The Domain of Divergence
For Complex Power Series

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February 5, 2022

The following is an attempt to give an account of observations and results related to a question originally asked by Daniel Savio in the Spring semester of 2022 regarding the behavior of divergent power series in complex analysis. The subject may be well-known to some, but I must confess it is a new direction of inquiry for me encompassing questions and constructions which I am quite sure I have not seen before. I'm not entirely sure of the overall depth of the subject, but at least superficially it seems worth considering if not somewhat exciting.

1 Introduction

Given a formal power series

$$\sum_{n=0}^{\infty} a_n z^n$$

determined by complex numbers/coefficients $a_0, a_1, a_2, a_3, \ldots$, the basic theorem about power series asserts that the series converges and determines a complex differentiable function $f : D_R(0) \to \mathbb{C}$ on a disk $D_R(0)$ determined by the **Hadamard radius**

$$R = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}. \tag{1}$$

Moreover the series **diverges** for $|z| > R$. Daniel Savio asked if the divergence occurring for $|z| > R$ satisfies

$$\sum_{n=0}^{\infty} a_n z^n = \infty \ ? \tag{2}$$

That is, given any $M > 0$, is it possible to find some $N \in \mathbb{N}$ for which $k > N$ implies

$$\left| \sum_{n=0}^{k} a_n z^n \right| > M \ ?$$

I initially suggested that the answer was "no," but I was unable to construct an example. A couple days later Katherine Booth did construct an example confirming my suspicion, and I hope to offer a detailed exposition of Katherine's example below as well as some additional observations.

It seems advisable to restrict to a particular class of formal series and introduce some terminology. We are interested in formal series

$$\sum_{n=0}^{\infty} a_n z^n$$

with **finite Hadamard radius** given by (1). In this case,

$$L = \limsup_{n\to\infty} |a_n|^{1/n}$$

is a positive extended real number with $L \in (0, \infty]$. Let us define the **Savio region** Ω_S of a given series by

$$\Omega_S = \left\{ z \in \mathbb{C}\backslash\overline{D_R(0)} : \lim_{N\to\infty} \sum_{n=0}^{N} a_n z^n = \infty \right\}.$$

Before proceeding further, I'd like to review the proof of the basic convergence result and observe what it tells us about behavior in the region of divergence $\mathbb{C}\backslash\overline{D_R(0)}$. The proof is also in Stein and Shakarchi, but I present it with minor notational differences—hopefully improvements.

Let us assume first that $|z| < R$. Notice that L must be a finite (positive) number in this case and we can write

$$|z| < \frac{1}{L}.$$

Consequently, there is some $\epsilon > 0$ for which it is also true that

$$|z| < \frac{1}{L+\epsilon}. \tag{3}$$

On the other hand, $\limsup_{n\to\infty} |a_n|^{1/n} = L$, so there is some N for which $n > N$ implies

$$|a_n|^{1/n} < L + \epsilon.$$

It follows then that for $K > N + 1$

$$\sum_{n=0}^{K} |a_n\, z^n| = \sum_{n=0}^{N} |a_n|\; |z|^n + \sum_{n=N+1}^{K} \left(|a_n|^{1/n}\; |z|\right)^n < \sum_{n=0}^{N} |a_n|\; |z|^n + \sum_{n=N+1}^{K} \left((L+\epsilon)\; |z|\right)^n.$$

Notice that N and $|z|$ are fixed here, so the first term/sum

$$\sum_{n=0}^{N} |a_n|\; |z|^n$$

is some fixed finite nonnegative number. Furthermore, by (3) we know

$$(L+\epsilon)\,|z| < 1,$$

so we can estimate the second term as follows:

$$\sum_{n=N+1}^{K} \left((L+\epsilon)\; |z|\right)^n \le \sum_{n=0}^{\infty} \left((L+\epsilon)\; |z|\right)^n < \infty.$$

The key observation here is that the number

$$\sum_{n=0}^{\infty} \left((L+\epsilon)\; |z|\right)^n$$

is fixed (and finite) independent of K. This means

$$\sum_{n=0}^{\infty} |a_n\, z^n| < \infty$$

and the series $\sum a_n z^n$ is absolutely summable/convergent.

