alpha} = -\frac{1}{2}(\alpha - 1)$$

is seen to have a unique positive root at a value $\alpha_0 \doteq 0.5936$. This corresponds to the unique values

$$y_0 = \frac{1}{2\pi}\sqrt{1 - \alpha_0^2} \doteq 0.12808 \qquad \text{and} \qquad \xi_0 = \frac{1 + \sqrt{1 - 4\pi^2 y_0^2}}{2\pi y_0} \doteq 1.98029$$

posited by the lemma. The second equation in (9) has the unique solution $\alpha = 1$ corresponding nominally to $y = 0$ and $\xi = 0$. This may be viewed as the degenerate case in which $\phi = \phi(\xi, 0) \equiv 1 \geq \psi(\xi)$ for all ξ, but indeed $\phi(0) = 1 = \psi(0)$ and $\phi'(0) = 0 = \psi'(0)$. At any rate, this does not lead to positive values for ξ_0 and y_0 as shown to exist based on our analysis of the sets $U(y)$ for $y > 0$.

Since we have characterized all possible values of $\xi_0 > 0$ and $y_0 > 0$ and found precisely one we have established the assertion of Lemma 1. $\square$

We have shown that for $y \geq y_0$ where

$$y_0 = \frac{1}{2\pi}\sqrt{1-\alpha_0^2} \doteq 0.12808$$

and $\alpha_0 \doteq 0.5936$ is the unique positive solution of

$$\frac{1}{e}\, e^{-\alpha_0} = -\frac{1}{2}(\alpha_0 - 1)$$

there holds

$$e^{-2\pi y|\xi|} \leq \frac{1}{a+\xi^2}. \tag{10}$$

Furthermore, we know equality holds in (10) for $\xi = 0$, and we have shown that aside from the equality at $\xi = 0$ the inequality is always strict unless $y = y_0$ and

$$\xi_0 = \frac{1+\sqrt{1-4\pi^2 y_0^2}}{2\pi y_0} \doteq 1.98029.$$

We recall a result from our consideration of the Fourier transform (or Stein and Shakarchi's Theorem 2.1 of Chapter 4):

Lemma 2 *(Theorem 2.1 in Stein and Shakarchi) For $f : \Omega \to \mathbb{C}$ holomorphic on the strip*

$$\Omega = \{x + iy \in \mathbb{C} : x \in \mathbb{R} \text{ and } 0 < y < b\}$$

and satisfying for some $A > 0$ the uniform decay estimate

$$|f(x+iy)| \leq \frac{A}{1+x^2} \qquad \text{for} \qquad x, y \in \mathbb{R} \quad \text{with} \quad x+iy \in \Omega, \tag{11}$$

we havefor each fixed y with $0 \leq y < b$

$$|\hat{f}(\xi)| \leq \pi A e^{-2\pi y|\xi|} \qquad \text{for} \qquad \xi \in \mathbb{R}. \tag{12}$$

We may now state a corollary of this result using (10):

Corollary 3 *For $f : \Omega \to \mathbb{C}$ holomorphic on the strip*

$$\Omega = \{x + iy \in \mathbb{C} : x \in \mathbb{R} \text{ and } 0 < y < b\}$$

for some $b \geq y_0$ and satisfying for some $A > 0$ the uniform decay estimate

$$|f(x + iy)| \leq \frac{A}{1 + x^2} \qquad \text{for} \qquad x, y \in \mathbb{R} \quad \text{with} \quad x + iy \in \Omega,$$

we have

$$|\hat{f}(\xi)| \leq \frac{\pi A}{1 + \xi^2} \qquad \text{for} \qquad \xi \in \mathbb{R}.$$

It remains to address the situation when $0 < y < y_0$. In this case, the set

$$U(y) = \{\xi \in (0, \infty) : \phi(\xi) > \psi(\xi)\}$$

is nonempty but satisfies $U = U(y) \subset\subset (0, \infty)$. We recall, however, that $2\pi y_0 < 1$. This means that for $0 < y < y_0$ we have

$$\phi(\xi) = \frac{1}{\sum_{n=0}^{\infty} \frac{(2\pi y \xi)^n}{n!}} < \frac{1}{1 + \frac{(2\pi y \xi)^2}{2}} \leq \frac{1}{2\pi y + 2\pi y \xi^2} = \frac{1}{2\pi y}\ \psi(\xi).$$

Therefore, for y fixed with $0 < y < y_0$, the set

$$\left\{\beta > 0 : e^{-2\pi y|\xi|} \leq \frac{\beta}{1 + \xi^2} \text{ for } \xi \in \mathbb{R}\right\}$$

is nonempty. Also, note that (trivially)

$$\frac{\partial}{\partial \beta} \frac{\beta}{1 + \xi^2} = \frac{1}{1 + \xi^2} > 0.$$

We conclude there is a unique function $B : (0, y_0) \to (1, \infty)$ given by

$$B(y) = \min \left\{\beta > 1 : e^{-2\pi y|\xi|} \leq \frac{\beta}{1 + \xi^2} \text{ for } \xi \in \mathbb{R}\right\}$$

giving the least value $B = B(y)$ for which

$$e^{-2\pi y|\xi|} \leq \frac{B}{1 + \xi^2} \qquad \text{for} \qquad \xi \in \mathbb{R}. \tag{13}$$

In order to show equality holds in (13) with $B = B(y)$ for precisely one value $\xi = \eta(y) > 0$ we prove a kind of second version of Lemma 1. It is striking that we also obtain completely explicit expressions for $B(y)$ and $\eta(y)$.

