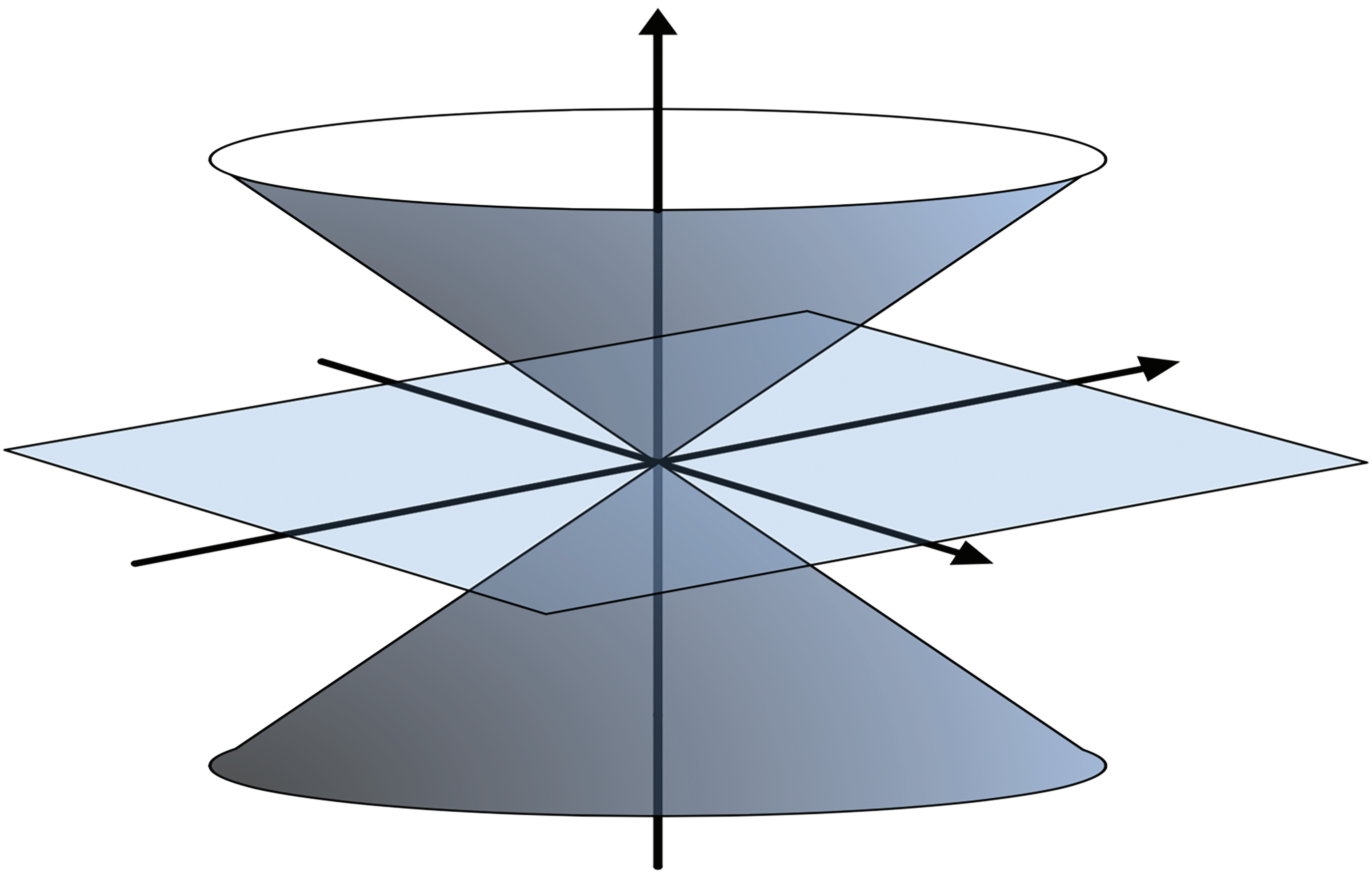
Geometric Measure Theory

Recent Applications



Tatiana Toro

GMT Introduction

Geometric Measure Theory (GMT) provides a framework to address questions in very different areas of mathematics, including calculus of variations, geometric analysis, potential theory, free boundary regularity, harmonic analysis, and theoretical computer science. Progress in different branches of GMT has led to the emergence of new challenges, making it a very vibrant area of research. In this note we will provide a historic background to some of the questions that gave rise to the field, briefly mention some of the milestones, and then focus on some of the recent developments at the intersection of GMT, potential theory, and harmonic analysis.

