Computational aspects of factorization theorems for generating special sums

Maxie D. Schmidt

Georgia Institute of Technology
School of Mathematics

maxieds@gmail.com
mschmidt34@gatech.edu

http://people.math.gatech.edu/~mschmidt34/

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Talk sections

1. Overview and goals of the talk
2. Motivating examples
3. Another more general class of sums
4. A second general class of sums
5. Matrix generated convolution sums
6. Concluding Remarks
7. References
Overview and goals of the talk

- Identify a general method for expanding out the generating functions of special sums (many examples)
- Give motivating examples of why these expansions are useful.
- Questions about the most natural ways of forming the generating-function-based expansions of these sums.
- Focus on computational tools and experimental mathematics methods to bring new insights
Motivating examples of generating function factorizations for special sums
So-termed Lambert series factorization theorems

- In some sense, the first and prototypical example
- Relates multiplicative number theoretic functions to more additive constructions in the theory of partitions.

For example,

\[
\sum_{n \geq 1} \frac{f(n)q^n}{1 \pm q^n} = \frac{1}{(\mp q; q)_{\infty}} \sum_{n \geq 1} \left( \sum_{k=1}^{n} (s_o(n, k) \pm s_e(n, k)) f(k) \right) q^n,
\]

where \(s_e(n, k)\) and \(s_0(n, k)\) respectively denote the the number of \(k\)s in all partitions of \(n\) into an even (odd) number of distinct parts, and \((a; q)_{\infty} = \prod_{m \geq 1} (1 - aq^{m-1})\) is the infinite \(q\)-Pochhammer symbol.
Re-write as

\[
\sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \left( \sum_{k=1}^{n} s_{n,k} f(k) \right) q^n.
\]

The matrices formed by the lower triangular sequence of \( s_{n,k} \) are invertible.

It appears by inspection that the inverse matrix, \( s_{n,k}^{-1} \), is also related to partition functions.
A table of the first few rows of the inverse matrices

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
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<tr>
<td>2</td>
<td>1</td>
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<td>1001</td>
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<td>2414</td>
</tr>
<tr>
<td>26</td>
<td>2907</td>
</tr>
<tr>
<td>27</td>
<td>3717</td>
</tr>
</tbody>
</table>

The first 29 rows of the function $s_{n,k}^{(-1)}$ where the values of Euler’s partition function $p(n)$ are highlighted in blue and the remaining values of the partition function $q(n)$ are highlighted in purple (in both sequences) or pink.
Motivating examples
Lambert series factorizations

A formula for the inverse matrices

We can prove exactly how $s_{n,k}^{-1}$ is related to $p(n)$:

$$s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right),$$

where $\mu(n)$ is the Möbius function.

The choice of scaling the RHS of the generating function expansion of the Lambert series over $f$ by $1/(q;q)_\infty$ brings forth a natural relation of both sequences of $s_{n,k}$ and $s_{n,k}^{-1}$ to partition theoretic functions. [!! IMPORTANT !!]

Important Point: Is this the most natural way to expand things in the context of other special sums? (Probably not, but is there a canonical way to expand things?)
Expansions of Lambert series: More connections to partitions

- Another partition theoretic connection
- For any integers $m \geq 1$, we have

$$\sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \sum_{n \geq 1} \left( \sum_{i=1}^{m-1} \sum_{j=1}^{\left\lfloor \frac{n}{i} \right\rfloor} p_{m-1}(n - ij)f(i) + \sum_{k=m}^{n} s_{n,k}^{(m-1)} f(k) \right) q^n,$$

where $p_m(n)$ denotes the number of partitions of $n$ that do not contain $1, 2, \ldots, m$ as a part and where $s_{n,k}^{(m)}$ denotes the number of $k$'s in all unrestricted partitions that do not contain $1, 2, \ldots, m$ as a part.
Key points about the definition of these factorizations

- Not one single "right", or canonical way to always expand the generating functions
- It’s important to experiment with Mathematica (or other CAS software) to find new interesting symbolic properties for creative new variations.

(Without these methods available most of the cited papers would not have been possible!)
Variants of these expansions of Lambert series I

Let $\tilde{f}_\gamma(n) := \sum_{d|n} f(d)(\gamma * 1)(n/d)$, where $(\gamma * 1)(n) = \sum_{d|n} \gamma(d)$.

