Defining canonically best factorization theorems for the generating functions of special convolution type sums

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February 10, 2021

1. No notes for this page

Goals of the talk

- Identify a general method for expanding out the generating functions of special sums (many examples)
- Questions about the most natural ways of forming the generating-function-based expansions of these sums

1. Motivate why generating function approaches to enumerating special sums is useful in applications
2. These expansions follow from results in published work on so-called *factorization theorems* for special GFs
3. We will cover the basics, including definitions of core sequences, from the beginning

Generating functions of sequences

- Recall that an *ordinary generating function* (or OGF) of a sequence (or arithmetic function) $\{f_n\}_{n\geq 0}$ is defined by $F(z) := \sum_{n\geq 0} f_n \cdot z^n$.
- We write the series coefficient extraction operator in the notation $[z^n]F(z) \equiv f_n$ for $n \geq 0$.
- Why generating functions are useful in studying sequence properties?

1. We can treat $F(z)$ as a formal power series object that enumerates, or *"pins up"* the terms of the sequence in powers of z like clothes on a clothesline.
2. Alternately, we can view $F(z)$ as an analytic function within its radius of convergence to justify properties like asymptotic expansions of the f_n.

Motivating examples

Motivating examples

1. In some sense, the first and prototypical example is the LGF case
2. The idea was based on some combinatorially themed work with these generating functions I published in *Acta Arithmetica* around 2017. Grew into collaborative work with M. Merca and later H. Mousavi at GA Tech published in the *Ramanujan Journal*.
3. Relates multiplicative number theoretic functions (e.g., divisor sums) to more additive constructions in the theory of partitions.

Review: Multiplicative functions in number theory

- Recall that an arithmetic function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ for all integers $a, b \geq 1$ such that $(a, b) = 1$.
- Examples of multiplicative functions:
  1. The *Möbius function* $\mu(n)$ is the signed indicator function of the squarefree integers.
  2. Euler's classical *totient function* $\phi(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} 1 = \sum_{d|n} d\mu\left(\frac{n}{d}\right)$.
  3. The generalized *sum-of-divisors functions* $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$ with the special cases $d(n) = \sigma_0(n)$ and $\sigma(n) = \sigma_1(n)$.
  4. The completely multiplicative *Liouville lambda function* $\lambda(n) = (-1)^{\Omega(n)}$.

1. Reminder: $(a, b) \equiv \gcd(a, b)$ denotes the GCD of a and b.
2. Reminder: $\Omega(n)$ counts the number of distinct prime factors of $n \geq 2$ (counting multiplicity). A related function is $\omega(n)$.

What is a Lambert series generating function (LGF)?

- Formally, given an arithmetic function $f : \mathbb{Z}^+ \to \mathbb{C}$ we define its *Lambert series generating function* (or LGF) to be

$$L_f(q) := \sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \sum_{m\geq 1} \left(\sum_{d|m} f(d)\right) q^m, |q| < 1.$$

- For $*$ denoting *Dirichlet convolution*, the RHS coefficients generated by $L_f(q)$ are $[q^n]L_f(q) = (f * 1)(n) = \sum_{d|n} f(d)$ for $n \geq 1$.
- Multiplicative functions tend to have nice expressions in terms of divisor sum convolutions of this type.

1. Hence, the LGF construction is a more classical number theoretic way of providing a OGF for multiplicative type functions.
2. We also have that if $F(z)$ is the OGF of $\{f(n)\}_{n\geq 1}$, then

$$L_f(q) = \sum_{n\geq 1} F(q^n),$$

and

$$F(q) = \sum_{n\geq 1} \mu(n) L_f(q^n).$$

Examples of Lambert series generating functions

$$\sum_{n\geq 1} \frac{\mu(n)q^n}{1-q^n} = q, \tag{1a}$$

$$\sum_{n\geq 1} \frac{\phi(n)q^n}{1-q^n} = \frac{q}{(1-q)^2}, \tag{1b}$$

$$\sum_{n\geq 1} \frac{n^\alpha q^n}{1-q^n} = \sum_{m\geq 1} \sigma_\alpha(n)q^n, \tag{1c}$$

$$\sum_{n\geq 1} \frac{\lambda(n)q^n}{1-q^n} = \sum_{m\geq 1} q^{m^2}, \tag{1d}$$

$$\sum_{n\geq 1} \frac{\mu^2(n)q^n}{1-q^n} = \sum_{m\geq 1} 2^{\omega(m)}q^m. \tag{1e}$$

1. Reminder of what each of these functions provides.
2. The RHS series terms for $L_f(q)$ each correspond to known identities for $f * 1$:
   (a) $(\mu * 1)(n) = \varepsilon(n) = \delta_{n,1}$;
   (b) $(\phi * 1)(n) = n$;
   (c) $\lambda * 1 = \chi_{\text{squares}}$;
   (d) $\sum_{d|n} \mu^2(n) = \sum_{d|n} |\mu(n)| = 2^{\omega(n)}$.