The origins of the field can be traced to the following question: *do the infinitesimal properties of a measure determine the structure of its support?*

In the late 1920s and early '30s Besicovitch was interested in understanding the structure of a set $E \subset \mathbb{R}^2$ satisfying $0 < \mathcal{H}^1(E) < \infty$ and such that for $\mathcal{H}^1$-a.e. $x \in E$,

$$\lim_{r \to 0^+} \frac{\mathcal{H}^1(B(x,r) \cap E)}{2r} = 1, \tag{1}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. (The formulation above is a modern version of the problem. Besicovitch, who most likely was unaware of the existence of the Hausdorff measure, formulated the question

Tatiana Toro is the Craig McKibben & Sarah Merner Professor in Mathematics at the University of Washington. She was partially supported by NSF grant DMS-1664867. Her email address is toro@uw.edu.

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in terms of the linear measure.) For $n \geq 1$, the n-dimensional Hausdorff measure $\mathcal{H}^n$ in $\mathbb{R}^m$ generalizes the notions of length of a curve ($n = 1$), surface area ($n = 2$), and volume ($n = 3$) to subsets of $\mathbb{R}^m$. Moreover $\mathcal{L}^n \llcorner \mathbb{R}^n = \mathcal{H}^n \llcorner \mathbb{R}^n$

Besicovitch showed that if E satisfies the hypothesis (1), then E is 1-rectifiable; that is, E is contained in a countable union of Lipschitz images of $\mathbb{R}$ union a set of 1-Hausdorff measure 0 (see [16], [17]). In GMT the notion of rectifiability is used to describe the structure (also the regularity) of a set or a measure in a way similar to how the degree of differentiability of charts is used to describe the smoothness of a manifold in differential geometry. A set $E \subset \mathbb{R}^m$ is *n-rectifiable* if

$$E \subset \bigcup_{i=1}^{\infty} f_j(\mathbb{R}^n) \cup E_0,$$

where $f_j : \mathbb{R}^n \to \mathbb{R}^m$ is a Lipschitz map and $\mathcal{H}^n(E_0) = 0$. Recall that f_j is *Lipschitz* if there exists $L_j > 0$ s.t. for $x, y \in \mathbb{R}^n$

$$|f_j(x) - f_j(y)| \leq L_j |x - y|.$$

In 1947, Federer [30] proved a general converse of Besicovitch's theorem: if $n < m$ and $E \subset \mathbb{R}^m$ is n-rectifiable, then for $\mathcal{H}^n$-a.e. $x \in E$

$$\lim_{r \to 0^+} \frac{\mathcal{H}^n(B(x,r) \cap E)}{\omega_n r^n} = 1, \tag{2}$$

where ω_n denotes the Lebesgue measure of the unit ball in $\mathbb{R}^n$.

We introduce some terminology that will help us set the framework. Let μ be a Radon measure in $\mathbb{R}^m$ (i.e. a Borel regular measure that is finite on compact sets). The *n-density* of μ at x

$$\theta^n(\mu, x) := \lim_{r \to 0} \frac{\mu(B(x,r))}{\omega_n r^n} \tag{3}$$

exists if the limit exists and $\theta^n(\mu, x) \in (0, \infty)$

A locally finite measure μ on $\mathbb{R}^m$ is *n-rectifiable* if μ is absolutely continuous with respect to $\mathcal{H}^n$ ($\mu \ll \mathcal{H}^n$, i.e $\mathcal{H}^n(F) = 0$ implies $\mu(F) = 0$) and

$$\mu(\mathbb{R}^m \setminus \bigcup_{j=1}^{\infty} f_j(\mathbb{R}^n)) = 0,$$

where each $f_j : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz. Recasting the results above in this light, we have in the case when $m = 2$ and $n = 1$, that by Besicovitch's work if $E \subset \mathbb{R}^m$ such that $0 < \mathcal{H}^n(E) < \infty$, and for $\mathcal{H}^n$-a.e. $x \in E$ the density of $\mu = \mathcal{H}^n \llcorner E$ exists and is 1, then E is n-rectifiable. By Federer's work the converse in any dimension is true, that is, if $E \subset \mathbb{R}^m$ is n-rectifiable then the density of $\mu = \mathcal{H}^n \llcorner E$ exists and is 1 for $\mathcal{H}^n$-a.e. $x \in E$. Two natural questions arise at this point: 1) does Besicovitch's result hold for any $n, m \in \mathbb{N}$ with $n < m$? 2) what happens if we replace $\mu = \mathcal{H}^n \llcorner E$ by a general Radon measure μ? Initially progress on these questions was slow.

