CHAPTER 4

The Black-Scholes Equation

We have stated that standard Brownian motion with drift is described by the stochastic differential equation

$$dx = \mu\, dt + \sigma\, dW$$

$$x(t_0) = x_0$$

or rigorously,

$$x(t) = x_0 + \int_{t_0}^{t} ds + \int_{t_0}^{t} dW(s).$$

Since

$$\int_{t_0}^{t} dW(s) = \lim_{\Delta s \to 0} \sum_{j=0}^{N-1} [W(s_{j+1}) - W(s_j)] = W(t) - W(t_0)$$

for $\Delta s = (t - t_0)/N$, we see that the Brownian motion is given explicitly by

$$x(t) = x_0 + \mu(t - t_0) + \sigma[W(t) - W(t_0)].$$

We may now ask the question how does a function change which depends smoothly on a stochastic variable. Let us illustrate the complications brought into the picture by the stochastic component.

The model for an equity asset is not the simple Brownian motion with drift but

$$dS = \mu S\, dt + \sigma S\, dW.$$

How does one solve this equation? One might think that the equation is equivalent to

$$du = \mu\, dt + \sigma\, dW$$

for $u = \ln S$ so that the problem is is reduced to simple Brownian motion. However, S is a stochastic variable and not differentiable so that the chain rule cannot be applied to conclude that

$$du(t) = \frac{dS(t)}{S(t)}\,.$$

Instead we need a new tool, called Itô's lemma, to determine how a function of a stochastic variable varies with changes of the independent variable.

We need the following estimates from the theory of Brownian motion for the increment dW over the infinitesimal time interval dt:

$$E(dt\,dW) = 0$$

$$\mathrm{var}(dt\,dW) = o(dt)$$

$$E((dW)^2) = dt$$

$$\mathrm{var}((dW)^2) = o(dt)$$

where we say that $f(t) = o(g(t))$ as $t \to 0$ if

$$\lim_{t\to 0} \frac{f(t)}{g(t)} = 0$$

i.e. $f(t)$ goes to zero faster than $g(t)$ as $t \to 0$. Note that these estimates say that $dW^2 \to dt$ and $dt\,dW \to 0$ as $dt \to 0$ so that in the limit these quantities are no longer stochastic. We can now state Itô's lemma:

Let X satisfy

$$dX(t) = a(X,t)dt + b(X,t)dW$$
$$X(0) = X.$$

Assume that $u(x,t)$ is a smooth function of the independent variables x and t. Then

$$du(t) = \left[\frac{\partial u}{\partial t} + a(X,t)\,\frac{\partial u}{\partial x} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\,b(X,t)^2\right] dt + b(X,t)\,\frac{\partial u}{\partial x}\,dW.$$

The proof of this lemma is based on the Taylor expansion

$$du = \frac{\partial u}{\partial x}\,dx \frac{\partial u}{\partial t}\,dt + \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\,dx^2 + 2\,\frac{\partial^2 u}{\partial t \partial x}\,dt\,dx + \frac{\partial^2 u}{\partial t^2}\,dt^2\right] + \cdots$$

We now take $dx = dX$, substitute for dX from the stochastic differential equation and collect terms of order $\sqrt{dt}$. Here we use $dW^2 = dt$ and that $dW\,dt$ is negligible. The dominant term is $b(X,t)\frac{\partial u}{\partial x}\,dW$ since dW behaves like $\sqrt{dt}$.

If we apply Itô's lemma to $u = \ln s$ where

$$dS(t) = \mu S\,dt + \sigma S\,dW$$

then (with $X = S$, $a(S,t) = \mu S$, $b(S,t) = \sigma S$) we find

$$\frac{\partial u}{\partial s} = \frac{1}{s}\,, \qquad \frac{\partial^2 u}{\partial s^2} = -\frac{1}{s^2}\,, \qquad \frac{\partial u}{\partial t} = 0$$

so that

$$du(t) = [\mu - \sigma^2/2]dt + \sigma\, dW.$$

This is the equation for Brownian motion which has the solution

$$u(t) = u(t_0) + (\mu - \sigma^2/2)(t - t_0) + \sigma[W(t) - W(t_0)]$$

so that

$$S(t) = S(t_0) \exp\left\{(\mu - \sigma^2/2)(t - t_0) + \sigma[W(t) - W(t_0)]\right\}.$$

While we shall need this representation of $S(t)$ later on in connection with the binomial method we are interested at this point in deriving the Black-Scholes equation for the value of an equity option where the underlying asset satisfies equation ().

Let V be the value of a put or call written on an underlying asset with value $S(t)$ at time t. We assume that V depends differentiably on the two independent variables S and t, where S itself moves randomly according to (). Then according to Itô's lemma, V changes over the infinitesimal time interval dt according to

$$dV = \left[\frac{\partial V}{\partial t} + \mu S\, \frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S\, \frac{\partial V}{\partial S}\, dW.$$

Let us now assume that we have a portfolio consisting of one option of value V and Δ shares of the underlying where Δ is as yet undetermined, with $\Delta > 0$ for shares held long and $\Delta < 0$ for shares held short. The value of the portfolio at any time t is

$$\pi = V(S,t) + \Delta S.$$

Over the time interval dt the gain in the value of the portfolio is

$$d\pi = dV(S,t) + \Delta\, dS$$

i.e.

$$d\pi = \left[\frac{\partial V}{\partial t} + \mu S\, \frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2}\right] dt + \sigma S\, \frac{\partial V}{\partial S}\, dW + \Delta[\mu S\, dt + \sigma S\, dW].$$

We now observe that if $\Delta = -\frac{\partial V}{\partial S}$ then the stochastic terms cancel so that the gain is deterministic. If the gain in the value of π is deterministic, then it cannot be more or less than the gain in the value of the portfolio were it invested at the risk free interest rate r. It follows that also

$$d\pi = r\pi\, dt = r\left[V - \frac{\partial V}{\partial S}\, S\right] dt.$$

Equating these two expressions for $d\pi$ we find that

$$\frac{1}{2}\,\sigma^2 S^2\, \frac{\partial^2 V}{\partial S^2} + rS\,\frac{\partial V}{\partial S} - rV + \frac{\partial V}{\partial t} = 0.$$

This is the famous Black Scholes equation for the value of an option. It provides quantitative information to continuously buy or sell assets to maintain a portfolio that grows at the riskless rate and thus provides insurance against downturns in the value of assets held long or protect against a rise in the value of assets held short. In other words, the portfolio is hedged against losses, so that the option serves as an insurance policy.