Lemma 4 *For* $0 < y < y_0$*, there is a unique* $B = B(y) > 1$ *and a unique* $\xi = \eta(y) > 0$ *giving a solution of the (transcendental) system*

$$\phi(\xi) = \phi(y,\xi) = e^{-2\pi y\xi} = \frac{B}{1+\xi^2} = B\ \psi(\xi) \tag{14}$$

$$\phi'(\xi) = \phi_\xi(y,\xi) = -2\pi y e^{-2\pi y\xi} = -\frac{2B\xi}{(1+\xi^2)^2} = B\ \psi'(\xi). \tag{15}$$

In fact,

$$\eta = \eta(y) = \frac{1}{\pi y}\sqrt{\frac{1 - 2\pi^2 y^2 + \sqrt{1-4\pi^2 y^2}}{2}}, \tag{16}$$

and

$$\begin{aligned} B(y) &= \min\left\{\beta > 1 : e^{-2\pi y|\xi|} \le \frac{\beta}{1+\xi^2} \text{ for } \xi \in \mathbb{R}\right\} \\ &= (1+\eta^2)e^{-2\pi y\eta} \\ &= \frac{1+\sqrt{1-4\pi^2y^2}}{2\pi^2y^2}\ e^{-1-\sqrt{1-4\pi^2y^2}}. \end{aligned} \tag{17}$$

Proof: We know the system (14-15) holds for at least one value $\xi = \eta$ for

$$B = B(y) = \min\left\{\beta > 1 : e^{-2\pi y|\xi|} \le \frac{\beta}{1+\xi^2} \text{ for } \xi \in \mathbb{R}\right\}.$$

Substituting as in the proof of Lemma 1, we have

$$-2\pi y \frac{B}{1+\eta^2} = -\frac{2B\eta}{(1+\eta^2)^2}$$

or

$$\pi y = \frac{\eta}{1+\eta^2}.$$

This equation is very similar to the equation in (6) except that y is now a given value instead of an unknown. Notice also that B, and our particular choice of B, does not play an essential role; we only need the existence of some B that corresponds to a solution. As in (6) we get a quadratic equation

$$piy\eta^2 - \eta + \pi y = 0$$

with solution(s)

$$\eta = \frac{1 \pm \sqrt{1 - 4\pi^2 y^2}}{2\pi y}.$$

Recognizing that at least one of these numbers (and possibly both) should yield a solution, we compute

$$\eta^2 = \frac{1 - 2\pi^2 y^2 \pm \sqrt{1 - 4\pi^2 y^2}}{2\pi^2 y^2}$$

and

$$1 + \eta^2 = \frac{1 \pm \sqrt{1 - 4\pi^2 y^2}}{2\pi^2 y^2} = 2\frac{1 \pm \alpha}{1 - \alpha^2}$$

where $\alpha = \sqrt{1 - 4\pi^2 y^2}$. As before $2\pi^2 y^2 = (1 - \alpha^2)/2$ and equation (14) becomes

$$e^{-1 \mp \alpha} = \frac{B}{2(1 \pm \alpha)}(1 - \alpha^2).$$

As in the proof of Lemma 1 we consider each choice of coordinated sign separately: Choosing the top sign gives

$$\frac{1}{e}\ e^{-\alpha} = \frac{B}{2}(1 - \alpha). \tag{18}$$

The function $g(\alpha) = e^{-\alpha}/e$ on the left is decreasing and convex for $\alpha > 0$ with

$$g(0) = \frac{1}{e} < \frac{1}{2} \qquad \text{and} \qquad \lim_{\alpha \nearrow \infty} g(\alpha) = 0.$$

The function $h(\alpha) = b(1 - \alpha)/2$ is decreasing and affine with

$$h(0) = \frac{B}{2} > \frac{1}{2}.$$

Therefore, there is a unique $\alpha > 0$ determined by (18), and because $g(\alpha) > 0$, it must be the case that $0 < \alpha < 1$. This value corresponds to the unique solution giving (16) in the statement of the lemma.

The alternative (bottom) choice of sign gives

$$\frac{1}{e}\ e^{\alpha} = \frac{B}{2}(1 + \alpha). \tag{19}$$

The function $g(\alpha) = e^{\alpha}/e$ on the left is increasing and convex for $\alpha > 0$ with

$$g(0) = \frac{1}{e} < \frac{1}{2}$$

while $h(\alpha) = b(1+\alpha)/2$ is increasing and affine with

$$h(0) = \frac{B}{2} > \frac{1}{2}.$$

Again, there is a unique solution $\alpha > 0$ of (19). In this case, however, $g(1) = 1 < B = h(1)$. Therefore, $\alpha > 1$, and this value is not

$$\alpha = \sqrt{1 - 4\pi^2 y^2} < 1.$$

The solution here is essentially extraneous. We have established the existence and uniqueness, and the value of $B = B(y)$ given in the statement of the lemma can be computed from (14). □

There are a number of additional aspects of the comparison between

$$\phi(\xi, y) = e^{-2\pi y|\xi|} \qquad \text{and} \qquad \psi(\xi) = \frac{1}{1+\xi^2}$$

which would be interesting to explore. The regularity and the monotonicity of the functions R and r appearing in Theorems 1 and 2 would be nice to understand. Can these functions be expressed in terms of solutions of transcendental equations (or explicitly)? It would also be nice to know the sets

$$U = U(y) = \{\xi \in (0, \infty) : \phi(\xi, y) > \psi(\xi)\}$$

with the nesting property established in (4) are intervals (when they are nonempty). I guess I will leave these considerations to someone else; I may have already found out more about this topic that anyone wants to know.