In 1944, Morse and Randolph [52] proved when $m = 2$ that if μ is a Radon measure on $\mathbb{R}^m$ for which the 1-density exists μ-a.e., then μ is 1-rectifiable. In 1950, Moore [51] showed that this result holds for any m. In 1961, Marstrand [48] showed that if $E \subset \mathbb{R}^3$ and the 2-density exists for $\mathcal{H}^2 \llcorner E$, $\mathcal{H}^2$-a.e $x \in \mathbb{R}^3$, then E is 2-rectifiable. In 1975, Mattila [49] proved that if $E \subset \mathbb{R}^m$ and the n-density exists for $\mathcal{H}^n \llcorner E$, $\mathcal{H}^n$-a.e $x \in \mathbb{R}^m$, then E is n-rectifiable, completing the study of the problem for measures that were defined as the restriction of Hausdorff measure to a subset of Euclidean space. This still left open the case of a general Radon measure.

In 1987, in a true masterpiece, Preiss [56] showed that if μ is a Radon measure on $\mathbb{R}^m$ for which the n-density exists for μ-a.e $x \in \mathbb{R}^m$, then μ is n-rectifiable. Preiss introduced a number of new tools and ideas whose applications are still being unraveled and play a central role in the results to be discussed later in this article. The question of rectifiability of a measure carries information about the fine structure of its measure-theoretic support. Motivated by this perspective, Preiss introduced the notion of tangent measures, which play the role that derivatives do when analyzing the regularity of a function. They are obtained by a limiting process of rescaled multiples of the initial measure. Preiss's argument includes a number of major steps, some of which have given rise to very interesting questions. Roughly speaking, a blow up procedure shows that when the n-density of μ exists μ-a.e., then at μ-a.e. point all tangent measures are n-uniform. A measure ν is *n-uniform* if there is a constant $C > 0$ so that for $r > 0$ and x in the support $\Lambda = \operatorname{spt} \nu = \{y \in \mathbb{R}^m : \nu(B(y,s)) > 0 \text{ for every } s > 0\}$ of ν, we have

$$\nu(B(x,r)) = Cr^n. \tag{4}$$

His argument now requires a detailed understanding of the structure and geometry of the support of n-uniform measures. By work of Kirchheim and Preiss [43], the support of an n-uniform measure ν is an analytic variety. Thus, using the fact that at μ-a.e point, tangent measures to tangent measures to μ are tangent measures to μ, he showed that at μ-a.e. point there is an n-flat tangent measure; that is, a measure that is a multiple of the n-dimensional Hausdorff measure restricted to an n-plane. Then he showed that either an n-uniform measure ν is an n-flat measure or its support is very far away from any n-plane. Using a deep result about the "cones" of measures, he proves that necessarily at μ-a.e point all tangent measures are n-flat. Then modulo showing that this implies that the measure-theoretic support of μ satisfies the hypothesis of the Marstrand-Mattila rectifiability criterion, one concludes that μ is n-rectifiable.