The more interesting part of the basic convergence result for our current considerations applies in the case $|z| > R$. Here we know $L > 0$, and that is a given assumption for our series of finite Hadamard radius. If L is finite, we can write

$$|z| > \frac{1}{L}.$$

Here the condition

$$\limsup_{n\to\infty} |a_n|^{1/n} > L$$

implies that for some $\epsilon > 0$ there exists a subsequence n_k, $k = 1, 2, 3, \ldots$ for which

$$|a_{n_k}|^{1/n_k} > (1+\epsilon)L.$$

Consequently,

$$|a_{n_k}|\,|z|^{n_k} = \left(|a_{n_k}|^{1/n_k}\,|z|\right)^{n_k} > (1+\epsilon)^{n_k} \to +\infty$$

as $k \nearrow \infty$. Thus, $\sum a_n z^n$ does not satisfy the **basic necessary condition for convergence** of a complex series. See section 1 of my notes on convergence of series. It is in this sense that the basic convergence result gives us divergence of the series for $z \in \mathbb{C}\backslash\overline{D_R(0)}$.

Let us briefly return to consider the case $L = \infty$ in which the condition $|z| > R$ becomes simply $|z| > 0$. In this case, there is a subsequence of indices n_k for $k = 1, 2, 3, \ldots$ with

$$|a_{n_k}|^{1/n_k} > k. \tag{4}$$

Then

$$|a_{n_k}|\,|z|^{n_k} = \left(|a_{n_k}|^{1/n_k}\,|z|\right)^{n_k} > (k|z|)^{n_k} \to +\infty,$$

and we reach the same conclusion. Note that it would be adequate to replace k on the right in (4) with $(1+\epsilon)/|z|$ for any $\epsilon > 0$. □

Also in my notes on convergence I gave an example (simply the obvious example of the geometric series) demonstrating that, in at least some instances, the Savio region associated with a formal complex power series is nonempty. I didn't double check the details in those notes, but I verified the assertion independently again and generalized it slightly. I will present the resulting family of simple examples before continuing to the more sophisticated examples of Katherine Booth.

Example (McCuan) Consider a series $\sum a_n z^n$ with coefficients satisfying

$$|a_n| = m > 0 \qquad \text{for all } n \text{ sufficiently large.}$$

That is, there is some N_0 such that $|a_n| = m$ for $n \geq N_0$. The geometric series

$$\sum_{n=0}^{\infty} z^n$$

is such a series with $N_0 = 0$ and $a_n = 1$ for all n. The Hadamard radius of any such series is $R = 1$ since

$$\lim_{n\to\infty} m^{1/n} \sim \exp\left(\lim_{n\to\infty} \frac{\log m}{n}\right) = e^0.$$

Let us assume first $N_0 = 0$. Let us also assume initially that $|z| > 1$. The triangle inequality gives

$$\begin{aligned} m\,|z|^N &= |a_N|\,|z|^N \\ &= \left|a_N\, z^N\right| \\ &= \left|\sum_{n=0}^{N} a_n\, z^n - \sum_{n=0}^{N-1} a_n\, z^n\right| \\ &\leq \left|\sum_{n=0}^{N} a_n\, z^n\right| + \left|\sum_{n=0}^{N-1} a_n\, z^n\right| \\ &\leq \left|\sum_{n=0}^{N} a_n\, z^n\right| + \sum_{n=0}^{N-1} |a_n|\,|z|^n \\ &= \left|\sum_{n=0}^{N} a_n\, z^n\right| + m\sum_{n=0}^{N-1} |z|^n \\ &= \left|\sum_{n=0}^{N} a_n\, z^n\right| + m\frac{1-|z|^N}{1-|z|}. \end{aligned}$$

It follows that

$$\left|\sum_{n=0}^{N} a_n\, z^n\right| \geq m\left(|z|^N - \frac{1-|z|^N}{1-|z|}\right) = m\frac{|z|^{N+1} - 2|z|^N + 1}{|z|-1} = m\frac{|z|^N(|z|-2)+1}{|z|-1}.$$

Thus, restricting further to $|z| > 2$ we have

$$\lim_{N\nearrow\infty} \sum_{n=0}^{N} a_n\, z^n = \infty,$$

and

$$\Omega_S \supset \mathbb{C}\backslash\overline{D_2(0)} \tag{5}$$

for all such power series.

