Defining canonically best factorization theorems for the generating functions of special convolution type sums

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Goals of the talk

- Identify a general method for expanding out the generating functions of special sums (many examples)
- Questions about the most natural ways of forming the generating-function-based expansions of these sums

Generating functions of sequences

- Recall that an *ordinary generating function* (or OGF) of a sequence (or arithmetic function) $\{f_n\}_{n\geq 0}$ is defined by $F(z) := \sum_{n\geq 0} f_n \cdot z^n$.
- We write the series coefficient extraction operator in the notation $[z^n]F(z) \equiv f_n$ for $n \geq 0$.
- Why generating functions are useful in studying sequence properties?

Motivating examples

Motivating examples

Review: Multiplicative functions in number theory

- Recall that an arithmetic function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ for all integers $a, b \geq 1$ such that $(a, b) = 1$.
- Examples of multiplicative functions:
  1. The *Möbius function* $\mu(n)$ is the signed indicator function of the squarefree integers.
  2. Euler's classical *totient function* $\phi(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} 1 = \sum_{d|n} d\mu\left(\frac{n}{d}\right)$.
  3. The generalized *sum-of-divisors functions* $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$ with the special cases $d(n) = \sigma_0(n)$ and $\sigma(n) = \sigma_1(n)$.
  4. The completely multiplicative *Liouville lambda function* $\lambda(n) = (-1)^{\Omega(n)}$.

What is a Lambert series generating function (LGF)?

- Formally, given an arithmetic function $f : \mathbb{Z}^+ \to \mathbb{C}$ we define its *Lambert series generating function* (or LGF) to be

$$L_f(q) := \sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \sum_{m\geq 1} \left(\sum_{d|m} f(d)\right) q^m, |q| < 1.$$

- For $*$ denoting *Dirichlet convolution*, the RHS coefficients generated by $L_f(q)$ are $[q^n]L_f(q) = (f * 1)(n) = \sum_{d|n} f(d)$ for $n \geq 1$.
- Multiplicative functions tend to have nice expressions in terms of divisor sum convolutions of this type.

Examples of Lambert series generating functions

$$\sum_{n\geq 1} \frac{\mu(n)q^n}{1-q^n} = q, \tag{1a}$$

$$\sum_{n\geq 1} \frac{\phi(n)q^n}{1-q^n} = \frac{q}{(1-q)^2}, \tag{1b}$$

$$\sum_{n\geq 1} \frac{n^{\alpha}q^n}{1-q^n} = \sum_{m\geq 1} \sigma_{\alpha}(n)q^n, \tag{1c}$$

$$\sum_{n\geq 1} \frac{\lambda(n)q^n}{1-q^n} = \sum_{m\geq 1} q^{m^2}, \tag{1d}$$

$$\sum_{n\geq 1} \frac{\mu^2(n)q^n}{1-q^n} = \sum_{m\geq 1} 2^{\omega(m)}q^m. \tag{1e}$$

Review: The partition function $p(n)$

- A partition of a positive integer n is a (finite) sequence of positive integers whose sum is n.
- More formally, we may partition n as a sum of integers $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 1$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.
- We denote the total number of partitions of n by $p(n)$
- For example, $p(5) = 7$ since the distinct partitions of 5 are given by

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

Examples of partition function OGFs

Table 14.1 Generating functions

Generating function	The number of partitions of n into parts which are	Generating function	The number of partitions of n into parts which are
$\prod_{m=1}^{\infty} \frac{1}{1-x^{2m-1}}$	odd	$\prod_{m=1}^{\infty} (1+x^{2m-1})$	odd and unequal
$\prod_{m=1}^{\infty} \frac{1}{1-x^{2m}}$	even	$\prod_{m=1}^{\infty} (1+x^{2m})$	even and unequal
$\prod_{m=1}^{\infty} \frac{1}{1-x^{m^2}}$	squares	$\prod_{m=1}^{\infty} (1+x^{m^2})$	distinct squares
$\prod_{p} \frac{1}{1-x^{p}}$	primes	$\prod_{p} (1+x^{p})$	distinct primes
$\prod_{m=1}^{\infty} (1+x^{m})$	unequal		

Work on Lambert series factorization theorems

- Let $s_e(n,k)$ and $s_0(n,k)$ respectively denote the the number of k's in all partitions of n into an even (odd) number of distinct parts.
- Let $(a;q)_\infty = \prod_{m\geq 1}(1-aq^{m-1})$ denote the infinite q-Pochhammer symbol.
- Then we have

$$\sum_{n\geq 1}\frac{f(n)q^n}{1\pm q^n} = \frac{1}{(\mp q;q)_\infty}\sum_{n\geq 1}\left(\sum_{k=1}^{n}(s_o(n,k)\pm s_e(n,k))\,f(k)\right)q^n,$$