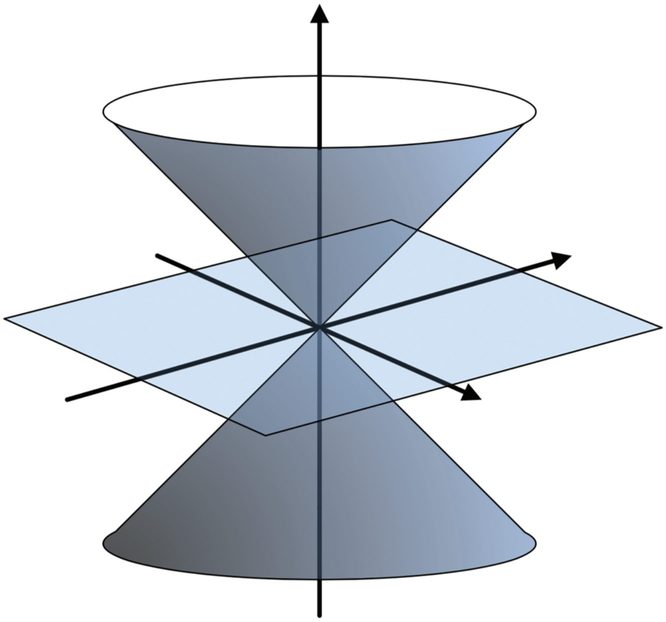

Figure 1. Kowalski-Preiss cone

A natural and easy-to-state question derived from this work is: what does the support of an n-uniform measure on $\mathbb{R}^m$ really looks like? Clearly the restriction of n-Hausdorff measure to an n-plane is n-uniform, but are there other examples? Work of Preiss [56] shows that if $n = 1, 2$ flat measures are the only examples of n-uniform measures. In 1987 Kowalski and Preiss [44] showed that if $m = n+1$ and $n \geq 3$ then, modulo rotation and translation, Λ the support of an n-uniform measure ν is either:

• $\Lambda = \mathbb{R}^n \times \{0\}$

or

• $\Lambda = \left\{(x_1, x_2, x_3, x_4, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} : x_4^2 = x_1^2 + x_2^2 + x_3^2\right\}$.

See figure above.

Thirty years later Nimer [55] produced the first examples of n-uniform measures in higher co-dimensions. His argument has an important combinatorial component and uses Archimedes' theorem, namely that, in $\mathbb{R}^3$ the surface measure of the intersection of the unit sphere with a ball of small radius r and centered on the sphere is exactly πr^2. He classifies up to isometry all conical 3-uniform measures in $\mathbb{R}^5$ and produces families of examples in any co-dimension.

Much time has elapsed between the initial work of Preiss and collaborators and the next set of examples. This illustrates a trend in this area. Indeed for years Preiss's work was perceived as somewhat impenetrable. In the early-to-mid 2000s several successful attempts were made to understand and apply some segments of the paper (see [25, 39, 46, 57]). De Lellis [28] produced a more comprehensive version of a special case of the argument, which has made this work more accessible.

Harmonic Measure

To illustrate how ideas from one field can have a profound impact in another, we will focus on some of the recent applications of Preiss's work to harmonic analysis and potential theory. The harmonic measure is a canonical measure associated to the Laplacian (see definition below). It plays a fundamental role in potential theory, constitutes the main building block for the solutions of the classical Dirichlet problem, and in non-smooth domains is the object that allows us to describe boundary regularity of the solutions to Laplace's equation. We recall some of the background. Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded domain, let f be a continuous function on the boundary of Ω, i.e. $f \in C(\partial\Omega)$. The classical Dirichlet problems asks whether there exists a function $u \in C(\overline{\Omega}) \cap W^{1,2}(\Omega)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u = f \text{ on } \partial\Omega. \end{cases} \tag{5}$$

Here $u \in W^{1,2}(\Omega)$ means that u and its weak derivatives are in $L^2(\Omega)$ and $\Delta u = 0$ is interpreted in the weak sense; that is, for any $\zeta \in C_c^1(\Omega)$,

$$\int \langle \nabla u, \nabla \zeta \rangle = 0.$$

The questions here are whether a solution u of (5) exists, if so how regular it is, and whether there is a formula in terms of f to describe it. We say that Ω is *regular* if for all $f \in C(\partial\Omega)$, any solution u of(5) is in $C(\overline{\Omega}) \cap W^{1,2}(\Omega)$. In 1923 Wiener [60] provided a remarkable characterization of regular domains using capacity. If Ω is regular, then for $x \in \Omega$ and $f \in C(\partial\Omega)$ if $u \in C(\overline{\Omega})$ is the solution to (5), by the Maximum Principle $|u(x)| \leq \max_{\partial\Omega} |f|$. Thus for $x \in \Omega$, $T_x : C(\partial\Omega) \to \mathbb{R}$ defined by $T_x(f) = u(x)$ is a bounded linear operator, with $\|T_x\| \leq 1$. Moreover $T_x(1) = 1$. Hence, by the Riesz Representation Theorem, there is a probability measure ω^x, the *harmonic measure* with pole at x, satisfying

$$u(x) = \int_{\partial\Omega} f(q) d\omega^x(q). \tag{6}$$

If Ω is regular and connected, the Harnack Principle implies that for $x, y \in \Omega$, ω^x and ω^y are mutually absolutely continuous. Thus we will often drop the pole dependence and simply refer to the harmonic measure ω rather than ω^x.