Review: The partition function $p(n)$

- A partition of a positive integer n is a (finite) sequence of positive integers whose sum is n.
- More formally, we may partition n as a sum of integers $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 1$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.
- We denote the total number of partitions of n by $p(n)$
- For example, $p(5) = 7$ since the distinct partitions of 5 are given by

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

1. By a combinatorial argument with products of geometric series, we have that the OGF for $p(n)$ is given by
$$p(n) = [q^n](q;q)_\infty^{-1} = [q^n]\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$
2. Another common partition function is $q(n)$ which counts the number of partitions of n into *distinct parts*. Here, we have that $q(5) = 3$.
3. In general, we term a *partition function* as a function that counts the number of ways to partition n into bins (or partitions) with some property.
4. Many partition function variants are typically associated with product type OGF expansions that rely on counting techniques to interpret their significance.

Examples of partition function OGFs

Table 14.1 Generating functions

Generating function	The number of partitions of n into parts which are	Generating function	The number of partitions of n into parts which are
$\prod_{m=1}^{\infty} \frac{1}{1-x^{2m-1}}$	odd	$\prod_{m=1}^{\infty}(1+x^{2m-1})$	odd and unequal
$\prod_{m=1}^{\infty} \frac{1}{1-x^{2m}}$	even	$\prod_{m=1}^{\infty}(1+x^{2m})$	even and unequal
$\prod_{m=1}^{\infty} \frac{1}{1-x^{m^2}}$	squares	$\prod_{m=1}^{\infty}(1+x^{m^2})$	distinct squares
$\prod_{p} \frac{1}{1-x^p}$	primes	$\prod_{p}(1+x^p)$	distinct primes
$\prod_{m=1}^{\infty}(1+x^m)$	unequal		

1. The table is copied from Apostol's book.

Work on Lambert series factorization theorems

- Let $s_e(n,k)$ and $s_0(n,k)$ respectively denote the the number of k's in all partitions of n into an even (odd) number of distinct parts.
- Let $(a;q)_\infty = \prod_{m\geq 1}(1-aq^{m-1})$ denote the infinite q-Pochhammer symbol.
- Then we have

$$\sum_{n\geq 1}\frac{f(n)q^n}{1\pm q^n} = \frac{1}{(\mp q;q)_\infty}\sum_{n\geq 1}\left(\sum_{k=1}^{n}(s_o(n,k)\pm s_e(n,k))\, f(k)\right) q^n,$$

1. How to think about this type of formula (intuition)?
2. This particular identity falls out of a natural algebraic structure for this series definition.

$$[q^n]L_f(q) = [q^n]\sum_{m=1}^{n}\frac{f(n)q^n}{1-q^n},$$

So can combine like denominators and see the limiting forms are suggested.

Lambert series factorization theorems (expressions by invertible matrices)

- Re-write the previous factorization theorem statement as

$$\sum_{n\geq 1}\frac{f(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\left(\sum_{k=1}^{n}s_{n,k}f(k)\right) q^n.$$

- The matrices formed by the lower triangular sequence of $s_{n,k}$ are invertible with ones on the diagonal.
- We can prove exactly how $s_{n,k}^{-1}$ is related to $p(n)$:

$$s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right).$$

1. How to think about this type of formula (intuition)? – More generally, how to look at the factorization theorem type results and parse them.
2. Suppose we have sums of the form

$$S_f(n) := \sum_{d\in\mathcal{A}_n} f(d).$$

Then

$$C_m(q) := [f(m)]\sum_{n\geq m} S_f(n)q^n = \sum_{n\geq 1}[m\in\mathcal{A}_n]_\delta\, q^n.$$

So it makes sense to write

$$S_f(q) = [q^n]\left(\mathcal{B}(q)\times\sum_{n\geq 1}\sum_{k=1}^{n} t_{n,k}f(k)\cdot q^n\right),$$

where

$$\mathcal{B}(q) = \lim_{n\to\infty}\prod_{i=1}^{n} C_i(q).$$

Relation of the matrices to partition functions brings up a natural question

- There is a very **natural** relation of both sequences of $s_{n,k}$ and $s_{n,k}^{-1}$ to *partition theoretic functions*.
- This is unusual in so much as LGFs typically generate *multiplicative* functions (product based properties), whereas partition function variants have a much more *additive* structure
- Brings up a natural question: Why did partitions fit so naturally with the multiplicative functions enumerated by the LGFs above?