Repeating the estimation above when $N_0 > 0$ and $|z| > 1$, we get

$$\begin{aligned} m\,|z|^N &= \left|a_N\, z^N\right| \\ &= \left|\sum_{n=0}^{N} a_n\, z^n - \sum_{n=0}^{N_0-1} a_n\, z^n - \sum_{n=N_0}^{N-1} a_n\, z^n\right| \\ &\le \left|\sum_{n=0}^{N} a_n\, z^n\right| + \left|\sum_{n=0}^{N_0-1} a_n\, z^n\right| + m\frac{1-|z|^N}{1-|z|}. \end{aligned}$$

The following estimate becomes

$$\left|\sum_{n=0}^{N} a_n\, z^n\right| \ge m\frac{|z|^N(|z|-2)+1}{|z|-1} - \left|\sum_{n=0}^{N_0-1} a_n\, z^n\right|.$$

Since the last term is fixed, we conclude the Savio region is again nonempty and contains an annular neighborhood of infinity according to (5).

2 Booth Series and Booth Region

Katherine considered specific series in the class of my examples above under the following assumptions:

1. The coefficients a_n satisfy $|a_n| = 1$ and
2. The coefficients repeat in a cycle of 3 with $a_{3k} = a_0$, $a_{3k+1} = a_1$, and $a_{3k+2} = a_2$ for $k = 0, 1, 2, 3, \ldots$.

Her objective was to choose the three complex coefficients a_0, a_1, and a_2 so that the sequence of partial sums determined by

$$S_k = \sum_{n=0}^{k} a_n z^n$$

contains a bounded subsequence for some fixed $z \in \mathbb{C}\backslash\overline{D_R(0)}$. As mentioned above the Hadamard radius is given by $R = 1$ for such a formal series, and my examples above also imply that z must satisfy

$$1 < |z| \le 2.$$

Katherine also made the crucial ansatz that not only does she want a bounded subsequence, but such a subsequence will be given by $S_2, S_5, S_8, \dots$ with $S_{3k+2} \equiv 0$ for $k = 0, 1, 2, 3, \dots$. This, she observes can be accomplished if

$$S_2 = a_0 + za_1 + z^2a_2 = 0, \tag{6}$$

since each succeeding triple of terms in the partial sum will have the form:

$$a_0 z^{3k} + a_1 z^{3k+1} + a_2 z^{3k+2} = z^{3k}\left(a_0 + za_1 + z^2 a_2\right).$$

Finally Katherine assumes $z = 3/2$, but let us generalize this slightly and simply assume $z = x \in (1, \infty)$ for the time being. Then given $x > 1$, we seek three complex numbers a_0, a_1, and a_2 satisfying

$$a_0 + xa_1 + x^2a_2 = 0. \tag{7}$$

Finally, with a view toward a geometric interpretation and later generalization, let me assume $a_0 = 1$.[1]

Writing also

$$a_1 = e^{i\theta_1} \qquad \text{and} \qquad a_2 = e^{i\theta_2}$$

we can see geometrically what the condition (6) means in this case, and the potential viability of determining the angles θ_1 and θ_2 so that it holds. More precisely, we want

$$1 + xe^{i\theta_1} + x^2e^{i\theta_2} = 0 \tag{8}$$

which means, as indicated in Figure 1, the following should hold:

1. the point xa_1 lies on a circle of radius x and on the horizontal line $\operatorname{Im} z = x \sin\theta_1$.

2. The point x^2a_2 lies on a circle of radius $x^2 > x$ but with $\operatorname{Im}(x^2a_2) = -\operatorname{Im}(xa_1) = -\sin\theta_1$ so that $\operatorname{Im}(xa_1 + x^2a_2) = 0$.

3. It follows from the previous condition and the fact that $x^2 > x$ because $x > 1$ that $|\operatorname{Re}(x^2a_2)| > |\operatorname{Re}(xa_1)|$.

4. Since we want $\operatorname{Re}(xa_1 + x^2a_2) = -1$, we need $\operatorname{Re}(x^2a_2 < 0$, i.e., a_2 is in the second or third quadrant. We have chosen a_2 to be in the third quadrant, and a_1 (must be) in the first quadrant.

[1] If I am understanding Katherine's construction correctly, she assumed instead $a_2 = -i$ and $x = 3/2$.