The question of whether the behavior of the harmonic measure on a given domain yields information about the structure of the boundary of the domain has attracted considerable interest over the last century, with a period of intense activity over the last two decades. The initial results in $\mathbb{R}^2$ are very satisfactory. For a simply-connected domain $\Omega \subset \mathbb{R}^2$, bounded by a Jordan curve, the boundary is a disjoint union, with the following properties:

$$\partial\Omega = G \cup S \cup N \tag{7}$$

1. In G, ω, and $\mathcal{H}^1$ are mutually absolutely continuous, which we denote by $\omega \ll \mathcal{H}^1 \ll \omega$,
2. Every point of G is the vertex of a cone in Ω. Moreover if C denotes the set of "cone points" of $\partial\Omega$, then $\mathcal{H}^1(C\backslash G) = 0$ and $\omega(C\backslash G) = 0$.
3. $\omega(N) = 0$ and $\mathcal{H}^1(S) = 0$.
4. S consists (ω a.e.) of "twist points" (see [31] for the definition).
5. For ω a.e. $q \in G$, the 1-density of ω exists and $\theta^1(\omega, q) \in (0, \infty)$ (see (3)).
6. At ω a.e. point $q \in S$ we have

$$\limsup_{r\to 0} \frac{\omega(B(q,r))}{r} = +\infty,$$
$$\liminf_{r\to 0} \frac{\omega(B(q,r))}{r} = 0.$$

These results are a combination of work of Makarov, McMillan, Pommerenke, and Choi. See [31] for the precise references.

Recall that the Hausdorff dimension of ω (denoted by $\mathcal{H} - \dim \omega$) is defined by

$$\begin{aligned} \mathcal{H} - \dim \omega = \inf \{k : \text{ there exists } E \subset \partial\Omega \\ \text{with } \mathcal{H}^k(E) = 0 \text{ and} \\ \omega(E \cap K) = \omega(\partial\Omega \cap K) \\ \text{for all compact sets } K \subset \mathbb{R}^{n+1}\} \end{aligned} \tag{8}$$

Important work of Makarov [47] shows that for simply connected domains in $\mathbb{R}^2$, $\mathcal{H} - \dim \omega = 1$, establishing Oksendal's conjecture (i.e. for what type of domains in $\mathbb{R}^{n+1}$ is $\mathcal{H} - \dim \omega = n$?) in dimension 2. Carleson [22], and Jones and Wolff [38] proved that, in general, for domains in $\mathbb{R}^2$ with a well defined harmonic measure ω, $\mathcal{H} - \dim \omega \leq 1$. Bourgain showed that there exists $\epsilon(n) > 0$ such that for domains in $\mathbb{R}^{n+1}$ with a well defined harmonic measure ω, $\mathcal{H} - \dim \omega \leq n + 1 - \epsilon(n)$, see [21]. Finding the optimal bound for this Hausdorff dimension is an important open question in the area.

T. Wolff [61] showed, by a deep example, that, for $n \geq 2$, Oksendal's conjecture (that $\mathcal{H} - \dim \omega = n$) fails. He constructed what are known as "Wolff snowflakes," domains in $\mathbb{R}^3$, for which $\mathcal{H} - \dim \omega > 2$ and others for which $\mathcal{H} - \dim \omega < 2$. In Wolff's construction, the domains have a certain weak regularity property. They are non-tangentially accessible domains (NTA) in the sense of [37]. In fact, they are 2-sided NTA domains (i.e. $\Omega^+ = \Omega$ and $\Omega^- = \mathrm{int}(\Omega^c)$ are both NTA) which plays an important role in his estimates. NTA domains are open, connected, and Wiener regular in a quantitative way. In [45], Lewis, Verchota, and Vogel reexamined Wolff's construction and were able to produce "Wolff snowflakes" in $\mathbb{R}^{n+1}$, $n \geq 2$ for which either $\mathcal{H} - \dim \omega^\pm < n$ or $\mathcal{H} - \dim \omega^\pm > n$, where $\omega^\pm$ denote the harmonic measure of $\Omega^\pm$. They also observed, as a consequence of the monotonicity formula in [2], that if $\omega^+ \ll \omega^- \ll \omega^+$, then $\mathcal{H} - \dim \omega^\pm \geq n$.