Let $C(q)$ be any generating function such that $C(0) \neq 0$, and consider expanding

$$\sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \frac{1}{C(q)} \sum_{n \geq 1} \left( \sum_{k=1}^{n} s_{n,k}(\gamma)\tilde{f}_\gamma(k) \right) q^n,$$

where $\gamma(n)$ is any prescribed arithmetic function.

Then we have that

$$s_{n,k}^{-1}(\gamma) = \sum_{d|n} [q^{d-k}] \frac{1}{C(q)} \cdot \gamma\left(\frac{n}{d}\right).$$

Note: In the first Lambert series factorization case, we had $\gamma \equiv \mu$, so that

$(\gamma * 1)(n) = \delta_{n,1} \implies \tilde{f}_\gamma(n) = f(n)$, and $C(q) \equiv (q; q)_{\infty}$ so that

$[q^{d-k}]1/C(q) = p(d - k)$. 
Let \( F(x) := \sum_{n \leq x} f(n) \) denote the summatory function of \( f(n) \).

**Variation 1:** (Corresponding matrix formulas are an exercise)

\[
\sum_{n \geq 1} \frac{F(n)q^n}{1 - q^n} = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{1,n,k} f(k) \cdot q^n.
\]

**Variation 2:** (Corresponding matrix formulas are an exercise)

\[
\sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{2,n,k} F(k) \cdot q^n.
\]
Variants and other possibilities III

▶ Variation 3: (Exercise)

\[ \sum_{n \geq 1} \frac{f(n)q^n}{1 - q^n} = \frac{1}{C(q)} \sum_{n \geq 1} \sum_{k=1}^{n} \tilde{s}_{n,k}(g) \left( \sum_{i=1}^{k} g(i)f(i) \right) \cdot q^n. \]

▶ Variation 4: (Exercise)

Find a formula for \( \hat{f}_\gamma(n) \) defined by

\[ \sum_{n \geq 1} \frac{\hat{f}_\gamma(n)q^n}{1 - q^n} = \frac{1}{C(q)} \sum_{n \geq 1} \sum_{k=1}^{n} \tilde{s}_{n,k}(\gamma)F(k) \cdot q^n, \]

where we require that the inverse matrices satisfy

\[ \tilde{s}_{n,k}^{-1}(\gamma) := \sum_{d \mid n} [q^{d-k}] \frac{1}{C(q)} \cdot \gamma \left( \frac{n}{d} \right), \]

for any fixed arithmetic function \( \gamma \).
Lambert series of Dirichlet convolutions

- The Dirichlet convolution of two arithmetic functions $f$ and $g$ is defined for $n \geq 1$ as

$$
(f \ast g)(n) := \sum_{d|n} f(d)g\left(\frac{n}{d}\right).
$$

- Consider factorizing the Lambert series generating function of $f \ast g$:

$$
\sum_{n \geq 1} \frac{(f \ast g)(n)q^n}{1 - q^n} = \frac{1}{(q; q)^{m}} \sum_{n \geq 1} \sum_{k=1}^{n} \tilde{s}_{n,k}(g)f(k) \cdot q^n.
$$

- Then we can prove that $\tilde{s}_{n,k}(g) = \sum_{j=1}^{n} s_{n,kj} \cdot g(j)$.

- What about the inverse matrices $\tilde{s}_{n,k}^{-1}(g)$?

(Hint: There is not an obvious solution without looking at example cases in detail, and being able to modify them to see what “works” experimentally, in real time.) [!! IMPORTANT !!]
Dirichlet convolutions – Computations with Mathematica