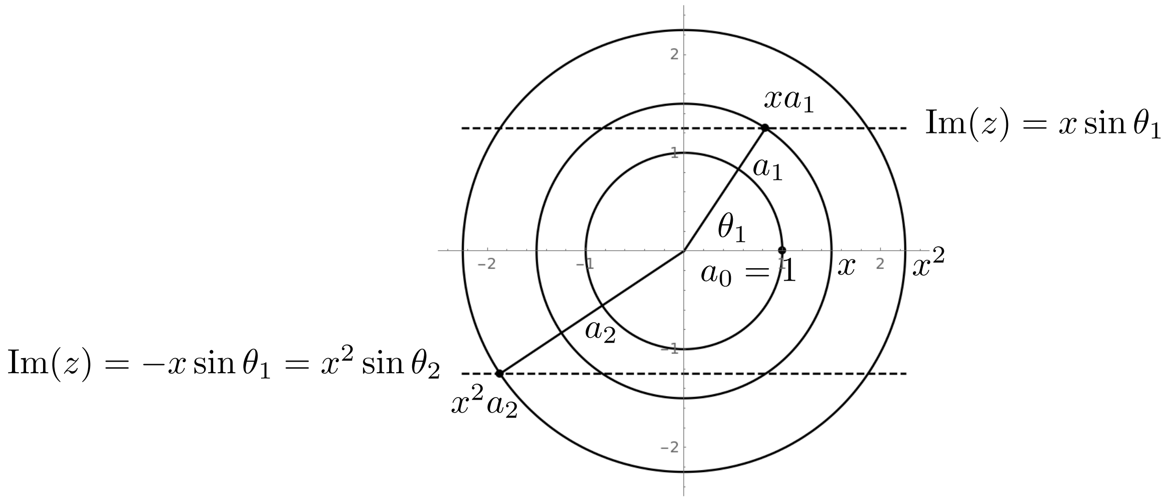

Figure 1: The three Booth points $a_0 = 1$, $a_1 = e^{i\theta_1}$, and $a_2 = e^{i\theta_2}$.

These geometric conditions do not give a solution (a_1, a_2) of the desired equation (6, 7, 8), but they show the existence is at least plausible. Furthermore, the choice of quadrants for a_1 and a_2 allows us to treat the equations more easily (and with more insight) as we now demonstrate.

We begin by writing (8) as

$$\begin{cases} x\cos\theta_1 + x^2\cos\theta_2 & = & -1 \\ x\sin\theta_1 + x^2\sin\theta_2 & = & 0. \end{cases}$$

With the substitutions

$$\cos\theta_2 = -\frac{x\cos\theta_1 + 1}{x^2} < 0,$$

and

$$\sin\theta_1 = \sqrt{1-\cos^2\theta_1} \qquad \text{and} \qquad \sin\theta_2 = -\sqrt{1-\cos^2\theta_2}$$

with the signs in the latter two substitutions chosen with the help of Figure 1, we can write

$$\cos^2\theta_2 = \frac{1 + 2x\cos\theta_1 + x^2\cos^2\theta_1}{x^4}$$

and

$$x\sqrt{1-\cos^2\theta_1} = \sqrt{x^4 - 1 - 2x\cos\theta_1 - x^2\cos^2\theta_1}.$$

Squaring both sides gives

$$x^2 = x^4 - 1 - 2x\cos\theta_1$$

or

$$\cos\theta_1 = \frac{x^4 - x^2 - 1}{2x}. \tag{9}$$

Let us pause at this point and consider the restrictions imposed on x by the necessary condition $0 < \cos\theta_1 < 1$ arising from the choice indicated in Figure 1. In Figure 2 we have plotted the quotient

$$\phi(x) = \frac{q(x)}{2x}$$

where $q(x) = x^4 - x^2 - 1$ is an even quartic polynomial. This function ϕ has some

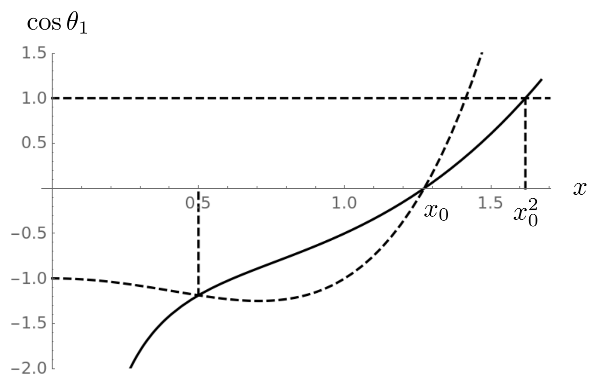

Figure 2: A function giving restrictions on the choice of the real parameter $x > 1$. A plot of the even quartic polynomial $q(x) = x^4 - x^2 - 1$ is indicated with a dashed curve.