Returning to the case of $n = 1$, when $\Omega \subset \mathbb{R}^2$ is again simply connected, and bounded by a Jordan curve, Bishop, Carleson, Garnett, and Jones [19] showed that if $E \subset \partial\Omega$, $\omega^\pm(E) > 0$, then ω^+ and ω^- are mutually singular (i.e. $\omega^+ \perp \omega^-$) on E if and only if $\mathcal{H}^1(Tn(\partial\Omega) \cap E) = 0$, where $Tn(\partial\Omega) \subset \partial\Omega$ is the set of points in $\partial\Omega$ where $\partial\Omega$ has a unique tangent line. Let $E \subset \partial\Omega$ be such that $\omega^+ \ll \omega^- \ll \omega^+$ on E and $\omega^\pm(E) > 0$. Then, because of [19], modulo sets of $\omega^\pm$ measure 0, $E \subset Tn(\partial\Omega)$. Using Beurling's inequality, i.e., the fact that for $q \in \partial\Omega$ and $r > 0$, $\omega^+(B(q,r))\omega^-(B(q,r)) \leq Cr^2$, and the characterization above where $\partial\Omega = G^\pm \cup S^\pm \cup N^\pm$ (see (7)), we conclude that $\omega^+ \ll \mathcal{H}^1 \ll \omega^- \ll \omega^+$ on E. Thus, the set of mutual absolute continuity of ω^-, ω^+ is a subset of $G^+ \cap G^-$ and hence of Hausdorff dimension 1.

In [18], motivated by this last result, Bishop asked if in the case of $\mathbb{R}^{n+1}$, $n \geq 2$, the fact that ω^-, ω^+ are mutually absolutely continuous on a set $E \subset \partial\Omega$, with $\omega^\pm(E) > 0$, implies that $\omega^\pm$ are mutually absolutely continuous with respect to $\mathcal{H}^n$ on E and hence $\dim_{\mathcal{H}}(E) = n$, where $\dim_{\mathcal{H}}$ denotes the Hausdorff dimension of a set.

Two Phase Case

While in [40] and [41] we had already used some of the properties of the tangent measures, Bishop's question plus the desire to understand the Wolff snowflakes better led us to dig deep into Preiss's work [56]. There we found the necessary tools to start tackling the problem of describing the boundary in terms of $\omega^\pm$. In [39, Kenig, Preiss and the author] prove the following result:

Theorem 1 ([39]). *For $n \geq 3$, if $\Omega \subset \mathbb{R}^{n+1}$ is a 2-sided NTA domain, then*

$$\partial\Omega = \Gamma \cup S \cup N, \tag{9}$$

1. $\omega^+ \perp \omega^-$ *on* S *and* $\omega^\pm(N) = 0$.
2. *On* Γ, $\omega^+ \ll \omega^- \ll \omega^+$, $\dim_{\mathcal{H}} \Gamma \leq n$.
3. *If* $\omega^\pm(\Gamma) > 0$, $\dim_{\mathcal{H}} \Gamma = n$.
4. *If* $\mathcal{H}^n \llcorner \partial\Omega$ *is a Radon measure then* Γ *is* n*-rectifiable, and* $\omega^- \ll \mathcal{H}^n \ll \omega^+ \ll \omega^-$ *on* Γ.

As a consequence there can be no Wolff snowflake for which ω^+, ω^- are mutually absolutely continuous. Theorem 1 answered Bishop's question under the assumption that $\mathcal{H}^n \llcorner \partial\Omega$ is a Radon measure. The general case was left open, and it was clear that a new idea was needed to deal with the main obstacle, namely the set of points for which the n-density of the harmonic measure is 0. A noteworthy issue in this branch of GMT is that difficulties often arise from either a measure that is not locally finite or a measure whose appropriate density is zero.

The proof of Theorem 1 uses tools from the theory of non-tangentially accessible domains (NTA) introduced by Jerison and Kenig [37], the monotonicity formula of Alt, Caffarelli, and Friedman [2], the theory of tangent measures introduced by Preiss [56], and the blow up techniques for harmonic measures at infinity for unbounded NTA domains due to Kenig and Toro [40, 41]. For additional results along these lines see [11–14, 29].