Conversely, a quoted option price may be inconsistent with the value of the option as predicted by the Black Scholes equation. In this case it is possible to construct a portfolio which is guaranteed to outperform a riskless investment of the same magnitude. This possibility is called arbitrage.

We note that a number of assumptions were made in the derivation of the Black Scholes equation:

i) The value of the asset can be described by the equation for geometric Brownian motion

$$dS = \mu S\, dt + \sigma S\, dW.$$

ii) Options and shares can be bought and sold at any time since Δ changes smoothly with time.

iii) $\partial V/\partial S$ is a smooth function of S; hence the number of shares in π is allowed to vary continuously with S which means that fractional shares can be bought.

iv) The change in the value of the portfolio is due solely to the change of V and S and does not include transactions costs or the spread between selling and buying prices for options and assets.

v) All options and assets can be freely bought and sold.

The resulting equation is a mathematical model for the value of an option. It holds for all options depending on S and t as long as S is modeled by the equation of geometric Brownian motion (). It has become the dominant model for the pricing of options.

The Black Scholes equation is an example of a diffusion equation. In order to guarantee that it has a unique solution one needs initial and boundary conditions. These will be determined by the specific option under consideration. We shall consider first the simplest case of a European put (and call) to indicate where the Black Scholes formula comes from.

Let us complete the model for a European put with strike price K and expiration at $t = T$ by deriving initial and boundary conditions. We shall denote the value of this option by $P(S,t)$. It must satisfy the Black Scholes equation. Moreover, at $t = T$ we know the value of the option. If $S(T) > K$ the option will not be exercised so it has no value, whereas if $S(T) < K$ then the option will be exercised and the gain to the holder is

$$K - S(T).$$

Hence

$$P(S,T) = \max\{0, K - S\}.$$

Moreover, it follows from the stochastic differential equation that if $S(t) = 0$ for any $t \in [0,T]$ then $S(T) = 0$ so that the option will be exercised at time T and return a gain of K. The present value of this gain is

$$P(0,t) = Ke^{-r(T-t)}.$$

Finally, if $S(t)$ becomes very large then it is unlikely that it will fall back below K at time T so the put would not be exercised. Hence

$$\lim_{S\to\infty} P(S,t) = 0.$$

Initial and boundary values for other options, e.g. for an American put, will be introduced later.

As we know, the value of this put is given in terms of the Black Scholes formula introduced in Chap. 1. Its appearance is not mysterious because the problem for a European option can be reduced to a pure initial value problem for the so-called heat equation:

$$u_{yy} - u_\tau = 0$$

$$u(y,0) = u_0(y), \qquad \infty - < y < \infty.$$

This problem arises frequently in science and engineering, and it has been known long before the advent of quantitative finance that it can be solved by the formula

$$u(y,\tau) = \int_{-\infty}^{\infty} s(y-z,\tau)u_0(z)dz$$

where

$$s(y,t) = \frac{1}{\sqrt{4\pi t}}\, e^{-\frac{y^2}{4t}}\,.$$

Let us outline the basic steps involved in deriving the Black Scholes formula from this integral.

We make the change of variable

$$y = \ln S/K,$$

then $0 < S < \infty$ corresponds to $-\infty < y < \infty$. Moreover,

$$\frac{\partial P}{\partial S} = \frac{\partial P}{\partial y}\frac{1}{S} \qquad \text{and} \qquad \frac{\partial^2 P}{\partial S^2} = \frac{\partial^2 P}{\partial y^2}\frac{1}{S^2} - \frac{\partial P}{\partial y}\frac{1}{S^2}\,.$$

Substitution into the Black Scholes equation produces the constant coefficient equation

$$\frac{1}{2}\,\sigma^2\,\frac{\partial^2 P}{\partial y^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial P}{\partial y} - rP + \frac{\partial P}{\partial t} = 0.$$

Next we shall introduce a new dependent variable $u(y,t)$ by writing

$$P(y,t) = Ke^{\alpha(T-t)+\beta y}u(y,t).$$

Differentiation and substitution into () shows that u satisfies the equation

$$\begin{aligned}\frac{1}{2}\,\sigma^2\left[\frac{\partial^2 u}{\partial y^2} + 2\beta\,\frac{\partial u}{\partial y} + \beta^2 u\right] + \left(r - \frac{1}{2}\,\sigma^2\right)\left[\frac{\partial u}{\partial y} + \beta u\right]&\\ - ru - \alpha u + \frac{\partial u}{\partial t} = 0&\end{aligned}$$

$$u(y,T) = e^{-\beta y}\max\{1 - e^y, 0\}.$$

The coefficients of $\frac{\partial u}{\partial y}$ and u vanish if we choose

$$\sigma^2\beta + \left(r - \frac{1}{2}\,\sigma^2\right) = 0$$

$$\frac{1}{2}\sigma^2\beta^2 + \beta\left(r - \frac{1}{2}\sigma^2\right) - r - \alpha = 0.$$

Then

$$\frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial t} = 0$$

$$\begin{aligned} u(y,T) &= e^{-\beta y}\max\{0, 1-e^y\} \\ &= \begin{cases} e^{-\beta y} - e^{(1-\beta)y} & y \le 0 \\ 0 & y > 0. \end{cases} \end{aligned}$$

