The Argument Principle and Rouche's Theorem

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Theorem 1 *(The argument principle[1]) Let $f : \Omega \to \mathbb{C}$ be meromorphic and assume $\alpha : [a, b] \to \Omega$ parameterizes a simple loop Γ homotopic to a point in Ω with*

$$(\{z \in \Omega : f(z) = 0\} \cup \{\zeta \in \Omega : f(\zeta) = \infty\}) \cap \Gamma = \phi. \tag{1}$$

If

(i) *$z_1, z_2, \dots, z_k$ are the distinct zeros of f circumnavigated by α and $\zeta_1, \zeta_2, \dots, \zeta_\ell$ are the distinct poles of f circumnavigated by α with*

(ii) *n_j is the order of the zero z_j for $j = 1, 2, \dots, k$ and m_j is the order of the pole ζ_j for $j = 1, 2, \dots, \ell$,*

then

$$\frac{1}{2\pi i}\int_\alpha \frac{f'}{f} = \sum_{j=1}^{k} n_j - \sum_{j=1}^{\ell} m_j. \tag{2}$$

The condition (1) says simply that there are no zeros nor poles of f on the loop Γ itself. Note that the quantity appearing in (2) may also be viewed as the **number of zeros of f circumnavigated by α (counted with multiplicities) minus the number of poles of f circumnavigated by α (counted with multiplicities)**. In the particular case when f is holomorphic this quantity is simply the number of zeros.

I won't review the proof of the argument principle, but recall simply that the "integer nature" of the value of the integral on the left in (2) arises from what one gets by calculating the logarithmic derivative f'/f has when $f(z)$ has the (local) form

$$(z - z_j)^{n_j}\, g \qquad \text{or} \qquad \frac{g}{(z-\zeta_j)^{m_j}}$$

[1]Theorem 18 of §5.2 in Ahlfors.

where g is a non-vanishing holomorphic function, along with the knowledge that

$$\int_{\zeta=\beta} \frac{1}{\zeta - z} = 2\pi i$$

for a small circle β around any point z.

Theorem 2 *(Rouche's Theorem[2]) Let $f, g : \Omega \to \mathbb{C}$ be holomorphic functions and assume α parameterizes a simple loop Γ homotopic to a point in Ω with*

$$\{z \in \Omega : f(z) = 0\} \cap \Gamma = \phi. \tag{3}$$

If

$$\left| \frac{g \circ \alpha}{f \circ \alpha} - 1 \right| < 1 \tag{4}$$

then f and g have the same number of zeros (counted with multiplicities) circumnavigated by α, in other words

$$\int_\alpha \frac{g'}{g} = \int_\alpha \frac{f'}{f}.$$

The condition (3) is that there are no zeros of f on the loop Γ. It may be noted that the condition (4) implies immediately that there are no zeros of the function g on the loop Γ either. In view of the Cauchy integral formula, we know the values of $f(z)$ and of $g(z)$ for z inside the bounded domain $W \subset \Omega$ with $\partial W = \Gamma$ are given by integrals

$$f(z) = \frac{1}{2\pi i} \int_{\zeta=\alpha} \frac{f(\zeta)}{\zeta - z} \qquad \text{and} \qquad g(z) = \frac{1}{2\pi i} \int_{\zeta=\alpha} \frac{g(\zeta)}{\zeta - z}.$$

Obviously, if f and g agree identically on Γ, then they have precisely the same zeros within W with precisely the same orders. It may be viewed as a perterbation result that the zeros of f and g in W counted with multiplicities still match if a condition

$$\left| \frac{g \circ \alpha}{f \circ \alpha} - 1 \right| < \epsilon$$

is imposed with ϵ small enough. The fact that ϵ may be taken as large as 1 is the striking assertion of Rouche's theorem.

[2]cf Theorem 4.3 of Chapter 3 in Stein and Shakarchi, Corollary of Theorem 18 of §5.2 in Ahlfors

1 Proof of Stein and Shakarchi

The proof of Stein and Shakarchi is based on the following elementary result:

Lemma 1 *If $\nu : [0,1] \to \mathbb{N}_0 = \{0,1,2,3,\ldots\}$ is continuous, then ν is constant, taking only one integer value.*

This, of course, follows from the intermediate value theorem for continuous real valued functions on $[0,1]$.
Proof of Stein and Shakarchi: For $0 \le \tau \le 1$ consider

$$h = h(z) = h(z;\tau) = (1-\tau)f(z) + \tau g(z).$$

For each fixed $\tau \in [0,1]$ the function $h : \Omega \to \mathbb{C}$ is holomorphic. The condition (3) which says there are no zeros of f on the loop Γ implies there are also no zeros of $h = h(z)$ on the loop Γ. In fact, if one assumes there is some $\tau \in (0,1)$ for which $h \circ \alpha = h \circ \alpha(t) = 0$ for some $t \in [a,b]$, then

$$(1-\tau)f \circ \alpha + \tau g \circ \alpha = 0,$$

so

$$\frac{g \circ \alpha}{f \circ \alpha} = 1 - \frac{1}{\tau} \qquad \text{and} \qquad \left|\frac{g \circ \alpha}{f \circ \alpha} - 1\right| = \frac{1}{\tau} > 1$$

which is a contradiction. For $\tau = 1$, $h = g$, and $g \circ \alpha = 0$ contradicts (3) immediately.