Lambert series factorization theorems (expressions by invertible matrices)

- Re-write the previous factorization theorem statement as

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k) \right) q^n.$$

- The matrices formed by the lower triangular sequence of $s_{n,k}$ are invertible with ones on the diagonal.
- We can prove exactly how $s_{n,k}^{-1}$ is related to $p(n)$:

$$s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right).$$

Relation of the matrices to partition functions brings up a natural question

- There is a very **natural** relation of both sequences of $s_{n,k}$ and $s_{n,k}^{-1}$ to *partition theoretic functions*.
- This is unusual in so much as LGFs typically generate *multiplicative* functions (product based properties), whereas partition function variants have a much more *additive* structure
- Brings up a natural question: Why did partitions fit so naturally with the multiplicative functions enumerated by the LGFs above?

More general constructions ($\mathcal{D}$-convolution type sums)

Generalized classes of convolution type sums

Examples: What other sum types are we interested in studying?

$$S_f(n) = \sum_{d|n} f(d) \qquad \text{(e.g., LGF cases)} \tag{2a}$$

$$S_f(n) = \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} f(d) \tag{2b}$$

$$S_f(n) = \sum_{\substack{d \in A_n \\ A_n \subseteq \{1,2,\ldots,n\}}} f(d) \qquad \text{(and weighted versions)} \tag{2c}$$

$$S_f(n) = \sum_{d=1}^{n} \begin{bmatrix} n \\ d \end{bmatrix} (-1)^{n-d} f(d) \qquad \text{(cf. Stirling transform)} \tag{2d}$$

$$S_{f,g}(n) = \sum_{d=1}^{n} \binom{d}{n} f(d) g(n+1-d) \qquad \text{(cf. binomial transform)} \tag{2e}$$

Generating functions for a more general class of sums

- Consider a fixed *kernel* function $\mathcal{D} : (\mathbb{Z}^+)^2 \to \mathbb{Z}$.
- Suppose that $\mathcal{D}(n, k)$ is lower triangular so that $\mathcal{D}(n, k) = 0$ whenever $k > n$.
- Suppose that $\mathcal{D}$ is invertible so that $\mathcal{D}(n, n) \neq 0$ for all $n \geq 1$.
- For any arithmetic functions f, g, we consider the class of *$\mathcal{D}$-convolution sums* of the form

$$(f \boxdot_{\mathcal{D}} g)(n) := \sum_{k=1}^{n} f(k) g(n+1-k) \mathcal{D}(n, k), n \geq 1.$$

Defining analogous factorization theorems to generate these sums (up to an undetermined OGF)

- We expand the generalized *factorization theorems* for any fixed $(\mathcal{C}, \mathcal{D})$ that uniquely determines the following expansions:

$$(f \boxdot_{\mathcal{D}} 1)(n) := [q^n]\left(\frac{1}{\mathcal{C}(q)} \times \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}(\mathcal{C}, \mathcal{D}) \cdot f(k) \cdot q^n\right), n \geq 1.$$

- Take $\mathcal{C}(q)$ any OGF with integer coefficients such that $\mathcal{C}(0) \neq 0$ (typically set to one up to normalization).
- For this fixed function, define the series coefficients $c_n(\mathcal{C}) := [q^n]\mathcal{C}(q)$ and $p_n(\mathcal{C}) := [q^n]\mathcal{C}(q)^{-1}$ for $n \geq 0$.

How do we choose a "canonically best" OGF $\mathcal{C}$?

- In the LGF case, the products for $\mathcal{C}(q) := (q;q)_\infty$ arise in arithmetic with these generating functions.
- The observation of how well the structurally revealing "natural choice" of sequences was from the LGF case is still very fuzzy and *qualitative*.
- **Big Question:** How do we *quantify* the notion of how well related the structures of the respective sequences $\{p_n(\mathcal{C})\}_{n\geq 0}$ and $\{\mathcal{D}(n,k)\}_{n\geq k\geq 1}$ are so that we can then prove the form of an optimal, or most structurally revealing, or say "*canonically best*" OGF $\mathcal{C}(q)$?

Idea: Define suitable cross-correlation statistics (and then sum them up)

- Consider a metric (statistic) that indicates the *cross-correlation* between these sequences.
- There are many ways to do this!
- A variant of the standard formula for a *Pearson correlation* statistic between any two N-tuples:

$$\mathrm{PearsonCorr}(N; \vec{a}, \vec{b}) := \frac{1}{N} \times \frac{\sum_{j=1}^{N} a_j \cdot b_j}{\sqrt{\sum_{1 \leq i,j \leq N} a_i^2 \cdot b_j^2}}.$$