We describe the main steps to emphasize the similarities with the train of thought present in Preiss's work. To accomplish our objective, we use the blow-up analysis developed in [41]. At $\omega^\pm$ a.e. point on the set where ω^+ and ω^- are mutually absolutely continuous, the tangent measures to $\omega^\pm$ (in the sense of [50], [56]) are harmonic measures associated to the Laplacian on domains where a harmonic polynomial is either positive or negative. The resulting harmonic measure is supported on the zero set of this harmonic polynomial. Using the fact that for almost every point a tangent measure to a tangent measure is a tangent measure (see [50]) and the fact that the zero set of a harmonic polynomial in $\mathbb{R}^{n+1}$ is smooth except for a set of Hausdorff dimension $n-1$ (see [33]), one shows that at $\omega^\pm$ a.e. point on this set, n-flat measures always arise as tangent measures to $\omega^\pm$. They correspond to linear harmonic polynomials. We then show and this is the crucial step, that if one tangent measure is flat on the set of mutual absolute continuity, then all tangent measures are flat. To accomplish this we use a connectivity argument similar to the one from [56]. The key point is that if a tangent measure is not flat, being the harmonic measure supported on the zero set of a harmonic polynomial of degree higher than 1, its tangent measure at infinity is far from flat, and a connectivity argument, as in [56], gives a contradiction. Modulo a set of $\omega^\pm$ measure 0, Γ as in (10), is the set of mutual absolute continuity for which one (and hence all) tangent measures are n-flat. An easy argument then shows that $\dim_{\mathcal{H}} \Gamma \le n$. To conclude that if $\omega^\pm(\Gamma) > 0$, $\dim_{\mathcal{H}} \Gamma = n$, one uses the Alt-Caffarelli-Friedman monotonicity formula of [2] as in [45]. This yields a version of Beurling's inequality in higher dimensions. If $\sigma = \mathcal{H}^n \llcorner \partial\Omega$, the surface measure to the boundary, is a Radon measure, we show that the n-density of σ is 1 a.e., which by Preiss's theorem ensures that Γ is n-rectifiable.

In a remarkable paper, Azzam, Mourgoglou, and Tolsa [8] answer Bishop's question completely. Although their result holds in greater generality, we state it here in the context of the discussion above for simplicity.

Theorem 2 ([8]). *For $n \ge 2$, if $\Omega \subset \mathbb{R}^{n+1}$ is a 2-sided NTA domain, then*

$$\partial\Omega = G \cup S \cup N', \tag{10}$$

1. *$\omega^+ \perp \omega^-$ on S and $\omega^\pm(N') = 0$.*
2. *G is n-rectifiable and $\omega^- \ll \mathcal{H}^n \ll \omega^+ \ll \omega^-$.*

The main innovation in [8] is the introduction of a new set of ideas involving the n-dimensional Riesz transform. In particular they use a result by Girela-Sarrión and Tolsa [32] concerning the connection between Riesz transforms and quantititative rectifiability for general Radon measures. This allows them to deal with the set of points for which the n-density of the harmonic measure is 0. The connection between the Riesz transform and harmonic measure stems from the fact that the Riesz kernel is the gradient of the Newtonian potential. The relationship between the Riesz transform and rectifiability has been an important component in the development of quantitative geometric measure theory, a field initiated by David and Semmes (see [26],[27]) in the early 1990s, and embraced by a large community. Quantitative GMT has developed into a vibrant area in which several important milestones have been accomplished in recent years (e.g. the solution of the David-Semmes conjecture by Nazarov, Tolsa, and Volberg [53, 54]). In a subsequent paper, Azzam, Mourgoglou, Tolsa, and Volberg significantly relax the hypothesis on the domains for which Theorem 2 holds [10].

We note that the narrative started with a question from potential analysis. The initial results were the product of a successful approach taking a GMT point of view. Once this work was in place, questions arose that required deep results in harmonic analysis and quantitative GMT to be tackled. The final outcome lies in the interface of potential theory and geometric measure theory. These results illustrate how the synergy between very distinct areas can produce truly unique and unexpected results.

