S&S Exercise 1.1(e)

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Here we are asked to describe geometrically the set

$$S = \{z \in \mathbb{C} : \text{Re}(az + b) > 0\}$$

where a and b are two fixed complex numbers. We have seen in the previous part the **open right half plane**

$$H_c = \{z \in \mathbb{C} : \text{Re}\, z > c\}$$

where $c \in \mathbb{R}$ as indicated on the left in Figure 1, and our guess is the set S too is some open half plane. In fact, we note that $S = \{z \in \mathbb{C} : az + b \in H_0\}$ and this is at least a partial motivation for our guess. As in part (a) of this problem, there is a kind of degenerate case which defies our guess: If $a = 0 \in \mathbb{C}$, then $S = \{z \in \mathbb{C} : \text{Re}\, b > 0\}$, and we find the following (two) preliminary cases.

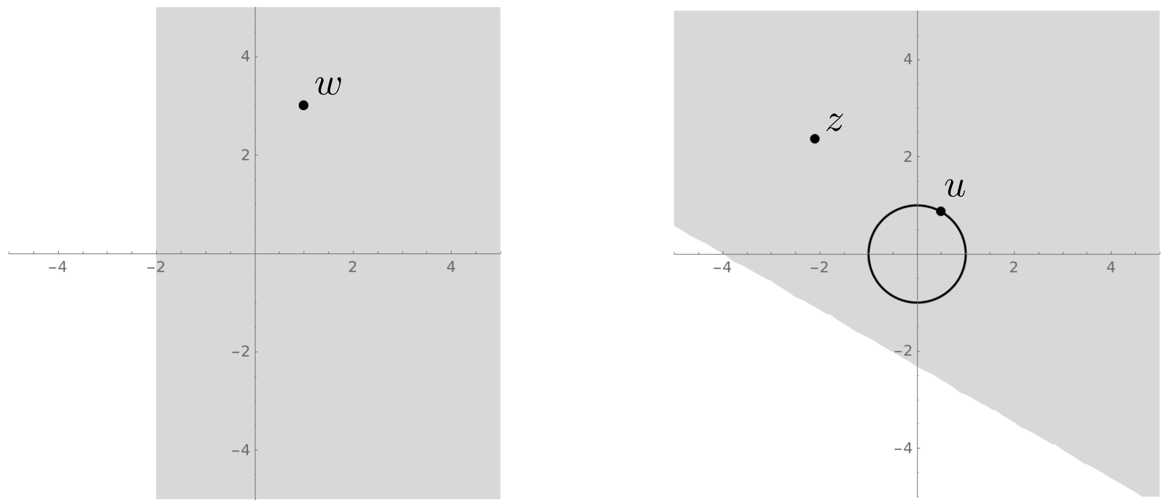
If $a = 0$ and $\text{Re}\, b \leq 0$, then $S = \phi$ is the empty set.

If $a = 0$ and $\text{Re}\, b > 0$, then $S = \mathbb{C}$ is the entire complex plane.

Henceforth we assume $a \neq 0$. In order to give a nice treatment (or at least one way to give a nice treatment) of this exercise is to introduce a kind of general form for open half planes in $\mathbb{C}$ generalizing the simple right half plane H_c. This may be done as follows: Given $u \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $c \in \mathbb{R}$, we set

$$H_{u,c} = \{z \in \mathbb{C} : \overline{u}z \in H_c\} = \{z \in \mathbb{C} : \text{Re}(\overline{u}z) > c\}. \tag{1}$$

We claim this expression represents the rotation of H_c counterclockwise by the angle $\theta = \text{Arg}(u)$ as indicated on the right in Figure 1. By this time, we should know



H_c with $c = -2$ $\qquad$ $H_{u,c}$ with $c = -2$

Figure 1: Open half planes.

any complex number $u \in \mathbb{S}^1$ in the unit circle[1] of $\mathbb{C}$ determines a unique principal argument $\theta \in [0, 2\pi)$ by

$$\cos\theta = \operatorname{Re} u \qquad \text{and} \qquad \sin\theta = \operatorname{Im} u, \tag{2}$$

and multiplication by u, i.e., $z \mapsto uz$, can be interpreted as counterclockwise rotation of z by the angle θ. More generally, any nonzero complex number z determines a principal argument $\theta \in [0, 2\pi)$ by

$$\cos\theta = \operatorname{Re} \frac{z}{|z|} \qquad \text{and} \qquad \sin\theta = \operatorname{Im} \frac{z}{|z|}.$$