Finally, let

$$\tau = \frac{1}{2}\sigma^2(T-t)$$

then the initial value problem in the y, τ coordinates becomes

$$\begin{aligned} \frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial \tau} &= 0 \\ u(y,0) &= \begin{cases} e^{-\beta y} - e^{(1-\beta)y} & y \le 0 \\ 0 & y > 0. \end{cases} \end{aligned}$$

According to (1) the solution is

$$\begin{aligned} u(y,t) = \frac{1}{\sqrt{4\pi\tau}} &\int_{-\infty}^{0} \exp\left[-\frac{(y-z)^2}{4\tau} - \beta z\right] dz \\ &- \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{0} \exp\left[-\frac{(y-z)^2}{4\tau} + (1-\beta)z\right] dz. \end{aligned} \tag{1}$$

The integrals can be converted to probability density integrals by completing the square. We write, for example for the first integral,

$$\frac{(y^2 - 2yz + z^2 + 4\beta\tau z)}{4\tau} = \frac{(z + (2\beta\tau - y))^2}{4\tau} - \beta^2\tau + \beta y$$

and make the change of variable

$$\frac{z + 2\beta\tau - y}{\sqrt{4\tau}} = \frac{s}{\sqrt{2}}$$

then the first integral of (1) becomes

$$e^{\beta^2\tau - \beta y}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\frac{2\beta\tau - y}{\sqrt{2\tau}}} e^{-s^2/2} ds.$$

It follows from the defining relationship for β and τ that

$$\beta^2\tau - \beta y = \beta^2\frac{\sigma^2}{2}(T-t) - \beta y = (-r-\alpha)(T-t) - \beta y$$

and that

$$\frac{2\beta\tau - y}{\sqrt{2\tau}} = -\frac{y + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}.$$

Let

$$d_2 = \frac{y + (r-\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

then

$$\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{0} \exp\left[-\frac{(y-z)^2}{4\tau} - \beta z\right] dz = e^{-r(T-t)}e^{-\alpha(T-t)}e^{-\beta y}N(-d_2).$$

The value of the second integral follows if we formally replace β by $\beta - 1$. Using the defining expressions for β and τ we find that

$$\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty}^{0} \exp\left[-\frac{(y-z)^2}{4\tau} + (1-\beta)z\right] dz = e^{-\alpha(T-t)}e^{-\beta y}e^{y}N(-d_1)$$

where

$$d_1 = \frac{y + (r+\sigma^2/2)(T-t)}{T-t}.$$

It follows that

$$P(S,t) = Ke^{-r(T-t)}N(-d_2) - SN(-d_1)$$

where $y = \ln(S/K)$ in the expressions for d_1 and d_2. We note that $N(-x) = 1 - N(x)$ so that also

$$P(S,t) = SN(d_1) - KN(d_2) - (S - Ke^{-r(T-t)})$$

or

$$P(S,t) = C(S,t) - (S - Ke^{-r(T-t)}) \tag{2}$$

where C is the price of a European call at time t. The formula for a call at $t = 0$ was already used in Chapter 1 for the implied volatility calculations. Equation (2) is known as the put-call parity. Since both P and C satisfy the Black-Scholes equation, their difference also must be a solution of the Black Scholes equation. It is straightforward to verify that

$$(S,t) = aS + be^{rt}$$

is a solution of the Black-Scholes equation for all constants a and b.

We observe that the initial and boundary conditions for a call consistent with the put-call parity are

$$C(S,T) = \max\{0, S-K\}$$

$$C(0,t) = Ke^{-r(T-t)}$$

$$C(S,t) \to S - Ke^{-r(T-t)} \qquad \text{as } S \to \infty.$$

We also note that any numerical evaluation of the Black-Scholes formula does involve numerical techniques since $N(x)$ cannot be found explicitly. However, very accurate numerical approximations for $N(x)$ have been developed and added to program libraries so that for all practical purposes $N(x)$ is available for all x.

The Black-Scholes equation for European and some related options can be solved by formula, but for many other options, notably American puts and calls, the boundary conditions cannot be satisfied by the Black-Scholes formula. Hence an analytic solution is not available and one has to resort to numerical methods. While such methods will always be applied to the Black Scholes equation in its original form, except perhaps for scaling out the strike price K, their analysis is usually carried out for a simple model problem. If methods fail to solve the model problem then they usually will fail in a more complex setting.

The model problem associated with the Black-Scholes equation is the pure initial value problem

$$Lu \equiv \rho u_{xx} - u_t = 0 \tag{4c1}$$

$$u(x,0) = e^{ikx} \qquad -\infty < x < \infty$$

where $\rho > 0$ and where k is an arbitrary but fixed integer. By inspection we see that

$$u(x,t) = e^{ikx - k^2 \rho t}$$

solves this problem. It follows that for all k the solution $u(x,t)$ is bounded for all x and t.

A numerical method for this problem typically involves a finite dimensional algebraic approximation to the problem. The approximation in general depends on a discretization parameter which we shall call h. In this exposition we shall consider only finite difference methods for (4c1) where h will be identified with the mesh sizes of the grid on which the differential equation is approximated. There are many different finite difference methods for this and related problems, but all must satisfy the same requirements.

I. The numerical approximation must be consistent so that one correctly approximates the given problem as $h \to 0$.

II. The numerical method must be convergent as $h \to 0$ so that the value of the numerical solution at a given point approaches that of the analytic solution as $h \to 0$.

III. The numerical method must be stable. This means that the value at an arbitrary fixed point in the domain of computation must remain bounded as $h \to 0$.

We shall make these notions more precise when we talk about specific numerical methods for (4c1).

