Computational aspects of factorization theorems for generating special sums

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Talk sections

Overview and goals of the talk

- Identify a general method for expanding out the generating functions of special sums (many examples)
- Give motivating examples of why these expansions are useful.
- Questions about the most natural ways of forming the generating-function-based expansions of these sums.
- Focus on computational tools and experimental mathematics methods to bring new insights

Motivating examples of generating function factorizations for special sums

Motivating examples

So-termed Lambert series factorization theorems

- In some sense, the first and prototypical example
- Relates multiplicative number theoretic functions to more additive constructions in the theory of partitions.

For example,

$$\sum_{n\geq 1}\frac{f(n)q^n}{1\pm q^n}=\frac{1}{(\mp q;q)_\infty}\sum_{n\geq 1}\left(\sum_{k=1}^{n}(s_o(n,k)\pm s_e(n,k))\,f(k)\right)q^n,$$

where $s_e(n,k)$ and $s_0(n,k)$ respectively denote the the number of ks in all partitions of n into an even (odd) number of distinct parts, and $(a;q)_\infty=\prod_{m\geq 1}(1-aq^{m-1})$ is the infinite q-Pochhammer symbol.

Lambert series factorization theorems (inverse matrices)

- Re-write as

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k)\right) q^n.$$

- The matrices formed by the lower triangular sequence of $s_{n,k}$ are invertible.
- It appears by inspection that the inverse matrix, $s_{n,k}^{-1}$, is also related to partition functions.

A table of the first few rows of the inverse matrices

1																												
	1																											
1	1	1																										
2	1	1	1																									
4	3	2	1	1																								
5	3	2	2	1	1																							
10	7	5	3	2	1	1																						
12	9	6	4	3	2	1	1																					
20	14	10	7	5	3	2	1	1																				
25	18	13	10	6	5	3	2	1	1																			
41	30	22	15	11	7	5	3	2	1	1																		
47	36	26	19	14	10	7	5	3	2	1	1																	
76	56	42	30	22	15	11	7	5	3	2	1	1																
90	69	51	39	28	21	14	11	7	5	3	2	1	1															
129	97	74	55	41	30	22	15	11	7	5	3	2	1	1														
161	124	94	72	53	40	29	21	15	11	7	5	3	2	1	1													
230	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1												
270	212	163	126	95	73	54	41	29	22	15	11	7	5	3	2	1	1											
384	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1										
458	362	281	219	169	130	98	75	55	41	30	22	15	11	7	5	3	2	1	1									
615	482	379	294	229	175	134	101	77	56	42	30	22	15	11	7	5	3	2	1	1								
750	596	468	370	286	224	171	132	99	76	55	42	30	22	15	11	7	5	3	2	1	1							
1001	792	627	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1						
1187	951	756	601	472	372	289	225	173	133	100	76	56	42	30	22	15	11	7	5	3	2	1	1					
1570	1252	1000	791	626	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1				
1881	1518	1213	972	770	612	479	378	292	228	174	134	100	77	56	42	30	22	15	11	7	5	3	2	1	1			
2414	1943	1564	1248	997	789	625	489	384	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1		
2907	2358	1901	1532	1225	980	777	616	483	380	294	229	175	134	101	77	56	42	30	22	15	11	7	5	3	2	1	1	
3717	3010	2436	1958	1575	1255	1002	792	627	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1

The first 29 rows of the function $s_{n,k}^{(-1)}$ where the values of Euler's partition function $p(n)$ are highlighted in blue and the remaining values of the partition function $q(n)$ are highlighted in purple (in both sequences) or pink.

A formula for the inverse matrices

- We can prove exactly how $s_{n,k}^{-1}$ is related to $p(n)$:

$$s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right),$$

  where $\mu(n)$ is the Möbius function.
- The choice of scaling the RHS of the generating function expansion of the Lambert series over f by $1/(q;q)_\infty$ brings forth a **natural** relation of both sequences of $s_{n,k}$ and $s_{n,k}^{-1}$ to *partition theoretic functions*. **[!! IMPORTANT !!]**
- **Important Point:** Is this the most natural way to expand things in the context of other special sums?
  (*Probably not, but is there a canonical way to expand things?*)

Expansions of Lambert series: More connections to partitions

- Another partition theoretic connection
- For any integers $m \geq 1$, we have

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \sum_{n\geq 1} \left(\sum_{i=1}^{m-1} \sum_{j=1}^{\left\lfloor \frac{n}{i} \right\rfloor} p_{m-1}(n-ij)f(i) + \sum_{k=m}^{n} s_{n,k}^{(m-1)} f(k)\right) q^n,$$

where where $p_m(n)$ denotes the number of partitions of n that do not contain $1, 2, \ldots, m$ as a part and where $s_{n,k}^{(m)}$ denotes the number of k's in all unrestricted partitions that do not contain $1, 2, \ldots, m$ as a part.