This of course doesn't work when $z = 0$. Naturally, multiplication by $1/u = \overline{u}$ corresponds to clockwise rotation by the argument of u. With this observation, we can see clearly the set $H_{u,c}$ defined in (1) represents the open half plane we have in mind. In fact, if $w \in H_c$ as illustrated on the left in Figure 1, then $z = uw$ is

[1]Stein and Shakarchi do not introduce this fairly standard notation for the unit circle on page 6 where various related notations are introduced. They do give a standard notation to the unit ball or disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and in terms of this notation, the unit circle $\mathbb{S}^1$ would be $\partial\mathbb{D}$. The notation $\mathbb{S}^1 = \partial B_1(\mathbf{0}) = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is also used for the unit circle in $\mathbb{R}^2$ where $B_r(p)$ is roughly equivalent to Stein's $D_r(z)$.

in $H_{u,c}$ since $\overline{u}z = \overline{u}uw = w$. And conversely, if $z \in H_{u,c}$, then $w = \overline{u}z$ satisfies $\operatorname{Re} w = \operatorname{Re}(\overline{u}z) > c$ straight from the definition in (1), so $w \in H_c$.

At this point, we make two simple but useful observations about right open half planes H_c. First, any right open half plane may be expressed as

$$H_c = \{z \in \mathbb{C} : \operatorname{Re} z > c\} = \{z \in \mathbb{C} : \operatorname{Re}(z + it) > c\}$$

where $it \in i\mathbb{R}$ is any purely imaginary number. Second, in the special case $c = 0$, a dilation may be introduced;

$$H_0 = \{z \in \mathbb{C} : \operatorname{Re} z > 0\} = \{z \in \mathbb{C} : \operatorname{Re}(\mu z) > c\}$$

where $\mu > 0$ is any fixed positive real number.

Let us now state clearly what we want (and maybe need) to do: We want to identify $u \in \mathbb{S}^1$ and $c \in \mathbb{R}$ so that $S = H_{u,c}$ (in the case where $a \neq 0$). I think we can now do that pretty directly:

$$\begin{aligned}
S &= \{z \in \mathbb{C} : \operatorname{Re}(az + b) > 0\} \\
&= \{z \in \mathbb{C} : az + b \in H_0\} \\
&= \{z \in \mathbb{C} : az + b - i \operatorname{Im} b \in H_0\} \\
&= \{z \in \mathbb{C} : az + \operatorname{Re} b \in H_0\} \\
&= \left\{z \in \mathbb{C} : \frac{az}{|a|} + \frac{\operatorname{Re} b}{|a|} \in H_0\right\} \\
&= \left\{z \in \mathbb{C} : \operatorname{Re}\left(\frac{az}{|a|} + \frac{\operatorname{Re} b}{|a|}\right) > 0\right\} \\
&= \left\{z \in \mathbb{C} : \operatorname{Re}\frac{az}{|a|} + \frac{\operatorname{Re} b}{|a|} > 0\right\} \\
&= \left\{z \in \mathbb{C} : \operatorname{Re}\frac{az}{|a|} > -\frac{\operatorname{Re} b}{|a|}\right\} \\
&= \left\{z \in \mathbb{C} : \frac{az}{|a|} \in H_c\right\}
\end{aligned}$$

where

$$c = -\frac{\operatorname{Re} b}{|a|} \in \mathbb{R}.$$

Notice we used $a \neq 0$ in the fifth line where we dilated by $\mu = 1/|a|$. Finally, then we have

$$S = \{z \in \mathbb{C} : \overline{u}z \in H_c\} = H_{u,c}$$

where

$$u = \frac{\overline{a}}{|a|} \in \mathbb{S}^1 \qquad \text{since} \qquad \overline{\frac{\overline{a}}{|a|}} = \frac{a}{|a|}.$$

According to my notes, these were the values I gave for u and c in the lecture, though I had not fully prepared the solution/discussion and the explanation left a great deal to be desired. Hopefully, the written explanation above is closer to clear and correct.