properties as indicated by the plot and easily verified. In particular, we are interested in values of ϕ corresponding to angles θ_1 in the first quadrant, that is, with $0 < \phi(x) < 1$. These values correspond to an interval $x_0 \leq x < x_0^2$ where x_0^2 is the positive root of the equation $\phi(x) = 1$ or the positive root of quartic polynomial

$$q(x) - 2x = x^4 - x^2 - 1 - 2x = x^4 - (x+1)^2 = (x^2 - x - 1)(x^2 + x + 1).$$

That is,

$$x_0^2 = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad x_0 = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

The endpoint $x = x_0^2$ where $\cos\theta_1 = 1$ and $\theta_1 = 0$ seems to correspond to a purely real solution a_1, which is interesting since presumably there is a corresponding value of $\theta_2 = \pi$, and it is possible to take $a_0 = 1$, a_1 and a_2 all real. The geometry suggests that the other endpoint $x = x_0$ may be artificial with possible values of θ_1 beyond the one corresponding to $\theta_1 = \pi/2$ with θ_1 in the second quadrant (and θ_2 still in the third quadrant). Indeed, it may be natural to consider a left endpoint determined by the positive root of $\phi(x) = -1$. This may lead to a different choice of real a_1 and a_2. I guess that is correct with the left endpoint being

$$\xi_0^2 = \frac{-1+\sqrt{5}}{2}.$$

For each x in this interval (9) determines a value of θ_1 on $[0, \pi]$ and consequently a value of a_1 in the closed upper half of the circle $\mathbb{S}^1$.

Returning to our system of equations, we find

$$\cos\theta_2 = -\frac{x^4 - x^2 + 1}{2x^2}. \tag{10}$$

In Figure 3 we have added the plot of $\cos\theta_2$ (which is seen to remain negative) along with the bounds $\cos\theta = \pm 1$.

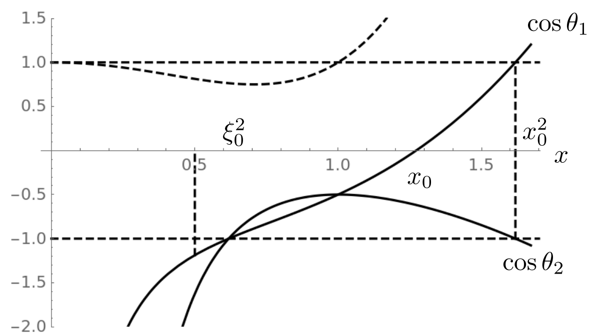

Figure 3: The function determining θ_2 as a function of x on the same interval $\xi_0^2 \le x \le x_0^2$ for the real parameter $x > 1$. A plot of the even quartic polynomial $q(x) = x^4 - x^2 + 1$ is indicated with a dashed curve.

The geometry strongly suggests we should be able to take $e^{i\alpha}a_0 = e^{i\alpha}$, $e^{i\alpha}a_1$, and $e^{i\alpha}a_2$ for any choices of a_1 and a_2 obtained above and any unit complex number $e^{i\alpha}$ to obtain a series with the desired property, namely that every partial sum $S_{3k+2} = 0$

for $k = 0, 1, 2, 3, \ldots$ and the appropriate choice of x. I suppose the algebra bears this assertion out. Presumably the choice $x = 3/2$ and $e^{i\alpha} = -i$ gives, up to a possible permutation/rearrangement of points, the example Katherine gave.

This shows of course, that the Savio region Ω_S is not always all of $\mathbb{C}\backslash\overline{D_R(0)}$. This example also shows that it is natural to define a complementary region, which we can call the **Booth region** Ω_B by

$$\Omega_B = \left\{ z \in \mathbb{C}\backslash\overline{D_R(0)} : \liminf_{N\to\infty} \left| \sum_{n=0}^{N} a_n z^n \right| < \infty \right\}.$$