Key points about the definition of these factorizations

- Not one single "right", or canonical way to always expand the generating functions
- It's important to experiment with *Mathematica* (or other CAS software) to find new interesting symbolic properties for creative new variations.
  (Without these methods available most of the cited papers would not have been possible!)

Variants of these expansions of Lambert series I

- Let $\widetilde{f}_\gamma(n) := \sum_{d|n} f(d)(\gamma * 1)(n/d)$, where $(\gamma * 1)(n) = \sum_{d|n} \gamma(d)$.
- Let $C(q)$ be any generating function such that $C(0) \neq 0$, and consider expanding

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k}(\gamma) \widetilde{f}_\gamma(k) \right) q^n,$$

  where $\gamma(n)$ is any prescribed arithmetic function.
- Then we have that

$$s_{n,k}^{-1}(\gamma) = \sum_{d|n} [q^{d-k}] \frac{1}{C(q)} \cdot \gamma\left(\frac{n}{d}\right).$$

Note: *In the first Lambert series factorization case, we had $\gamma \equiv \mu$, so that $(\gamma * 1)(n) = \delta_{n,1} \implies \widetilde{f}_\gamma(n) = f(n)$, and $C(q) \equiv (q;q)_\infty$ so that $[q^{d-k}]1/C(q) = p(d-k)$.*

Other possibilities for Lambert-series-like factorizations II

- Let $F(x) := \sum_{n \leq x} f(n)$ denote the summatory function of $f(n)$.
- **Variation 1:** (Corresponding matrix formulas are an exercise)

$$\sum_{n \geq 1} \frac{F(n) q^n}{1 - q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{1,n,k} f(k) \cdot q^n.$$

- **Variation 2:** (Corresponding matrix formulas are an exercise)

$$\sum_{n \geq 1} \frac{f(n) q^n}{1 - q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{2,n,k} F(k) \cdot q^n.$$

Variants and other possibilities III

- **Variation 3:** (Exercise)

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \sum_{k=1}^{n} \widetilde{s}_{n,k}(g) \left(\sum_{i=1}^{k} g(i)f(i) \right) \cdot q^n.$$

- **Variation 4:** (Exercise)
  Find a formula for $\widehat{f}_{\gamma}(n)$ defined by

$$\sum_{n\geq 1} \frac{\widehat{f}_{\gamma}(n)q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \sum_{k=1}^{n} \widehat{s}_{n,k}(\gamma)F(k) \cdot q^n,$$

  where we require that the inverse matrices satisfy

$$\widehat{s}_{n,k}^{-1}(\gamma) := \sum_{d\mid n} [q^{d-k}] \frac{1}{C(q)} \cdot \gamma\left(\frac{n}{d}\right),$$

  for any fixed arithmetic function γ.

Lambert series of Dirichlet convolutions

- The Dirichlet convolution of two arithmetic functions f and g is defined for $n \geq 1$ as

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

- Consider factorizing the Lambert series generating function of $f * g$:

$$\sum_{n \geq 1} \frac{(f * g)(n) q^n}{1 - q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} \widetilde{s}_{n,k}(g) f(k) \cdot q^n.$$

- Then we can prove that $\widetilde{s}_{n,k}(g) = \sum_{j=1}^{n} s_{n,kj} \cdot g(j)$.
- What about the inverse matrices $\tilde{s}_{n,k}^{-1}(g)$?
(*Hint: There is not an obvious solution without looking at example cases in detail, and being able to modify them to see what "works" experimentally, in real time.*) **[!! IMPORTANT !!]**

Dirichlet convolutions – Computations with Mathematica

We can fire up *Mathematica* to compute as many terms of the sequence $\tilde{s}_{n,k}^{-1}$ as we need using the following code:

DirichletConvolutionMatrices.nb

```
Clear[ComputeMatrixSequence];
ComputeMatrixSequence[upperN_, f_, g_, invertMatrix_:True, simplifyFunc_:Null] :=
     Module[{lhsLSeries,rhsGFSeries,seriesDiff,zeroCoeffs,
             snkEqns,snkVars,snkSols,snkTable},
     lhsLSeries=Sum[DivisorSum[n,f[#]g[n/#]&] Power[q,n]/(1-Power[q,n]),{n,1,upperN}];
     rhsGFSeries=1/QPochhammer[q,q]Sum[s[n,k]f[k]Power[q,n], {n,1,upperN}, {k,1,n}];
     seriesDiff=Collect[Series[lhsLSeries-rhsGFSeries, {q,0,upperN}],f[_],Factor];
     seriesDiff=(seriesDiff) /. (f[ni_]->Power[z, ni]);
     zeroCoeffs=Flatten[CoefficientList[Normal[seriesDiff], {z,q}]];
     snkEqns=LogicalExpand[zeroCoeffs==ConstantArray[0, Length[zeroCoeffs]]];
     snkVars=Flatten[Table[s[n,k], {n,1,upperN}, {k,1,n}]];
     snkSols=Solve[snkEqns,snkVars][[1]];
     snkTable=Table[If[k <= n, s[n,k], 0] /. snkSols, {n,1,upperN},{k,1,upperN}];
     If[invertMatrix,
          snkTable=Inverse[snkTable];
     ];
     If[simplifyFunc != Null,
          snkTable=simplifyFunc[snkTable];
     ];
     Return[snkTable];
];
Clear[f, g];
upperN=6;
ComputeMatrixSequence[upperN, f, g, True, Simplify] // TableForm
```

Dirichlet convolutions – Examples of the inverse matrices

n \ k	1	2	3	4	5	6
1	$\frac{1}{g(1)}$	0	0	0	0	0
2	$-\frac{g(2)}{g(1)^2}$	$\frac{1}{g(1)}$	0	0	0	0
3	$\frac{g(1)^5-g(1)^4g(3)}{g(1)^6}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$	0	0	0
4	$\frac{2g(1)^5-g(4)g(1)^4+g(2)^2g(1)^3}{g(1)^6}$	$\frac{g(1)^5-g(1)^4g(2)}{g(1)^6}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$	0	0
5	$\frac{4g(1)^5-g(1)^4g(5)}{g(1)^6}$	$\frac{3}{g(1)}$	$\frac{2}{g(1)}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$	0
6	$\frac{5g(1)^5-g(2)g(1)^4-g(6)g(1)^4+2g(2)g(3)g(1)^3}{g(1)^6}$	$\frac{3g(1)^5-g(2)g(1)^4-g(3)g(1)^4}{g(1)^6}$	$\frac{2g(1)^5-g(1)^4g(2)}{g(1)^6}$	$\frac{2}{g(1)}$	$\frac{1}{g(1)}$	$\frac{1}{g(1)}$

It's hard to make out the precise pattern from the data above, but looking closely suggests that the terms are partition-scaled multiple (m-fold) Dirichlet convolutions related to the indices (n,k).

A suggestion: Look at Dirichlet inverse functions

- The function $h^{-1}(n)$ is the *Dirichlet inverse* of $h(n)$ if $h^{-1} * h = h * h^{-1} = \varepsilon$, where $\varepsilon(n) = \delta_{n,1}$ is the multiplicative identity with respect to Dirichlet convolution.
- The function h has a Dirichlet inverse iff $h(1) \neq 0$.
- If h^{-1} exists, then it can be computed recursively by the formula

$$h^{-1}(n) = \begin{cases} \frac{1}{h(1)}, & n = 1; \\ -\frac{1}{h(1)} \sum\limits_{\substack{d|n \\ d>1}} h(d) h^{-1}\left(\frac{n}{d}\right), & n \geq 2. \end{cases}$$

n	$f^{-1}(n)$	**n**	$f^{-1}(n)$	**n**	$f^{-1}(n)$
1	$\frac{1}{h(1)}$	**4**	$-\frac{h(1)h(4)-h(2)^2}{h(1)^3}$	**7**	$-\frac{h(7)}{h(1)^2}$
2	$-\frac{h(2)}{h(1)^2}$	**5**	$-\frac{h(5)}{h(1)^2}$	**8**	$-\frac{h(2)^3-2h(1)h(4)h(2)+h(1)^2h(8)}{h(1)^4}$
3	$-\frac{h(3)}{h(1)^2}$	**6**	$-\frac{h(1)h(6)-2h(2)h(3)}{h(1)^3}$	**9**	$-\frac{h(1)h(9)-h(3)^2}{h(1)^3}$