Since I have the better part of a whole page blank below at this point, maybe I'll go ahead and type up the solution to the next part.

Part (f) Describe geometrically the set

$$S = \{z \in \mathbb{C} : |z| = \operatorname{Re} z + 1\}.$$

For this, I'm going to write $z = x + iy$. Then the condition $|z| = \operatorname{Re} z + 1$, which involves only real numbers, becomes

$$\sqrt{x^2 + y^2} = x + 1.$$

Squaring both sides, we have $x^2 + y^2 = x^2 + 2x + 1$ or $x = y^2/2 - 1/2$. This condition, I recognize right away defines a parabola

$$P = \left\{ x + iy \in \mathbb{C} : x = \frac{1}{2}y^2 - \frac{1}{2} \right\}.$$

I'm inclined to guess that $S = P$, but I'm a little worried I might have introduced extra extraneous points in P when I squared the relation, so I had better check that. There are a couple ways to do this. One way is to go ahead and draw the parabola as I've done in Figure 2 and remember that a parabola is the set of points equidistant from a fixed point called the focus and a fixed line called the directrix. In this case, one can figure out pretty quickly, by checking the vertex $(-1/2, 0)$ and the points $(0, \pm 1)$, that the focus is the origin and the directrix is $\operatorname{Re}(z) = -1$ as indicated in Figure 2. Thus, taking an arbitrary point in this parabola, the geometric condition defining the parabola is that the distance from the origin of a point w is the same as the distance from w to the vertical line $\operatorname{Re} z = -1$. That is,

$$|z| = |\operatorname{Re} z - (-1)| = \operatorname{Re} z + 1. \tag{3}$$

Here, of course, we need to know $\operatorname{Re} z > -1$, but since the vertex is at $(-1/2, 0)$, this is clear. Furthermore, the condition (3) is precisely the condition determining S, so we are done. That is, we have verified

$$P = \{z \in \mathbb{C} : |z| = \operatorname{Re} z + 1\} = S.$$

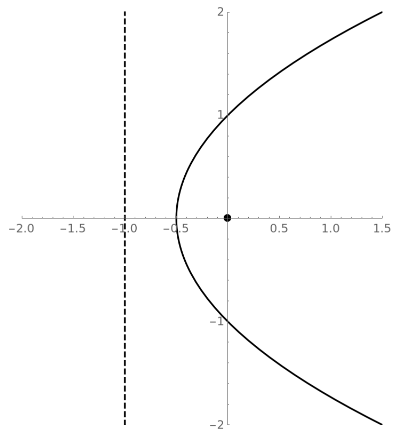


Figure 2: The parabola $x = y^2/2 - 1/2$.

Now, if you don't like to draw pictures, i.e., you are a stuffy algebraist instead of a happy-go-lucky geometer, then you might want to look somewhat more critically at the squared condition

$$x^2 + y^2 = x^2 + 2x + 1 = (x+1)^2$$

and justify taking the square root of both sides. You'll get

$$|z| = \sqrt{x^2 + y^2} = |x+1|.$$

But since

$$x = \frac{y^2}{2} - \frac{1}{2} \geq -\frac{1}{2}$$

we do know $x + 1 \geq 1 - 1/2 = 1/2 > 0$, so $|x+1| = x+1$, so you get done (and get the same answer) this way too.

Part (g) Describe geometrically the set

$$\{z \in \mathbb{C} : \operatorname{Im} z = c\}$$

where $c \in \mathbb{R}$ is a fixed real number. This part is rather dissappointing. Of course, this is a horizontal line. The only somewhat amusing thing I can think to do with it is show that this line can be written as the counterclockwise rotation by $\theta = \pi/2$ of the vertical line $\{z \in \mathbb{C} : \operatorname{Re} z = c\}$ of the sorts considered in parts (c) and (d). That is,

$$\{z \in \mathbb{C} : \operatorname{Im} z = c\} = \{iw \in \mathbb{C} : \operatorname{Re} w = c\} = \{z \in \mathbb{C} : \operatorname{Re}(-iw) = c\}.$$

But this is sort of all (painfully) obvious.

It also puts me back in the position of looking at the better part of a blank page below.