1. Additive structure: E.g., they count the numbers of ways to put decompositions of n items into bins.
2. Is this the most natural way to expand things in the context of other special sums?
3. How else can we see special relations like this for more general sum types?

More general constructions ($\mathcal{D}$-convolution type sums)

Generalized classes of convolution type sums

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Examples: What other sum types are we interested in studying?

$$S_f(n) = \sum_{d|n} f(d) \qquad \text{(e.g., LGF cases)} \qquad (2a)$$

$$S_f(n) = \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} f(d) \qquad (2b)$$

$$S_f(n) = \sum_{\substack{d \in A_n \\ A_n \subseteq \{1,2,\ldots,n\}}} f(d) \qquad \text{(and weighted versions)} \qquad (2c)$$

$$S_f(n) = \sum_{d=1}^{n} \begin{bmatrix} n \\ d \end{bmatrix} (-1)^{n-d} f(d) \qquad \text{(cf. Stirling transform)} \qquad (2d)$$

$$S_{f,g}(n) = \sum_{d=1}^{n} \binom{d}{n} f(d) g(n+1-d) \qquad \text{(cf. binomial transform)} \qquad (2e)$$

1. Walk through each sum type.

Generating functions for a more general class of sums

- Consider a fixed *kernel* function $\mathcal{D} : (\mathbb{Z}^+)^2 \to \mathbb{Z}$.
- Suppose that $\mathcal{D}(n, k)$ is lower triangular so that $\mathcal{D}(n, k) = 0$ whenever $k > n$.
- Suppose that $\mathcal{D}$ is invertible so that $\mathcal{D}(n, n) \neq 0$ for all $n \geq 1$.
- For any arithmetic functions f, g, we consider the class of *$\mathcal{D}$-convolution sums* of the form

$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{k=1}^{n} f(k) g(n+1-k) \mathcal{D}(n, k), n \geq 1.$$

1. Invertible kernel function: For any fixed $N \geq 1$ $\det[(\mathcal{D}(n, k))_{1 \leq n, k \leq N}] \neq 0$.
2. The LGFs generate the special case $f \boxdot_{\mathcal{D}} 1$ where $\mathcal{D}(n, k) = [k|n]_\delta$ and $\mathcal{D}^{-1}(n, k) = \mu\left(\frac{n}{k}\right) [k|n]_\delta$ (e.g., we recover Möbius inversion).
3. Special cases: Set $g \equiv 1$, or use $\mathcal{D}(n, k) \in \{0, 1\}$ to denote inclusion in another set.

Defining analogous factorization theorems to generate these sums (up to an undetermined OGF)

- We expand the generalized *factorization theorems* for any fixed $(\mathcal{C}, \mathcal{D})$ that uniquely determines the following expansions:

$$(f \boxdot_{\mathcal{D}} 1)(n) := [q^n]\left(\frac{1}{\mathcal{C}(q)} \times \sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k}(\mathcal{C},\mathcal{D}) \cdot f(k) \cdot q^n\right), n \geq 1.$$

- Take $\mathcal{C}(q)$ any OGF with integer coefficients such that $\mathcal{C}(0) \neq 0$ (typically set to one up to normalization).
- For this fixed function, define the series coefficients $c_n(\mathcal{C}) := [q^n]\mathcal{C}(q)$ and $p_n(\mathcal{C}) := [q^n]\mathcal{C}(q)^{-1}$ for $n \geq 0$.