Dirichlet convolutions – Examples of the inverse matrices simplified

A Simplifying Question: What happens if we set $g(1) \mapsto 1$?

n \ k	1	2	3	4	5	6	7	8	9	10
2	$-g(2)$	1	0	0	0	0	0	0	0	0
3	$1-g(3)$	1	1	0	0	0	0	0	0	0
4	$g(2)^2-g(4)+2$	$1-g(2)$	1	1	0	0	0	0	0	0
5	$4-g(5)$	3	2	1	1	0	0	0	0	0
6	$2g(3)g(2)-g(2)-g(6)+5$	$-g(2)-g(3)+3$	$2-g(2)$	2	1	1	0	0	0	0
7	$10-g(7)$	7	5	3	2	1	1	0	0	0
8	$-g(2)^3+2g(4)g(2)-2g(2)-g(8)+12$	$g(2)^2-g(2)-g(4)+9$	$6-g(2)$	$4-g(2)$	3	2	1	1	0	0
9	$g(3)^2-g(3)-g(9)+20$	$14-g(3)$	$10-g(3)$	7	5	3	2	1	1	0
10	$2g(5)g(2)-4g(2)-g(10)+25$	$-3g(2)-g(5)+18$	$13-2g(2)$	$10-g(2)$	$6-g(2)$	5	3	2	1	1

Observation: These inverse matrix entries now definitely look like quasi-partition function scaled sums of the Dirichlet inverse functions!

Returning to the Lambert series factorization over $f * g$

- Define $g_{\pm}(n) = g(n)\,[n > 1]_{\delta} - g(1)\,[n = 1]_{\delta}$, or equivalently, $g_{\pm}(n) = g(n) - 2\varepsilon(n)g(1)$.
- Define the j-fold multiple convolution of g as

$$\mathrm{ds}_{j,g}(n) := \begin{cases} g_{\pm}(n), & j = 1; \\ \sum\limits_{\substack{d|n \\ d>1}} g(d)\,\mathrm{ds}_{j-1,g}(n/d), & j \geq 2. \end{cases}$$

- Set $D_{u,g}(n) := \sum_{j=1}^{u} \mathrm{ds}_{2j,g}(n)$.
- Let $p_k(n) := p(n-k)$ for $k \geq 1$. Then we can prove that

$$\tilde{s}_{n,k}^{-1}(g) = (p_k * \mu)(n) + (p_k * D_{\Omega(n),g} * \mu)(n).$$

A class of Type I and Type II sums

- Analogous factorization theorems for *type I sums* of the form
$$\sum_{\substack{1\leq d\leq n \\ (d,n)=1}} f(d).$$
- Factorization theorems for *type II sums* (Anderson-Apostol sums) of the form
$$\sum_{d|(m,n)} f(d)g\left(\frac{n}{d}\right), \forall n \geq 1; \text{fixed } 1 \leq m \leq n.$$
- **Motivation:** The summatory functions are given by compositions of these two types of sums:
$$\begin{aligned}\sum_{d\leq x} f(d) &= \sum_{\substack{d=1 \\ (d,x)=1}}^{x} f(d) + \sum_{d|x} f(d) + \sum_{\substack{d=1 \\ 1<(d,x)<x}}^{x} f(d) \\ &= \sum_{m|x} \sum_{\substack{k=1 \\ (k,m)=1}}^{m} f\left(\frac{kx}{m}\right).\end{aligned}$$

More general classes of generating function factorizations for special sums

More general expansions of special sums

Generating functions for a more general class of sums

- More generally, we can consider factorization theorems of the form

$$\sum_{n\geq 1}\left(\sum_{\substack{k\in\mathcal{A}_n \\ \mathcal{A}_n\subseteq[1,n]}} f(k)\right) q^n = \frac{1}{\mathcal{C}(q)}\sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k}(\mathcal{A},\mathcal{C}) f(k)\cdot q^n.$$