Intuitively, convergence and stability would seem closely related. This is indeed the case for the so-called "well-posed" problems. A problem is well-posed if it has a unique solution which depends continuously on the data of the problem. The connection between consistency, convergence and stability is given by the famous

Lax Equivalence Theorem: *Given a well-posed (linear) initial value problem and a consistent finite difference approximation to it, stability is necessary and sufficient for convergence.*

The practical importance of the theorem is due to the fact that consistency and stability are often easy to establish while convergence of a method may require more work. For example, look at the amount of work required to establish convergence of Euler's method discussed in Chapter 3.

It can be shown that the initial value problem for (4c1) is indeed well posed so that we need only be concerned with consistency and stability of the numerical method.

For the numerical integration of (4c1) over a time interval $[0, T]$ we shall use a three point finite difference method known as the explicit Euler method. Let

$$t_n = n\Delta t, \qquad n = 0, \ldots, N$$

$$x_j = -X + j\Delta x \qquad j = 0, \ldots, M$$

where

$$\Delta t = T/N$$

$$\Delta x = 2X/M$$

and where $\{X, M, N\}$ are chosen so that the mesh points $\{x_j, t_n\}$ cover the region over which we need a numerical solution.

We now approximate (4c1) with the explicit finite difference formula

$$L_h u_j^n \equiv \rho \, \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \frac{u_j^{n+1} - u_j^n}{\Delta t} = 0. \tag{4c2}$$

This formula is indeed explicit since given values at $j-1$, j and $j+1$ at time level n one can solve explicit for

$$u_j^{n+1} = u_j^n + \frac{\Delta t}{\Delta x^2} \left[u_{j+1}^n + u_{j-1}^n - 2u_j^n \right]. \tag{4c3}$$

Note that for a pure initial value problem the solution is computed on a narrowing mesh since at each new time level the right and left endpoints of the x-mesh have to be moved inward one Δx step from the endpoints at the preceding time level. In particular, if the solution at $t = T$ is desired at only one point x_j one can restrict the computation to a triangular domain with apex at x_j.

In our finite difference approximation there are two mesh parameters Δt and Δx. We shall henceforth make the assumption that

$$\Delta x = g(\Delta t)$$

where $g(r) \to 0$ as $r \to 0$, i.e., Δx goes to zero as Δt goes to zero. Thus the mesh parameter h can be identified with Δt. The finite difference approximation (4c2) is a consistent approximation to (4c1) if

$$\lim_{h \to 0} [L\phi(x_j, t_n) - L_h \phi(x_j, t_n)] = 0$$

for an arbitrary infinitely differentiable function $\phi(x, t)$. This property is easy to verify since Taylor's theorem yields

$$\frac{\phi(x_j, t_{n+1}) - \phi(x_j, t_n)}{\Delta t} = \phi_t(x_j, t_n) + O(\Delta t)$$

and

$$\frac{\phi(x_{j+1},t_n)+\phi(x_{j-1},t_n)-2\phi(x_j,t_n)}{\Delta x^2}=\phi_{xx}(x,t)+O(\Delta x).$$

We see that L_h correctly approximates L provided only that $\Delta t \to 0$ and $\Delta x \to 0$. Let us now turn to stability and observe the behavior of

$$u_j^n$$

at a given fixed point (x_j, t_n) where n is the number of time steps it took to reach the fixed value t_n as $\Delta t = T/N \to 0$. Note that $n \to \infty$ as $N \to \infty$ but $n\Delta t \leq T$.

We take as initial condition

$$u_j^0 = e^{ikx_j}, \qquad j = 0, \ldots, M.$$

Substitution into (4c3) shows that for $n = 1$ we have

$$u_j^1 = \left[1+\rho\,\frac{\Delta t}{\Delta x^2}\left[e^{ik\Delta x}+e^{-ik\Delta x}-2\right]e^{ikx_j}\right]$$

so that

$$u_j^1 = A(k,\Delta t)e^{ikx_j} \qquad \text{for all } j$$

where

$$A(k,\Delta t)=1+\rho\,\frac{\Delta t}{\Delta x^2}\left[e^{ik\Delta x}+e^{-ik\Delta x}-2\right].$$

If we proceed from time level to time level we find that

$$u_j^n = A(k,\Delta t)^n e^{ikx_j}.$$

It follows that u_j^n will remain bounded if

$$|A(k,\Delta t)| \leq 1.$$

Since

$$A(k,\Delta t)=1+\rho\frac{2\Delta t}{\Delta x^2}[\cos k\Delta x-1]$$

we see that

$$1-\rho\frac{4\Delta t}{\Delta x^2}\leq A(k,\Delta t)\leq 1$$

so that

$$|A(k, \Delta t)| \leq 1$$

whenever

$$\rho \, \frac{4\Delta t}{\Delta x^2} \leq 2$$

or

$$\rho \frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}\,. \tag{4c4}$$

The inequality (4c4) is sufficient for stability. One might argue that for certain values of k the condition can be relaxed because our estimates are based on the worst case scenario of

$$\cos k\Delta x - 1 = -2.$$

However, in applications the initial condition will be a general function $u_0(x)$. Many such functions can be approximated by the complex Fourier series

$$u_0(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}.$$

The numerical solution now will be a superposition of the solutions for each k. Hence one must expect that for some k and Δx one can attain $\cos k\Delta x - 1 = -2$. Thus we cannot allow

$$\rho \, \frac{\Delta t}{\Delta x^2} \geq \frac{1}{2} + \epsilon \qquad \text{for a fixed } \epsilon > 0$$

for all Δt as $\Delta t \to 0$.