1. As in the references, it is natural to parameterize the factorizations of the OGFs that generate, or formally enumerate, these sums by another indeterminate OGF.
2. How to think about the form of the factorization theorem:
   - The kernel function $\mathcal{D}(n,k)$ denotes a weight on the inclusion of the summand $f(k)$ in some set A_n.
   - Gives a matrix-based quasi-GF that enumerates the (weighted) $n \geq m$ that have a summand term of $f(m)$:

$$\sum_{n\geq m} \mathcal{D}(n,m)q^n = [f(m)]\left(\frac{1}{\mathcal{C}(q)} \times \sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k}(\mathcal{C},\mathcal{D}) \cdot f(k) \cdot q^n\right)$$

   - The matrix whose $(i,j)^{th}$ entries are $s_{i,j}(\mathcal{C},\mathcal{D})$ is invertible.
   - This allows us to invert and write $f(n)$ as a predictable sum over the $(f \boxdot_{\mathcal{D}} 1)(j)$ for $j \leq n$ (meaningful for some $\mathcal{C}(q)$).

How do we choose a "canonically best" OGF $\mathcal{C}$?

- In the LGF case, the products for $\mathcal{C}(q) := (q;q)_\infty$ arise in arithmetic with these generating functions.
- The observation of how well the structurally revealing "natural choice" of sequences was from the LGF case is still very fuzzy and *qualitative*.
- **Big Question:** How do we *quantify* the notion of how well related the structures of the respective sequences $\{p_n(\mathcal{C})\}_{n\geq 0}$ and $\{\mathcal{D}(n,k)\}_{n\geq k\geq 1}$ are so that we can then prove the form of an optimal, or most structurally revealing, or say "*canonically best*" OGF $\mathcal{C}(q)$?

1. LGF Algebra: That is, in taking common denominators of the terms in the partial sums that generate the factors of m in the coefficient sums: $\frac{q^m}{1-q^m}$ leads to a natural algebraic suggestion of this $\mathcal{C}(q)$ (cf. *Acta Aritmetica*).
2. Question: How to mimic the nice sequence relationships we recognized for LGFs in the more general $\mathcal{D}$-convolution sum types?
3. That is, find the most relevant (or revealing) choice of the $\mathcal{C}(q)$ given any fixed lower triangular, invertible kernel function $\mathcal{D}$.

Idea: Define suitable cross-correlation statistics (and then sum them up)

- Consider a metric (statistic) that indicates the *cross-correlation* between these sequences.
- There are many ways to do this!
- A variant of the standard formula for a *Pearson correlation* statistic between any two N-tuples:

$$\operatorname{PearsonCorr}(N; \vec{a}, \vec{b}) := \frac{1}{N} \times \frac{\sum_{j=1}^{N} a_j \cdot b_j}{\sqrt{\sum_{1\leq i,j\leq N} a_i^2 \cdot b_j^2}}.$$

1. Well correlated sequences have a very high cross-correlation statistic, whereas lower values of the statistic should indicate that these sequences are less compatible.
2. In general, theoretical upper and lower bounds on the possible range of the associated cross-correlation statistics will still very much depend on the precise way it is defined.
3. Optimal values of $\operatorname{PearsonCorr}(N; \vec{a}, \vec{b}) = \pm 1$ roughly correspond to linear data. The centralized form yields values in $[-1, 1]$. The non-centralized version defined here is still bounded depending on the means and variances of the input vectors.
4. This correlation statistic is invariant for linear combinations of the input vectors. That is, for $\alpha, \beta, \gamma, \rho \in \mathbb{R}$ with $\beta, \rho > 0$, $\operatorname{PearsonCorr}(N; \vec{a}, \vec{b}) = \operatorname{PearsonCorr}(N; \alpha + \beta\vec{a}, \gamma + \rho\vec{b})$.
5. There is some geometric interpretation about the angle between two vectors in Euclidean space: $\cos(\vartheta) = \frac{x \cdot y}{||x|| \, ||y||}$. For centrally shifted N-tuples (so that the sum of all entries is zero), this coefficient is the angle ϑ between the two vectors.
6. Other possibilities for correlation statistics exist, but I have not spent

More general cross-correlation statistic formulas to consider

- Let's look at finding $\mathcal{C}(q)$ such that the following statistic is maximized (minimized):

$$\mathrm{Corr}(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1}\frac{1}{n}\times\frac{\sum_{k=1}^{n}|c_k(\mathcal{C})\mathcal{D}^{-1}(n,k)|}{\sqrt{\left(\sum_{k=1}^{n}c_k(\mathcal{C})^2\right)\left(\sum_{k=1}^{n}\mathcal{D}^{-1}(n,k)^2\right)}}.$$