- The matrix entries are generated by

$$s_{n,k}(\mathcal{A},\mathcal{C}) = [q^n]\mathcal{C}(q)\times\sum_{m\geq 1}[k\in\mathcal{A}_m]_{\delta}\cdot q^m.$$

- **Natural question:** What is a "good" choice of the generating function $\mathcal{C}$ given the definitions of the sets $\{\mathcal{A}_n\}_{n\geq 1}$ that leads to natural formulas for $s_{n,k}(\mathcal{A},\mathcal{C})$ and its inverse matrix?
- The inverse matrices are very closely tied up to generalized forms of Möbius inversion by special Möbius functions defined by the problem.

An example: A-convolutions (restricted Dirichlet convolutions and divisor sums)

- For each $n \geq 1$, let $A(n) \subseteq \{d : 1 \leq d \leq n, d|n\}$ be a subset of the divisors of n.
- We can define the *A-convolution* of f and g by
$$(f *_A g)(n) := \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right).$$
- We say that a natural number $n \geq 1$ is *A-primitive* if $A(n) = \{1, n\}$.
- Under a list of assumptions so that the resulting A-convolutions are *regular convolutions*, we get a generalized multiplicative Möbius function:
$$\mu_A(p^\alpha) = \begin{cases} 1, & \alpha = 0; \\ -1, & p^\alpha > 1 \text{ is } A\text{-primitive}; \\ 0, & \text{otherwise.} \end{cases}$$
- This construction leads to a generalized form of Möbius inversion between the A-convolutions.

Generating functions for a general class of K-convolutions

- Let the function $K : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be defined on all ordered pairs (n, d) such that $n \geq 1$ and $d|n$.
- We define the K-convolution of two arithmetic functions f, g to be

$$(f \circ_K g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right) K(n, d).$$

- We can define generating function factorizations of these sums of the form

$$(f \circ_K g)(n) = [q^n] \frac{1}{\mathcal{C}(q)} \sum_{m \geq 1} \sum_{k=1}^{m} e_{n,k}(K, \mathcal{C}; g) f(k) \cdot q^m.$$

- **Same questions arise:** Natural choices of $\mathcal{C}(q)$ that lead to pleasing formulas for the sequences of $e_{n,k}(K, \mathcal{C}; g)$; And what are the corresponding forms of the inverse matrices, $e_{n,k}^{-1}(K, \mathcal{C}; g)$?

Example: B-convolutions

- Let $\nu_p(n)$ denote the maximum exponent of the prime p in the factorization of n.
- Define $B(n,d) := \prod_{p|n} \binom{\nu_p(n)}{\nu_p(d)}$ where the product runs over all prime divisors of n.
- Then we have an inversion formula of the form

$$f(n) = \sum_{d|n} g(d) B(n,d) \iff g(n) = \sum_{d|n} f(d)\lambda(d) B(n,d),$$

where $\lambda(n) = (-1)^{\Omega(n)}$ is the *Liouville lambda function*.

Matrix generated convolution sums

- Let $G := (g_{i,j})$ be an infinite dimensional matrix containing only elements in $\{0, 1\}$.
- Suppose that $g_{ij} = 1$ if $i = j$ and $g_{ij} = 0$ if $i > j$.
- Furthermore, let the 1's in column n of G appear in rows $n_1, n_2, \ldots, n_k$ with $n_1 < n_2 < \cdots < n_k = n$.
- We can defing the convolution of two arithmetic functions f, g by

$$(f *_G g)(n) := \sum_{i=1}^{k} f(n_i)g(n_{k+1-i}).$$

Examples and special cases

- Let $D = (d_{ij})$ where $d_{ij} = [i|j]_\delta$. Then the convolution operation generated by $*_D$ is the same as the standard Dirichlet convolution of two arithmetic functions.
- Let $T = (t_{ij})$ where $t_{ij} = [(i,j) = 1]_\delta [i \leq j]_\delta$. Then we get a convolution variant of the type I relatively prime GCD sums we defined before by
$$\sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} f(d).$$

Concluding remarks and discussion

The End

Questions?

Comments?

Feedback?

Thank you for attending!

References I

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