Each triple of numbers $\{1, a_1, a_2\}$ or more generally

$$\left\{e^{i\alpha}, e^{i\alpha}a_1, e^{i\alpha}a_2\right\} \qquad \text{for which} \qquad e^{i\alpha} + xe^{i\alpha}a_1 + x^2e^{i\alpha}a_2 = 0 \tag{11}$$

is also quite interesting and worthy of a name. Take, for example the cube roots of $w = 1 \in \mathbb{C}$. These are

$$\left\{1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}\right\},$$

and you will notice that the sum of the three roots is zero. Of course, these points all lie on the unit circle $\mathbb{S}^1$. I think it would be quite interesting to understand in what sense a triple satisfying (11) which we might call a **Booth triple** generalizes a unit rotation of the roots of unity

$$\left\{e^{i\alpha}, e^{i\alpha}\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right), e^{i\alpha}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\right\}.$$

Finally, it might be interesting to understand if this construction can be extended to cycles of coefficients of a longer length (with a view toward generalizing properties of the higher order roots of unity corresponding to $x = 1$.)

3 Open Problems

Katherine tells me there is numerical evidence to suggest that the Savio region for the geometric series

$$\sum_{n=0}^{\infty} z^n$$

with $a_n = 1$ for all n is $\Omega_S = \mathbb{C}\backslash\overline{B_1(0)}$. This would mean the Booth region is empty. But we have no proof either way even for the geometric series. In particular, there is no proof that the Booth region is always non-empty, and the numerics suggest this can happen.

Update:[2] Actually, we know the explicit formula for the partial sums of the geometric series. Thus, we can say the following: For $|z| > 1$,

$$S_N = \sum_{n=0}^{N} z^n = \frac{1 - z^{N+1}}{1 - z}.$$

In particular,

$$|S_N| \geq \frac{|z|^{N+1} - 1}{|1 - z|} \to \infty \qquad \text{as} \qquad N \nearrow \infty.$$

This gives two interesting conclusions:

> *The geometric series is an example in which the Savio region is the entire domain of divergence:*
>
> $$\textit{For} \quad \sum_{n=0}^{\infty} z^n, \qquad \Omega_S = \mathbb{C}\backslash\overline{D_1(0)}.$$

This observation tells us something also about Katherine's examples above: If

$$\sum_{n=0}^{\infty} a_n z^n = \sum_{k=0}^{\infty} (a_0 + a_1 z + a_2 z^2) z^k$$

with $|a_0| = |a_1| = |a_2| = 1$ and $a_0 + a_1 x + z_2 x^2 = 0$ for some x with $-1 + \sqrt{5} \leq 2x \leq 1 + \sqrt{5}$, then

$$\sum_{n=0}^{\infty} a_n z^n = (a_0 + a_1 z + a_2 z^2) \sum_{k=0}^{\infty} z^k. \tag{12}$$

The polynomial $a_0 + a_1 z + a_2 z^2$ has two roots one of which is x and $x > 1$. It can be shown that the other root ζ satisfies $|\zeta| < 1$. Therefore, for $|z| > 1$, we know $a_0 + a_1 z + a_2 z^2$ is nonzero, and the partial sums for this Booth series tend to ∞ because of the geometric series appearing in (12).

[2] Original document February 2, 2022. Update: February 4, 2022.

For a Booth series corresponding to the parameter x the Savio region is given precisely by

$$\Omega_S = \mathbb{C}\backslash \left(\overline{D_1(0)} \cup \{x\}\right) \tag{13}$$

and the Booth region consists of a single isolated point $\Omega_B = \{x\}$.

I will discuss these observations in a little more detail in a separate document which brings in boundary behavior to the discussion.

To end the update, I might add that, especially in view of the fact that Ω_S is open in every case in which we know the set explicitly namely in the cases of a geometric series and in (13), it seems reasonable to suggest Ω_S is always open. That might be a reasonable thing to try to prove.

Unfortunately, my example does not seem to generalize easily to imply the Savio region is always nonempty. It would be interesting to prove there is always an annular neighborhood of infinity in the Savio region, but it does not appear to be easy.

So far we do have an example with both Ω_S and Ω_B non-empty, but it's not the geometric series.

I feel like I had isolated another, fundamentally different, question about the relation between the divergence region, Ω_B and Ω_S, but I can't think of it at the moment, and it's late, so I'm going to stop.

Another Update: I'm not sure this is the question I had in mind, but it might be.

> Given a point in the domain of divergence $\mathbb{C}\backslash\overline{D_R(0)}$, is it possible to show there always exists a **subsequence** of partial sums tending to ∞?