- **Conjecture.** If $\mathrm{Corr}(\mathcal{C},\mathcal{D})$ is optimized by a particular OGF $\mathcal{C}(q)$, then so is the alternate statistic

$$\mathrm{Corr}_*(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1}\frac{1}{n}\times\frac{\sum_{k=1}^{n}|p_k(\mathcal{C})\mathcal{D}(n,k)|}{\sqrt{\left(\sum_{k=1}^{n}p_k(\mathcal{C})^2\right)\left(\sum_{k=1}^{n}\mathcal{D}(n,k)^2\right)}}.$$

1. The absolute value is somewhat non-standard, but leads to easier bounds in the prototypical LGF cases.
2. Intuition about why care: Compare to the revealing structure we saw in the LGF case. Want to replicate this type of special and unusual algebraic relationship with the more general sums.
3. Why chose these particular forms of the matrix-to-OGF-coefficients to correlate?
   Well, because the closed-form expressions for the denominator variances and absolute values of the $c_n(\mathcal{C})$ sequence in the LGF cases were nicest – In short, a *heuristic "guess"* (or ansatz of sorts) on what to look at.

Back to the LGF expansion cases

- For the LGF case, we have that $\mathcal{D}(n,k) := [k|n]_\delta$ and $\mathcal{D}^{-1}(n,k) = \mu(n/k)\,[k|n]_\delta$.
- This leads to the explicit formula for $\mathrm{Corr}(\mathcal{C},\mathcal{D})$ given by

$$\mathrm{Corr}_{LGF}(\mathcal{C}) := \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\mu^2(k)}{k(\sqrt{2})^{\omega(k)}}\times\sum_{j\leq\left\lfloor\frac{j}{k}\right\rfloor}\frac{|c_j(\mathcal{C})|}{j\cdot\rho_{\mathcal{C}}(jk)(\sqrt{2})^{\omega\left(\frac{j}{(j,k)}\right)}}.$$

where we define the partial variance of $\mathcal{C}$ to be

$$\rho_{\mathcal{C}}(N) := \sqrt{\sum_{1\leq i\leq N}c_i(\mathcal{C})^2}, N\geq 1.$$

1. We might as well refer to this special case since we already have good intuition as to what we expect the optimal OGF to be.
2. Here, $\sum_{k\leq n}\mathcal{D}^{-1}(n,k)^2 = \sum_{d|n}\mu^2(d) = 2^{\omega(n)}$.
3. I wanted to bound the correlation statistics for the LGF case to see how close the OGF $\mathcal{C}(q) := (q;q)_\infty$ comes to attaining theoretical upper (lower) bounds.

Back to the LGF expansion cases

Theorem. Fix any $0<\delta<+\infty$. Suppose that $\mathcal{C}(q)$ is an OGF whose series coefficients are integer valued so that $c_0(\mathcal{C}) = 1$ and where

$$A_0(\mathcal{C},\delta) := \lim_{N\to\infty}\frac{1}{N^{\frac{\delta}{2}}}\times\sqrt{\sum_{1\leq n\leq N}c_n(\mathcal{C})^2}\in[1,+\infty).$$

We have that

$$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C})^{-1}\geq\left(1+\frac{1}{2A_0(\mathcal{C},\delta)}\left(\frac{\zeta\left(\frac{2+\delta}{2}\right)}{\zeta(2+\delta)}-1\right)\mathrm{DGF}\,|\mathcal{C}|\left(\frac{2+\delta}{2}\right)+\widehat{\omega}_\ell(\mathcal{C})\right)^{-1},$$

and

$$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C})^{-1}\leq\left(\frac{1}{A_0(\mathcal{C},\delta)}\cdot\frac{\zeta\left(\frac{3+\delta}{2}\right)}{\zeta(3+\delta)}\cdot\mathrm{DGF}\,|\mathcal{C}|\left(\frac{3+\delta}{2}\right)-\widehat{\omega}_u(\mathcal{C})\right)^{-1},$$

where the constants $\widehat{\omega}_\ell(\mathcal{C})$ and $\widehat{\omega}_u(\mathcal{C})$ can be explicitly bounded given any fixed $(\delta,\mathcal{C}(q))$.