We can fire up Mathematica to compute as many terms of the sequence $\tilde{s}_{n,k}$ as we need using the following code:

```
DirichletConvolutions.nb

Clear[ComputeMatrixSequence];
ComputeMatrixSequence[upperN_, f_, g_, invertMatrix_:True, simplifyFunc_:Null] :=
  Module[{lhsLSeries, rhsGFSeries, seriesDiff, zeroCoeffs,
    snkEqns, snkVars, snkSols, snkTable},
    lhsLSeries=Sum[DivisorSum[n,f[#]g[n/#]&] Power[q,n]/(1-Power[q,n]),{n,1,upperN}];
    rhsGFSeries=1/QPochhammer[q,q]Sum[s[n,k]f[k]Power[q,n], {n,1,upperN}, {k,1,n}];
    seriesDiff=Collect[Series[lhsLSeries-rhsGFSeries, {q,0,upperN}],f[_],Factor];
    seriesDiff=(seriesDiff) /. (f[ni_]->Power[z, ni]);
    zeroCoeffs=Flatten[CoefficientList[Normal[seriesDiff], {z,q}]];
    snkEqns=LogicalExpand[zeroCoeffs==ConstantArray[0, Length[zeroCoeffs]]];
    snkVars=Flatten[Table[s[n,k], {n,1,upperN}, {k,1,n}]];
    snkSols=Solve[snkEqns,snkVars][[1]];
    snkTable=Table[If[k <= n, s[n,k], 0] /. snkSols, {n,1,upperN},{k,1,upperN}];
    If[invertMatrix,
      snkTable=Inverse[snkTable];
    ];
    If[simplifyFunc != Null,
      snkTable=simplifyFunc[snkTable];
    ];
    Return[snkTable];
  ];
Clear[f, g];
upperN=6;
ComputeMatrixSequence[upperN, f, g, True, Simplify] // TableForm
```
Motivating examples

Dirichlet convolutions

Dirichlet convolutions – Examples of the inverse matrices

<table>
<thead>
<tr>
<th>n \ k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \frac{1}{g(1)} )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>(- \frac{g(2)}{g(1)^2})</td>
<td>(\frac{1}{g(1)})</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>(g(1)^5 - g(1)^4 g(3))</td>
<td>(\frac{1}{g(1)})</td>
<td>(\frac{1}{g(1)})</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>(\frac{2g(1)^5 - g(4)g(1)^4 + g(2)^2 g(1)^3}{g(1)^6})</td>
<td>(g(1)^5 - g(1)^4 g(2))</td>
<td>(\frac{1}{g(1)})</td>
<td>(\frac{1}{g(1)})</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>(\frac{4g(1)^5 - g(1)^4 g(5)}{g(1)^6})</td>
<td>(\frac{3}{g(1)})</td>
<td>(\frac{2}{g(1)})</td>
<td>(\frac{1}{g(1)})</td>
<td>(\frac{1}{g(1)})</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>(\frac{5g(1)^5 - g(2)g(1)^4 - g(6)g(1)^4 + 2g(2)g(3)g(1)^3}{g(1)^6})</td>
<td>(\frac{3g(1)^5 - g(2)g(1)^4 - g(3)g(1)^4}{g(1)^6})</td>
<td>(\frac{2g(1)^5 - g(1)^4 g(2)}{g(1)^6})</td>
<td>(\frac{2}{g(1)})</td>
<td>(\frac{1}{g(1)})</td>
<td>(\frac{1}{g(1)})</td>
</tr>
</tbody>
</table>

It’s hard to make out the precise pattern from the data above, but looking closely suggests that the terms are partition-scaled multiple \((m\text{-fold})\) Dirichlet convolutions related to the indices \((n, k)\).
A suggestion: Look at Dirichlet inverse functions

- The function $h^{-1}(n)$ is the Dirichlet inverse of $h(n)$ if $h^{-1} \ast h = h \ast h^{-1} = \varepsilon$, where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution.

- The function $h$ has a Dirichlet inverse iff $h(1) \neq 0$.