Let us examine the implication of this stability restriction for the numerical solution of the Black-Scholes equation. As we have seen, the equation is formally equivalent to

$$\frac{1}{2}\,\sigma^2 u_{yy} - u_t = 0.$$

The numerical integration of this equation with the explicit Euler method on a uniform grid is subject to the stability condition

$$\sigma^2 \frac{\Delta t}{\Delta y^2} \leq 1.$$

Note that a uniform grid on the y-axis corresponds to a non-uniform grid on the x-axis for the original scaled Black-Scholes equation, with

$$\Delta x_i = x_i(e^{\Delta y} - 1) \approx x_i \Delta y.$$

The stability restriction in terms of x and t then is

$$x_i^2 \sigma^2 \frac{\Delta t}{\Delta x_i^2} \leq 1.$$

While in principle the Black-Scholes equation can be transformed into a constant coefficient equation it is in general safer to avoid such transformations and discretize the equation in its original form (although we shall always scale out the strike price K). The explicit Euler method for the Black-Scholes equation () on a uniform (x,t) grid is

$$\frac{1}{2}\sigma^2 x_j^2 \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{\Delta x^2} + r x_j \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} - u_j^n + \frac{u_j^{n+1} - u_j^n}{\Delta t} = 0.$$

For an equation with variable coefficients, it is in general not easy to give a stability analysis. However, theory and experience suggest that

i) the first derivative term rxu_x and the linear term ru may be ignored in a stability analysis;

ii) the stability condition known for the constant coefficient equation should hold locally at all x.

Thus the stability condition imposed on the explicit Euler method for the Black-Scholes equation is

$$\sigma^2 x_j^2 \frac{\Delta t}{\Delta x^2} \leq 1$$

which is consistent with the stability condition derived from the transformed Black-Scholes equation. Failure to heed this constraint on Δx and Δt will lead to non-sensical numerical results.

The explicit Euler method is of interest because of its relationship to the binomial method for option pricing as will be discussed in the next chapter. However, much more effective and more generally applicable is the implicit Euler method on a fixed uniform grid placed on the bounded set $x_0 \leq x \leq X$, $0 \leq t \leq T$. As before we use

$$x_j = x_0 + j\Delta x, \qquad \Delta x = (X - x_0)/M$$

$$t_n = n\Delta t, \qquad \Delta t = T/N$$

and write the finite difference approximation to the Black-Scholes equation in the form

$$\begin{aligned} L_h u_j^n \equiv \frac{1}{2}\,\sigma^2 x_j^2 \frac{u_{j+1}^n + u_{j-1}^n - 2u_j^n}{\Delta x^2} + r x_j \frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} \\ - u_j^n + \frac{u_j^{n+1} - u_j^n}{\Delta t} = 0 \qquad j = 1, \ldots, M-1. \end{aligned}$$

Here

$$u_0^n \quad \text{and} \quad u_M^n$$

are assumed known. For example, for a European put

$$\begin{aligned} u_0^n &= e^{-r(T-t_n)} && \text{for } x_0 = 0 \\ u_M^n &= 0 && \text{for } x_M = X \text{ sufficiently large.} \end{aligned}$$

Boundary conditions for a call can be read off the put-call parity.

The essential difference to the explicit method is that at each time level t_n a linear system of equations needs to be solved. To be specific, let U^n be the vector

$$U^n = (u_1^n, \ldots, u_{M-1}^n)$$

then the system to be solved is

$$AU^n = b^n$$

where A is the $(M-1) \times (M-1)$ tridiagonal matrix with entries

$$\begin{aligned} A_{j,j-1} &= \frac{1}{2}\,\sigma^2 x_j^2 \frac{1}{\Delta x^2} - r x_j \frac{1}{2\Delta x} \\ A_{j,j} &= -\sigma^2 x_j^2 \frac{1}{\Delta x^2} - r - \frac{1}{\Delta t} \\ A_{j,j+1} &= \frac{1}{2}\,\sigma^2 x_j^2 \frac{1}{\Delta x^2} + r x_j \frac{1}{2\Delta x} \end{aligned}$$

for $j = 1, \ldots, M-1$, and where b^n has the components

$$b_j^n = \begin{cases} -\frac{1}{\Delta t} u_1^{n+1} - A_{10} u_0^n, & j = 1 \\ -\frac{1}{\Delta t}\, u_j^{n+1}, & 1 < j < M-1 \\ -\frac{1}{\Delta t}\, u_{M-1}^{n+1} - A_{M-1,M} u_M^n, & j = M-1. \end{cases}$$

It is straightforward to verify that $L_h\phi(x_j, t_n)$ is a consistent approximation to $L\phi(x_n, t_n)$ for any smooth function ϕ. It remains to establish stability of the method. We shall now make the assumption that

$$A_{j,j-1} \geq 0 \qquad \text{for all } j \geq 1. \tag{4d1}$$

Then it follows by inspection that

$$|A_{jj}| > A_{j,j-1} + A_{j,j+1} \qquad \text{for all } j.$$

This inequality implies that A is strictly diagonally dominant which can be shown to imply that A is invertible. Hence if (4d1) holds then the values $\{u_j^n\}$ can be computed. Suppose now that at some interior mesh point (x_k, t_p) with $0 < k < M$, $0 \leq p < N$ the value u_k^p is bigger than all the other values, i.e.

$$u_k^p \geq u_j^n \qquad \text{for all } j \text{ and all } n.$$

Since the initial and boundary data are non-negative we have necessarily that

$$u_k^p \geq 0.$$

Then it follows from $A_{k,k-1}$, $A_{k,k+1} \geq 0$ that

$$\begin{aligned} A_{kk}u_k^p &+ A_{k,k-1}u_{k-1}^p + A_{k,k+1}u_{k+1}^p \\ &\geq A_{kk}u_k^p + A_{k,k-1}u_k^p + A_{k,k+1}u_k^p \\ &= -ru_k^p - \frac{1}{\Delta t}\, u_k^p < -\frac{1}{\Delta t}u_k^{p+1} \end{aligned}$$

which is inconsistent with the kth equation of

$$AU^p = b^p.$$

Hence there cannot be a value u_k^p which exceeds the maximum value of u at x_0, at $x_M = X$ or $t_N = T$. Since u at $x = 0$, $x = X$ and $t = T$ is bounded and independent of Δt and Δx it follows that u_k^p is uniformly bounded above for all Δx and Δt. An analogous argument shows that the minimum of $\{u_j^n\}$ is bounded below by the minimum of the data functions.