More general cross-correlation statistic formulas to consider

- Let's look at finding $\mathcal{C}(q)$ such that the following statistic is maximized (minimized):

$$\operatorname{Corr}(\mathcal{C}, \mathcal{D}) := \sum_{n \geq 1} \frac{1}{n} \times \frac{\sum_{k=1}^{n} \left| c_k(\mathcal{C}) \mathcal{D}^{-1}(n,k) \right|}{\sqrt{\left(\sum_{k=1}^{n} c_k(\mathcal{C})^2 \right) \left(\sum_{k=1}^{n} \mathcal{D}^{-1}(n,k)^2 \right)}}.$$

- **Conjecture.** If $\operatorname{Corr}(\mathcal{C}, \mathcal{D})$ is optimized by a particular OGF $\mathcal{C}(q)$, then so is the alternate statistic

$$\operatorname{Corr}_*(\mathcal{C}, \mathcal{D}) := \sum_{n \geq 1} \frac{1}{n} \times \frac{\sum_{k=1}^{n} \left| p_k(\mathcal{C}) \mathcal{D}(n,k) \right|}{\sqrt{\left(\sum_{k=1}^{n} p_k(\mathcal{C})^2 \right) \left(\sum_{k=1}^{n} \mathcal{D}(n,k)^2 \right)}}.$$

Back to the LGF expansion cases

- For the LGF case, we have that $\mathcal{D}(n,k) := [k|n]_\delta$ and $\mathcal{D}^{-1}(n,k) = \mu(n/k)\,[k|n]_\delta$.
- This leads to the explicit formula for $\mathrm{Corr}(\mathcal{C},\mathcal{D})$ given by

$$\mathrm{Corr}_{LGF}(\mathcal{C}) := \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\mu^2(k)}{k(\sqrt{2})^{\omega(k)}} \times \sum_{j \leq \lfloor \frac{j}{k} \rfloor} \frac{|c_j(\mathcal{C})|}{j \cdot \rho_{\mathcal{C}}(jk)(\sqrt{2})^{\omega\left(\frac{j}{(j,k)}\right)}}.$$

where we define the partial variance of $\mathcal{C}$ to be

$$\rho_{\mathcal{C}}(N) := \sqrt{\sum_{1\leq i\leq N} c_i(\mathcal{C})^2}, N \geq 1.$$

Back to the LGF expansion cases

Theorem. Fix any $0 < \delta < +\infty$. Suppose that $\mathcal{C}(q)$ is an OGF whose series coefficients are integer valued so that $c_0(\mathcal{C}) = 1$ and where

$$A_0(\mathcal{C}, \delta) := \lim_{N \to \infty} \frac{1}{N^{\frac{\delta}{2}}} \times \sqrt{\sum_{1 \leq n \leq N} c_n(\mathcal{C})^2} \in [1, +\infty).$$

We have that

$$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C})^{-1} \geq \left(1 + \frac{1}{2A_0(\mathcal{C}, \delta)} \left(\frac{\zeta\left(\frac{2+\delta}{2}\right)}{\zeta(2+\delta)} - 1\right) \mathrm{DGF}\,|\mathcal{C}| \left(\frac{2+\delta}{2}\right) + \widehat{\omega}_\ell(\mathcal{C})\right)^{-1},$$

and

$$\mathrm{Corr}_{\mathrm{LGF}}(\mathcal{C})^{-1} \leq \left(\frac{1}{A_0(\mathcal{C}, \delta)} \cdot \frac{\zeta\left(\frac{3+\delta}{2}\right)}{\zeta(3+\delta)} \cdot \mathrm{DGF}\,|\mathcal{C}| \left(\frac{3+\delta}{2}\right) - \widehat{\omega}_u(\mathcal{C})\right)^{-1},$$

where the constants $\hat{\omega}_\ell(\mathcal{C})$ and $\hat{\omega}_u(\mathcal{C})$ can be explicitly bounded given any fixed $(\delta, \mathcal{C}(q))$.

Back to the LGF expansion cases (some numerical data)

- Numerically, we find that the upper and lower bounds from my theorem yield a theoretical range of $\mathrm{Corr}_{\mathsf{LGF}}(\mathcal{C}) \in [0.169825, 0.7491]$.
- The actual sums for the ideal (q-Pochhammer) OGF for the LGF expansions yield that $\mathrm{Corr}_{\mathsf{LGF}}((q;q)_{\infty})^{-1} \approx 0.195349$.
- This is pretty close to actually maximizing the correlation up to some error terms (up to some error that may not be attainable).

Lingering questions and request for algebra audience feedback

- Is it possible to do better for the LGF case?
- That is, can we define a more natural statistic to optimize so that $\mathcal{C}(q) := (q;q)_\infty *IS*$ actually going to yield the theoretical best possible correlation?
- What about constructions for the more general $\mathcal{D}$-convolution sums (many special sum types are wrapped into this definition)?

Concluding remarks and discussion

The End

Questions?

Comments?

Feedback?

Thank you for attending!

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