- If $h^{-1}$ exists, then it can be computed recursively by the formula

$$h^{-1}(n) = \begin{cases} \frac{1}{h(1)}, & n = 1; \\ -\frac{1}{h(1)} \sum_{d|n, d>1} h(d)h^{-1}\left(\frac{n}{d}\right), & n \geq 2. \end{cases}$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$f^{-1}(n)$</th>
<th>$n$</th>
<th>$f^{-1}(n)$</th>
<th>$n$</th>
<th>$f^{-1}(n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{h(1)}$</td>
<td>4</td>
<td>$-\frac{h(1)h(4)-h(2)^2}{h(1)^3}$</td>
<td>7</td>
<td>$-\frac{h(7)}{h(1)^2}$</td>
</tr>
<tr>
<td>2</td>
<td>$-\frac{h(2)}{h(1)^2}$</td>
<td>5</td>
<td>$-\frac{h(5)}{h(1)^2}$</td>
<td>8</td>
<td>$\frac{h(2)^3-2h(1)h(4)h(2)+h(1)^2h(8)}{h(1)^4}$</td>
</tr>
<tr>
<td>3</td>
<td>$-\frac{h(3)}{h(1)^2}$</td>
<td>6</td>
<td>$-\frac{h(1)h(6)-2h(2)h(3)}{h(1)^3}$</td>
<td>9</td>
<td>$\frac{h(1)h(9)-h(3)^2}{h(1)^3}$</td>
</tr>
</tbody>
</table>
### Dirichlet convolutions – Examples of the inverse matrices simplified

#### A Simplifying Question: What happens if we set $g(1) \mapsto 1$?

<table>
<thead>
<tr>
<th>n</th>
<th>k</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td></td>
<td>$-g(2)$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>$1 - g(3)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td></td>
<td>$g(2)^2 - g(4) + 2$</td>
<td>$1 - g(2)$</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td></td>
<td>$4 - g(5)$</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td></td>
<td>$2g(3)g(2) - g(2) - g(6) + 5$</td>
<td>$-g(2) - g(3) + 3$</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td></td>
<td>$10 - g(7)$</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td></td>
<td>$-g(2)^3 + 2g(4)g(2) - 2g(2) - g(8) + 12$</td>
<td>$g(2)^2 - g(2) - g(4) + 9$</td>
<td>6</td>
<td>4</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td></td>
<td>$g(3)^2 - g(3) - g(9) + 20$</td>
<td>$14 - g(3)$</td>
<td>10</td>
<td>7</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td></td>
<td>$2g(5)g(2) - 4g(2) - g(10) + 25$</td>
<td>$-3g(2) - g(5) + 18$</td>
<td>13</td>
<td>10</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>2</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

#### Observation: These inverse matrix entries now definitely look like quasi-partition function scaled sums of the Dirichlet inverse functions!
Returning to the Lambert series factorization over $f \ast g$

- Define $g_\pm(n) = g(n)\lfloor n > 1 \rfloor \delta - g(1)\lfloor n = 1 \rfloor \delta$, or equivalently, $g_\pm(n) = g(n) - 2\varepsilon(n)g(1)$.

- Define the $j$-fold multiple convolution of $g$ as

$$ds_{j,g}(n) := \begin{cases} g_\pm(n), & j = 1; \\ \sum_{d \mid n, \ d > 1} g(d) \, ds_{j-1,g}(n/d), & j \geq 2. \end{cases}$$

- Set $D_{u,g}(n) := \sum_{j=1}^{u} ds_{2j,g}(n)$.

- Let $p_k(n) := p(n - k)$ for $k \geq 1$. Then we can prove that

$$\tilde{s}_{n,k}^{-1}(g) = (p_k \ast \mu)(n) + (p_k \ast D_{\Omega(n),g} \ast \mu)(n).$$
A class of Type I and Type II sums

- Analogous factorization theorems for *type I sums* of the form
  \[\sum_{1 \leq d \leq n, (d,n)=1} f(d)\].

- Factorization theorems for *type II sums* (Anderson-Apostol sums) of the form
  \[\sum_{d|(m,n)} f(d)g\left(\frac{n}{d}\right), \forall n \geq 1; \text{fixed } 1 \leq m \leq n\].

- **Motivation**: The summatory functions are given by compositions of these two types of sums:
  \[\sum_{d \leq x} f(d) = \sum_{d=1}^{x} f(d) + \sum_{d|x} f(d) + \sum_{1<d<x} f(d)\]
  \[= \sum_{m|x} \sum_{k=1}^{m} f\left(\frac{kx}{m}\right)\].
Motivating examples

GCD and relatively prime divisor sums

More general classes of generating function factorizations for special sums

More general expansions of special sums
More generally, we can consider factorization theorems of the form

$$\sum_{n \geq 1} \left( \sum_{k \in A_n} f(k) \right) q^n = \frac{1}{C(q)} \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}(A, C) f(k) \cdot q^n.$$ 

The matrix entries are generated by

$$s_{n,k}(A, C) = [q^n]C(q) \times \sum_{m \geq 1} [k \in A_m] \delta \cdot q^m.$$ 

**Natural question:** What is a “good” choice of the generating function $C$ given the definitions of the sets $\{A_n\}_{n \geq 1}$ that leads to natural formulas for $s_{n,k}(A, C)$ and its inverse matrix?