Hence, if condition (4d1) holds then the numbers $\{|u_j^n|\}$ are uniformly bounded so that the implicit Euler method is stable for all Δx and Δt.

Let us conclude the discussion of Euler's method by examining a little closer the requirement (4d1). We need

$$\frac{1}{2}\sigma^2 x_j^2 \frac{1}{\Delta x^2} - r x_j \frac{1}{2\Delta x} \geq 0, \quad \text{i.e.}$$
$$\sigma^2 \frac{x_j}{\Delta x} - r \geq 0 \qquad \text{for } j = 1, \ldots$$

This condition is most severe at $j = 1$. If for a given Δx we choose

$$x_0 = m\Delta x \quad \text{ then } x_1 = (m+1)\Delta x$$

and the condition is satisfies if

$$\sigma^2(m+1) - r > 0.$$

We shall see that for an American put the computational domain stays well away from $x = 0$ so choosing $x_0 > 0$ will not add an additional error. For a call it may be necessary to replace the condition

$$u(0,t) = 0$$

by the down and out barrier condition

$$u(x_0,t) = 0$$

for a suitably chosen $x_0 > 0$ in order to insure (4d1). Numerical experiments show that the influence of the barrier on call prices around $x = 1$ is not noticeable.

The unconditional stability of the implicit Euler method is bought at the expense of having to solve the linear system

$$Au^n = b^n$$

at each time level. Fortunately, the tridiagonal structure of A makes this solution extremely simple and rapid if a special version of Gaussian elimination known as the Thomas algorithm is employed. The argument is simple. A tridiagonal matrix is factored into lower and upper triangular matrices L and U so that

$$A = LU.$$

This can be achieved by setting

$$L_{ij} = U_{ij} = 0 \qquad \text{for } |i-j| > 2$$

$$U_{i,i-1} = L_{i,i+1} = 0$$

$$U_{ii} = 1$$

$$L_{i,i-1} = A_{i,i-1}.$$

The remaining elements L_{ii} and $U_{i,i+1}$ are found for $i = 1$ from

$$L_{11} = A_{11}$$

$$L_{11}U_{12} = A_{12}$$

and for $i = 2, \ldots, M-1$ from

$$L_{i,i-1}U_{i-1,i} + L_{ii} = A_{ii}$$

$$L_{ii}U_{i,i+1} = A_{i,i+1}.$$

It is straightforward to verify from

$$(LU)_{ij} = \sum_{k=1}^{M-1} L_{ik}U_{kj} = L_{i,i-1}U_{i-1,j} + L_{ii}U_{ij}$$

that

$$A = LU,$$

i.e. that A has been factored into a lower and upper triangular matrix. The linear system

$$Au^n = b^n$$

can now be solved by simple substitution. First we find the solution y^n of

$$Ly^n = b^n$$

and then obtain u^n from

$$Uu^n = y^n.$$

It is clear from $Au^n = LUu^n = Ly^n = b^n$ that u^n is the desired solution. For any European option, including those for variable interest rates and volatilities an implicit numerical discretization coupled with a factorization of the tridiagonal system would be hard to beat.

An entirely different numerical method for the solution of a linear system like

$$Aw = b$$

is an iterative method known as successive overrelaxation (SOR). This method will only succeed if A has specific properties. Fortunately, the strict diagonal dominance of A obtained from the implicit Euler method for the Black-Scholes equation makes the SOR method feasible.

We introduce new upper and lower triangular matrices by writing

$$A = D + L + U$$

where D is the diagonal of A, L is equal to A below the diagonal and zero on and above the diagonal, and U is zero on and below the diagonal and equal to A above it. In other words, we have split A into three matrices. Let us make an initial guess

$$w^0$$

for the solution of $Aw = b$. (For options the initial guess for prices at t_n might be the prices already available at time t_{n+1} but in general a good initial guess is not too important.) We then compute $k = 1, 2, \dots$ a sequence $\{w^k\}$ of vectors with components $\{w_i^k\}$ by solving for $i = 1, \dots, M-1$ the scalar equation

$$A_{ii}\tilde{w}_i + \sum_{j=1}^{i-1} A_{ij}w_j^k + \sum_{j=i+1}^{M-1} A_{ij}w_j^{k-1} = b_i$$

and setting

$$w_i^k = w_i^{k-1} + \omega[\tilde{w}_i - w_i^{k-1}]$$

where ω is a relaxation factor at our disposal. Typically, $\omega \in [1, 2)$; for the special case of $\omega = 1$ we see that

$$w_i^k = \tilde{w}_i.$$

This variant of the SOR method is known as the Gauss-Seidel iteration method for solving

$$Aw = b$$

and was known for some time before the parameter ω was introduced. The rationale for introducing ω will be apparent from the following brief discussion of the SOR method. We note that the above $M-1$ equations can be written conveniently in matrix form as

$$D\tilde{w} + Lw^k = -Uw^{k-1} + b$$

$$\tilde{w} = w^{k-1} + \frac{1}{\omega}\left[w^k - w^{k-1}\right]$$

which can be combined to yield

$$(D + \omega L)w^k = [(1-\omega)D - \omega U]w^{k-1} + \omega b. \tag{4d1}$$

If the iteration converges to a vector w^* then it follows that

$$(D + \omega L)w^* = [(1-\omega)D - \omega U]w^* + \omega b. \tag{4d2}$$

or

$$Aw^* = \omega b.$$

Since A is strictly diagonally dominant the solution w^* is uniquely defined. Let us introduce the error vector

$$e^k = w^* - w^k$$

then by subtracting (4d1) from (4d2) we find that

$$(D + \omega L)e^{k+1} = [(1-\omega)D - \omega U]e^k$$

and hence

$$e^k = [(D + \omega L)^{-1}((1-\omega)D - \omega U)]^k e^0$$

where e^0 is fixed once the initial guess w^0 has been chosen. The matrix

$$\mathcal{L}[\omega] = [(D + \omega L)((1-\omega)D - \omega U]$$

is known as the iteration matrix of the SOR method for $Aw = b$, and we see that convergence is assured if and only if