1. We define $\mathrm{DGF}\,|\mathcal{C}| := \sum_{n\geq 1}\frac{|c_n(\mathcal{C})|}{n^s}$.
2. In particular, explicit bounds are given by:

$$\widehat{\omega}_\ell(\mathcal{C}) := \left|\sum_{m\geq 1}\binom{\frac{1}{2}}{m}\frac{(-1)^m(2m-1)}{2A_0(\mathcal{C},\delta)^{2m+1}}\left(\frac{\zeta\left(1+\left(m+\frac{1}{2}\right)\delta\right)}{\zeta\left(2+2\left(m+\frac{1}{2}\right)\delta\right)}-1\right)\mathrm{DGF}[|\mathcal{C}|]\left(1+\left(m+\frac{1}{2}\right)\delta\right)\right|,$$

$$\widehat{\omega}_u(\mathcal{C}) := \left|\sum_{m\geq 1}\binom{\frac{1}{2}}{m}\frac{(-1)^m(2m-1)}{A_0(\mathcal{C},\delta)^{2m+1}}\cdot\frac{\zeta\left(\frac{3}{2}+\left(m+\frac{1}{2}\right)\delta\right)}{\zeta\left(3+2\left(m+\frac{1}{2}\right)\delta\right)}\cdot\mathrm{DGF}[|\mathcal{C}|]\left(\frac{3}{2}+\left(m+\frac{1}{2}\right)\delta\right)\right|.$$

3. Since the $c_n(\mathcal{C})$ are the *pentagonal numbers*, we have explicitly $\delta := \frac{1}{2}$ with $\mathcal{C}(q) := (q;q)_\infty$.

Back to the LGF expansion cases (some numerical data)

- Numerically, we find that the upper and lower bounds from my theorem yield a theoretical range of $\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C}) \in [0.169825, 0.7491]$.
- The actual sums for the ideal (q-Pochhammer) OGF for the LGF expansions yield that $\mathrm{Corr}_{\mathrm{LGF}}((q;q)_\infty)^{-1} \approx 0.195349$.
- This is pretty close to actually maximizing the correlation up to some error terms (up to some error that may not be attainable).

Back to the LGF expansion cases (some numerical data)
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- This is pretty close to actually maximizing the correlation up to some error terms (up to some error that may not be attainable).

1. Note that here we have the exact component DGF expansion
$$\mathrm{DGF}\,|\mathcal{C}|(s) \equiv \sum_{b=\pm 1}\sum_{j\geq 1}\left(\frac{j(3j+b)}{2}\right)^{-s}, \Re(s) > \frac{1}{2}.$$
2. Numerical computations of some variants for the LGF case:
$$\mathrm{Corr}(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1}\frac{1}{n}\times\frac{\sum_{k=1}^n c_k(\mathcal{C})\mathcal{D}^{-1}(n,k)}{\sqrt{\left(\sum_{k=1}^n c_k(\mathcal{C})^2\right)\left(\sum_{k=1}^n \mathcal{D}^{-1}(n,k)^2\right)}} \approx -0.469859.$$
$$\mathrm{Corr}(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1}\frac{1}{n}\times\frac{\sum_{k=1}^n c_k(\mathcal{C})\mathcal{D}(n,k)}{\sqrt{\left(\sum_{k=1}^n c_k(\mathcal{C})^2\right)\left(\sum_{k=1}^n \mathcal{D}(n,k)^2\right)}} \approx -2.65493.$$
$$\mathrm{Corr}(\mathcal{C},\mathcal{D}) := \sum_{n\geq 1}\frac{1}{n}\times\frac{\sum_{k=1}^n |c_k(\mathcal{C})\mathcal{D}(n,k)|}{\sqrt{\left(\sum_{k=1}^n c_k(\mathcal{C})^2\right)\left(\sum_{k=1}^n \mathcal{D}(n,k)^2\right)}} \approx 3.31356.$$

Lingering questions and request for algebra audience feedback

- Is it possible to do better for the LGF case?
- That is, can we define a more natural statistic to optimize so that $\mathcal{C}(q) := (q;q)_\infty$ *IS* actually going to yield the theoretical best possible correlation?
- What about constructions for the more general $\mathcal{D}$-convolution sums (many special sum types are wrapped into this definition)?

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- What about constructions for the more general $\mathcal{D}$-convolution sums (many special sum types are wrapped into this definition)?

1. How do you go about optimizing a particular correlation statistic formula (like we defined above) over all formal power series with integer coefficients and $\mathcal{C}(0) := 1$?
2. Any other thoughts or suggestions on this problem type?

Concluding remarks and discussion

The End

Questions?

Comments?

Feedback?

Thank you for attending!

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