The inverse matrices are very closely tied up to generalized forms of Möbius inversion by special Möbius functions defined by the problem.
An example: A-convolutions (restricted Dirichlet convolutions and divisor sums)

- For each $n \geq 1$, let $A(n) \subseteq \{ d : 1 \leq d \leq n, d|n \}$ be a subset of the divisors of $n$.

- We can define the A-convolution of $f$ and $g$ by
  \[
  (f \ast_A g)(n) := \sum_{d \in A(n)} f(d)g\left(\frac{n}{d}\right).
  \]

- We say that a natural number $n \geq 1$ is A-primitive if $A(n) = \{1, n\}$.

- Under a list of assumptions so that the resulting A-convolutions are regular convolutions, we get a generalized multiplicative Möbius function:
  \[
  \mu_A(p^\alpha) = \begin{cases} 
  1, & \alpha = 0; \\
  -1, & p^\alpha > 1 \text{ is A-primitive}; \\
  0, & \text{otherwise}.
  \end{cases}
  \]

- This construction leads to a generalized form of Möbius inversion between the A-convolutions.
A second general class of sums

Definitions

Generating functions for a general class of $K$-convolutions

Let the function $K : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be defined on all ordered pairs $(n, d)$ such that $n \geq 1$ and $d | n$.

We define the $K$-convolution of two arithmetic functions $f, g$ to be

$$(f \circ_K g)(n) := \sum_{d | n} f(d)g\left(\frac{n}{d}\right)K(n, d).$$

We can define generating function factorizations of these sums of the form

$$(f \circ_K g)(n) = [q^n] \frac{1}{C(q)} \sum_{m \geq 1} \sum_{k=1}^{m} e_{n,k}(K, C; g)f(k) \cdot q^m.$$

Same questions arise: Natural choices of $C(q)$ that lead to pleasing formulas for the sequences of $e_{n,k}(K, C; g)$; And what are the corresponding forms of the inverse matrices, $e_{n,k}^{-1}(K, C; g)$?
Example: \( B\)-convolutions

- Let \( \nu_p(n) \) denote the maximum exponent of the prime \( p \) in the factorization of \( n \).
- Define \( B(n, d) := \prod_{p|n} \left(\frac{\nu_p(n)}{\nu_p(d)}\right) \) where the product runs over all prime divisors of \( n \).
- Then we have an inversion formula of the form

\[
 f(n) = \sum_{d|n} g(d)B(n, d) \iff g(n) = \sum_{d|n} f(d)\lambda(d)B(n, d),
\]

where \( \lambda(n) = (-1)^{\Omega(n)} \) is the Liouville lambda function.
Let $G := (g_{i,j})$ be an infinite dimensional matrix containing only elements in $\{0, 1\}$.

Suppose that $g_{ij} = 1$ if $i = j$ and $g_{ij} = 0$ if $i > j$.

Furthermore, let the 1’s in column $n$ of $G$ appear in rows $n_1, n_2, \ldots, n_k$ with $n_1 < n_2 < \cdots < n_k = n$.

We can define the convolution of two arithmetic functions $f, g$ by

$$(f \ast_G g)(n) := \sum_{i=1}^{k} f(n_i)g(n_{k+1-i}).$$
Examples and special cases

Let $D = (d_{ij})$ where $d_{ij} = [i|j]_\delta$. Then the convolution operation generated by $*_D$ is the same as the standard Dirichlet convolution of two arithmetic functions.

Let $T = (t_{ij})$ where $t_{ij} = [(i,j) = 1]_\delta [i \leq j]_\delta$. Then we get a convolution variant of the type I relatively prime GCD sums we defined before by

$$
\sum_{1 \leq d \leq n} f(d).
\sum_{(d,n) = 1}
$$
The End

Questions?
Comments?
Feedback?

Thank you for attending!
References