$$\mathcal{L}[\omega]^k = [(D + \omega L)((1 - \omega)D - \omega U] \to 0 \qquad \text{(the zero matrix)}$$

as $k \to \infty$. It is known from linear algebra that given a square $n \times n$ matrix C one can assert that

$$C^k \to 0 \quad \text{as } k \to \infty$$

if and only if each eigenvalue λ of C has absolute value less than 1. Let us define the so-called spectral radius $\rho(C)$ of the the matrix C by

$$\rho(C) = \max_i |\lambda_i|$$

then the smaller the spectral radius the faster the convergence will be. This is readily apparent if C can be diagonalized. Indeed, if

$$C = X\Lambda X^{-1}$$

where Λ is a diagonal matrix, then its entries are the eigenvalues of C while the columns of X are the corresponding eigenvectors. It is easy to see that

$$C^k = X\Lambda^k X^{-1}$$

where

$$\Lambda^k = \operatorname{diag}\{\lambda_1^k, \ldots, \lambda_n^k\}$$

so that

$$\max_{i,j} \left|C_{ij}^k\right| = K\rho(C)^k$$

for some constant K.

If we consider the error for the special case of the Gauss-Seidel method then we need to worry about the spectral radius of the matrix

$$\mathcal{L}[1] = (D + L)^{-1}U.$$

If λ is any eigenvalue of this matrix with corresponding eigenvector x then the eigenvalue equation

$$\mathcal{L}[1]x = \lambda x$$

can be rewritten as

$$\lambda(D + L)x = Ux.$$

Since x is not a zero vector we can divide by the largest component, say x_k, and obtain a new eigenvector y with the property that

$$y_k = 1, \qquad |y_i| \leq 1.$$

If we now look at the kth equation of $\lambda(D + L)y = Uy$ we find

$$\lambda(A_{kk} + A_{k,k-1}y_{k-1}) = A_{k,k+1}y_{k+1}$$

so that necessarily

$$|\lambda| = \frac{|A_{k,k+1}y_{k+1}|}{|A_{kk} + A_{k,k-1}y_{k-1}|}$$

and thus

$$|\lambda| \leq \frac{|A_{k,k+1}|}{|A_{kk}| - |A_{k,k-1}|}\,.$$

But we agreed to choose our mesh parameters such that A was strictly diagonally dominant, i.e.

$$|A_{kk}| > |A_{k,k-1}| + |A_{k,k+1}|$$

which implies that

$$|\lambda| < 1.$$

Hence the Gauss-Seidel method applied to the matrix obtained from the implicit Euler method for the Black-Scholes equation will converge because all eigenvalues of the iteration matrix $\mathcal{L}[1]$ and hence its spectral radius have absolute values less than 1. Now the eigenvalues of the matrix $\mathcal{L}[\omega]$ depend continuously on the elements of the matrix. The goal is to find that for which $\rho(\mathcal{L}[\omega])$ is minimized. In general this problem cannot be solved analytically so that this ω must be found by trial and error. One makes a guess for ω, counts the number of iterations required for convergence of the SOR method, and then observes the influence of changes in the value of this guess on the iteration count. Typically $\omega \in (1.5, 1.8)$ would be a good choice. The influence of a good value for ω on the number of iterations required for convergence is pronounced.

Let us now introduce a third method for the numerical solution of European options which is in fact closely related to the LU decomposition discussed above but relies on a semi-discrete approximation of the Black-Scholes equation and the theory of ordinary differential equations rather than linear algebra. While of little significance for European options this method is quite useful for the American equivalents as will be shown later.

To be specific let us consider a European call

$$\frac{1}{2}\sigma^2 x^2 u_{xx} + rxu_x - ru + u_t = 0 \tag{4e1}$$

$$u(x_0, t) = 0, \qquad u(X, t) = X - e^{r(T-t)}$$

$$u(x, T) = \max\{0, x-1\}.$$

For technical reasons the condition at $x = 0$ is replaced by a down and out barrier at $x_0 > 0$ for some $x_0 \ll 1$. As before we shall discretize time by setting

$$\Delta t = T/N, \qquad t_n = n\Delta t.$$

At time t_n we shall write a time implicit approximation to the Black-Scholes equation in which x remains a continuous variable. To be specific, let $u_n(x)$ denote the approximation to $u(x, t_n)$. Then

$$\frac{1}{2}\sigma^2 x^2 u_n''(x) + rxu_n'(x) - ru_n(x) + \frac{u_{n+1}(x) - u_n(x)}{\Delta t} = 0 \tag{4e2}$$

$$u_n(x_0) = 0 \qquad u_n(X) = X - e^{-r(T-t_n)}$$

with

$$u_N(x) = \max\{0, x-1\}$$

is a consistent time discrete approximation of the European call. Rather than a matrix system we now have to solve a second order differential equation at each time level. The heart of the method lies in the solution method for this differential equation.