One Phase Case

In the Harmonic Measure section, we started by asking whether the behavior of the harmonic measure of a domain yields information about the structure of its boundary (one phase case). The discussion very quickly turned to the situation where we consider the harmonic measures of a set and its complement (two phase case). The rationale was that the two phase case was more clearly related to GMT. We now return to the one phase case, where both the quantitative and qualitative questions have sparked an incredible amount of interest, generating lots of activity that has culminated in truly optimal results.

In 1916 F. and M. Riesz proved that for a simply connected domain in the complex plane with a rectifiable boundary, harmonic measure is absolutely continuous with respect to arc length measure on the boundary [58].

Bishop and Jones [20] have shown that in this type of domain, if only a portion of the boundary is rectifiable, then harmonic measure is absolutely continuous with respect to arc length on that portion. They also showed that the result of [58] may fail in the absence of some topological hypothesis (e.g., simple connectedness). Examples constructed in [62] and [63] show that, in higher dimensions, some topological restrictions, even stronger than those needed in the planar case, are required for the absolute continuity of ω with respect to surface measure to the boundary.

Higher dimensional analogues of this question have played a central role in the development of the study of partial differential equations in non-smooth domains. In 1982 Dahlberg [23] showed the harmonic measure of a Lipschitz domain and the surface measure to its boundary are mutually absolutely continuous (in a quantitative scale-invariant way, namely $\omega \in A_\infty(\sigma)$). Similar results hold on chord arc domains (these are NTA domains for which the surface measure to the boundary is Ahlfors regular; that is, the surface measure of a ball centered on the boundary and of radius r grows like r^n) (see [24], [59]). The relationship between quantitative absolute continuity properties of harmonic measure with respect to surface measure and the regularity of the boundary (also expressed in quantitative terms) is now very well understood, see for example [3, 6, 7, 15, 34–36].

Here we only focus on the optimal qualitative result that provides a complete answer to Bishop's question. In a very interesting piece of work, Azzam, Hofmann, Martell, Mayboroda, Mourgoglou, Tolsa, and Volberg show a converse to the results in [20, 58] in all dimensions. See [4, 5]

Theorem 3 ([4], [5]). *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open connected set and let ω be the harmonic measure in Ω. Let $E \subset \partial\Omega$ with Hausdorff measure $\mathcal{H}^n(E) < \infty$.*

- *If ω is absolutely continuous with respect to $\mathcal{H}^n$ on E, then ω restricted to E is an n-rectifiable measure.*
- *If $\mathcal{H}^n$ is absolutely continuous with respect to ω on E, then E is an n-recitfiable set.*

This theorem can be understood as a free boundary regularity problem for the harmonic measure (an initial example of this type of question appears in [42]) where the goal is to obtain regularity of the boundary. The authors appeal to the *magic* of the Riesz transform, which for a measure μ in $\mathbb{R}^{n+1}$ is defined as follows:

$$\mathcal{R}\mu(x) = \int \frac{x-y}{|x-y|^{n+1}}\, d\mu(y). \tag{11}$$

They manage to exploit the absolute continuity hypothesis to obtain a series of estimates on the harmonic measure and the corresponding Green function that allow them to show that the Riesz transform is a bounded operator. Then they appeal the work of Nazarov, Tolsa, and Volberg [53, 54] where the authors prove the David-Semmes conjecture in co-dimension 1; that is, they show that the boundedness of the Riesz transform of a measure implies its rectifiability.

Note that the description above of the results in the area does not include a qualitative version of the F. and M. Riesz type result in higher dimensions. At this stage, works of [1, 9, 15] indicate that obtaining an optimal condition on a domain to ensure that rectifiability of the boundary implies absolute continuity of harmonic measure with respect to the surface measure is challenging.

This field has been evolving in several interesting new directions. They all fit under the umbrella of understanding the structure of the support of a measure associated to a differential operator in a canonical way. One direction concerns understanding questions similar to those discussed in the three previous sections for the elliptic measure of a uniformly elliptic second order divergence form operator. Another one looks at the problems analogous to those appearing in the Harmonic Measure and One Phase Case sections for the elliptic measure corresponding to a degenerate elliptic operator on a domain in $\mathbb{R}^{n+1}$ whose boundary has dimension strictly less than n. The unifying trait is the beautiful cross-pollination between geometric measure theory, potential theory, harmonic analysis, and partial differential equation. The expectation is that this synergy will continue to uncover unsuspected connections, leading to the development of the field in ways that are only possible thanks to the contributions from a diverse group of analysts.

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Tatiana Toro

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