Thus we have shown the restriction

$$h_* = h\Big|_{[0,1]\times\Gamma} : [0,1] \times \Gamma \to \mathbb{C}$$

is a non-vanishing function. In particular, the hypotheses of the argument principle are satisfied for the holomorphic function h (for each fixed τ). In particular, for each fixed $\tau \in [0,1]$

$$\nu(\tau) = \frac{1}{2\pi i}\int_\alpha \frac{h'}{h}$$

is well-defined and gives the number of zeros of h circumnavigated by α. This means $\nu : [0,1] \to \mathbb{N}_0$ is integer valued with

$$\nu(1) = \frac{1}{2\pi i}\int_\alpha \frac{g'}{g} \qquad \text{and} \qquad \nu(0) = \frac{1}{2\pi i}\int_\alpha \frac{f'}{f}.$$

It remains to verify that h is continuous.

The continuity of h follows from the fact that h_* considered as a function on the compact metric space $[0,1]\times\Gamma$, is continuous and non-vanishing. More precisely, we have

$$h_* \in C^0\left([0,1]\times\Gamma \to \mathbb{C}\right)$$

and

$$\delta = \min\left\{|(1-\tau)f\circ\alpha(t) + \tau g\circ\alpha(t)| : (\tau,t)\in[0,1]\times[a,b]\right\} > 0.$$

Also,

$$M = \max\left\{|f'g - g'f| : z\in\Gamma\right\} < \infty.$$

Consequently,

$$\begin{aligned}
\nu(\tau)-\nu(\sigma) &= \frac{1}{2\pi i}\int_\alpha \left[\frac{(1-\tau)f'+\tau g'}{(1-\tau)f+\tau g} - \frac{(1-\sigma)f'+\sigma g'}{(1-\sigma)f+\sigma g}\right] \\
&= \frac{1}{2\pi i}\int_\alpha \frac{(1-\tau)\sigma(f'g-fg') + (1-\sigma)\tau(g'f-gf')}{[(1-\tau)f+\tau g][(1-\sigma)f+\sigma g]} \\
&= \frac{1}{2\pi i}\int_\alpha (f'g-gf')\frac{(1-\tau)\sigma-(1-\sigma)\tau}{[(1-\tau)f+\tau g][(1-\sigma)f+\sigma g]} \\
&= \frac{1}{2\pi i}\int_\alpha (f'g-gf')\frac{\sigma-\tau}{[(1-\tau)f+\tau g][(1-\sigma)f+\sigma g]}.
\end{aligned}$$

Therefore,

$$|\nu(\tau)-\nu(\sigma)| \le \frac{M}{\delta}\,\mathrm{length}(\Gamma)|\tau-\sigma|. \qquad \square$$

2 Proof of Ahlfors

Ahlfors' proof is rather different. Since the quotient $q = g/f$ is well-defined on Γ, the condition (4) may be interpreted to mean the curve $\beta : [a, b] \to \mathbb{C}$ defined by

$$\beta(t) = \frac{g(t)}{f(t)}$$

has image Δ lying in $D_1(1) = \{w \in \mathbb{C} : |w - w| < 1\}$. Since the zeros of f cannot accumulate on Γ, there is an open subdomain W satisfying

$$\Gamma \subset W \subset\subset \Omega$$

for which the quotient q extends as

$$q = \frac{g\big|_W}{f\big|_W}$$

to a holomorphic fuction $q : W \to D_1(1)$. On $D_1(1)$ there is a well-defined branch of the logarithm. In fact, the principle branch of the logarithm $\log_0 : \mathcal{L}_0 \to \mathbb{C}$ is well-defined there with image in the strip $\{z : 0 < \operatorname{Re} z < 2\pi\}$. Consequently, $h : W \to \mathbb{C}$ by

$$h(z) = \log_0 q(z)$$

is a well-defined holomorphic function with

$$h'(z) = \frac{q'(z)}{q(z)}.$$

Globally, the quotient $q : \Omega \to \mathbb{C} \cup \{\infty\}$ is well-defined as a meromorphic function, and the argument principle may be applied. As a meromorphic function

$$q' = \left(\frac{g}{f}\right)' = \frac{fg' - gf'}{f^2} \tag{5}$$

so that

$$\frac{q'}{q} = \frac{fg' - gf'}{fg} = \frac{g'}{g} - \frac{f'}{f} \tag{6}$$

and

$$\frac{1}{2\pi i}\int_\alpha \frac{q'}{q} = \frac{1}{2\pi i}\int_\alpha \frac{g'}{g} - \frac{1}{2\pi i}\int_\alpha \frac{f'}{f}. \tag{7}$$

The final step is to note that h is a primitive for the logarithmic derivative q'/q on W where, in fact, q'/q is holomorphic. Therefore,

$$\frac{1}{2\pi i}\int_{\alpha}\frac{q'}{q} = \frac{1}{2\pi i}[h(\alpha(b)) - h(\alpha(a))] = 0. \qquad \square$$

It might be added/remarked that $h(\alpha(b)) - h(\alpha(a)) = \log_0 \beta(b) - \log_0 \beta(a)$ as well.