For simplicity let us drop the subscript n and simply remember that the discussion applies to the problem at time t_n. If we define

$$u'(x) = v(x)$$

then the second order equation (4e2) is equivalent to the first order system

$$u'(x) = v(x)$$

$$v'(x) = \frac{2}{\sigma^2 x^2}\left[\left(r + \frac{1}{\Delta t}\right)u(x) - rxv(x) - \frac{u_{n+1}(x)}{\Delta t}\right] \tag{4e3}$$

which we write with the obvious identifications simply as

$$u' = v$$
$$v' = c(x)u + d(x)v + g(x).$$

The boundary data are

$$u(x_0) = 0, \qquad u(X) = X - e^{-r(T-t_n)}.$$

The boundary value problem for the first order system will be solved with a variant of the LU decomposition known as a sweep method. We shall verify that this first order system can be solved by assuming that u and v are related through a transformation of the form

$$u(x) = R(x)v(x) + w(x). \tag{4e4}$$

In the context of ordinary differential equations this transformation is known as a Riccati transformation. The defining relations for R and w result by differentiation and substitution of the differential equations. We find

$$u'(x) = R'(x)v(x) + R(x)v'(x) + w'(x)$$

and from the differential equations

$$v(x) = R'(x)v(x) + R(x)[c(x)(R(x)v(x) + w(x)) + d(x)v(x) + g(x)] + w'(x).$$

We can rewrite this expression as

$$[R' + RcR + Rd - 1]v + [w' + Rcw + Rg] = 0.$$

If R and w are chosen such that each bracket vanishes, i.e.

$$\text{(4e5)} \qquad \begin{aligned} R' &= 1 - Rd - RcR, & R(X) &= 0 \\ w' &= -Rcw - Rg(x), & w(X) &= X - e^{-r(T-t_n)} \end{aligned}$$

then we have an explicit representation linking u and v at time level t_n which is consistent with the second order differential equation and the boundary condition at $x = X$.

The differential equation for R is known as a Riccati differential equation because of its quadratic right hand side. The equation for w is a linear inhomogeneous equation in w once R has been found. As solutions of well defined initial value problems R and w may be assumed known either from the theory of ordinary differential equations or from a numerical integration of the equations.

At $x = x_0$ it follows from the Riccati transformation that $v(x_0)$ must be chosen such that

$$u(x_0) = 0 = R(x_0)v(x_0) + w(x_0).$$

Since $R(x_0)$ and $w(x_0)$ are assumed known we see that $v(x_0)$ is now known. The solution $u(x)$ is then given by (4e4) if we find $v(x)$ such that

$$\text{(4e6)} \qquad \begin{aligned} v'(x) &= c(x)(R(x)v + w(x)) + d(x)v + g(x) \\ v(x_0) &= (u(x_0) - w(x_0))/R(x_0). \end{aligned}$$

The numerical solution proceeds as follows. Given $u_{n+1}(x)$ (and thus $g(x)$) we integrate (4e5) from $x = X$ to $x = x_0$ to obtain $\{R(x), w(x)\}$ over $[x_0, X]$. Then we integrate (4e6) over $[x_0, X]$ and substitute $v(x)$ into (4e4) to complete the solution at time t_n. We note that for the integration of (4e6) we need $R(x)$ and $w(x)$ over $[x_0, X]$. In general we cannot assume that these functions are available in closed form. More likely, a numerical method must be used to find R, w and v. One can, of course, use the explicit Euler method for this task, but the accuracy of the method is poor compared to competing numerical methods. On the other hand, a very accurate (and expensive) numerical method for this integration likewise is not justified because we already have introduced a sizeable error in the time discretization of the Black-Scholes equation. A good compromise between accuracy, time of execution and stability is provided by the trapezoidal rule for ordinary differential equations discussed in Chapter 2. We recall, for a general first order equation of the form

$$\varphi'(x) = F(\varphi(x), x)$$

we obtain an approximation at meshpoints $\{x_j\}$ with $\Delta x = x_{j+1} - x_j$ from

$$\frac{\varphi_{j+1} - \varphi_j}{\Delta x} = \frac{F(\varphi_{j+1}, x_{j+1}) + F(\varphi_j, x_j)}{2}.$$

If φ_j is given we can go forward in x and solve for φ_{j+1}. If φ_{j+1} is given we can go backward in x and solve for φ_j. If we apply the trapezoidal rule to the Riccati equation on the grid $x_0 < x_1 < \ldots < x_M = X$ with constant step size Δx then we need to solve the quadratic equation

$$\frac{R_{j+1} - R_j}{\Delta x} = \frac{[1 - d(x_j)R_j - c(x_j)R_j^2] + [1 - d(x_{j+1})R_{j+1} - c(x_{j+1})R_{j+1}^2]}{2}$$

for R_j. This is a quadratic equation which can be solved by formula. The choice of the correct algebraic sign in the quadratic formula is dictated by the requirement that $R(x) < 0$ on $[x_0, X)$ which follows from the Riccati equation by observing that

$$R'(x) = 1 \quad \text{whenever } R(x) = 0.$$

In particular, since $R(X) = R_M = 0$ we see that $R(x)$ is decreasing as x is decreasing from X. We shall assume that $\{R(x_j)\}$ has been found and stored. The computation of w is simpler since the equation is linear. The trapezoidal rule states that

$$\frac{w_{j+1} - w_j}{\Delta t} = \frac{[-c(x_j)R_j w_j - R_j g(x_j)] + [-c(x_{j+1})R_{j+1}w_{j+1} - R_{j+1}g(x_{j+1})]}{2}$$

must be solved for w_j for $j = M-1, \ldots, 0$ with $w_M = X - e^{-r(T-t_n)}$. Now we have available $\{R_j, w_j\}$ at all mesh points. It remains to find $\{v_j\}$. The initial condition is $v_0 = \frac{-w(x_0)}{R(x_0)}$. The trapezoidal rule requires that we solve

$$\frac{v_{j+1} - v_j}{\Delta x} = \frac{[c(x_j)(R_j v_j + w_j) + d(x_j)v_j + g(x_j)]}{2} + \frac{[c(x_{j+1})(R_{j+1}v_{j+1} + w_{j+1}) + d(x_{j+1})v_{j+1} + g(x_{j+1})]}{2}$$

for v_{j+1}. Then $u_{j+1} = R_{j+1}v_{j+1} + w_{j+1}$. Alternatively, one could have solved (4e5) forward from $x = x_0$ to X subject to

$$R(x_0) = 0, \qquad w(x_0) = 0$$

and (4e6) backward from $x = X$ to x_0 subject to

$$v(X) = (X - e^{-r(T-t_n)} - w(X))/R(X)$$

and implemented this computation with the trapezoidal rule.