Analysis Comprehensive Exam Review Notes

Revised: August 18, 2018

Topics Index

1 SET THEORY AND MEASURES

1.1 Definitions and key properties

Definition 1.1 (σ-algebras)**.** A *sigma algebra* is a collection of sets A in some universal set $\mathcal{U}$ such that

(i) $\varnothing \in A$;

(ii) $\{X_n\} \subseteq A \implies \cup_i X_i \in A$;

(iii) $X \in A \implies X^c \in A$.

Note that (ii) and (iii) imply that countable intersections of sets in A are also in A. For example, we can take $\mathcal{U} = \{1, 2\}$ and $A = \{\varnothing, \{1, 2\}\}$. For every set X the *power set* 2^X is a σ-algebra. We can form the *Borel σ-algebra* on $\mathbb{R}$ by considering all open intervals (a, b) and then taking all possible combinations of unions, intersections, and complements of these intervals.

Definition 1.2 (Measures)**.** A (*positive*) *measure* is a function $\mu : A \to [0, \infty]$ (or onto $\mathbb{R} \cup \{\infty\}$) which satisfies

(i) $\mu(\varnothing) = 0$;

(ii) (*Disjoint Unions*) $\mu(\cup_i X_i) = \sum_i \mu(X_i)$;

(iii) (*Non-Negativity*) $\mu \geq 0$.

Notice that these axioms imply monotonicity. A *finite measure* is a measure such that $\mu(\mathcal{U}) < \infty$. A *$\sigma$-finite measure space* is such that there exist finite measure sets, $\{X_i\}$, such that $\mathcal{U} = \cup_{i=1}^{\infty} X_i$. The *Borel measure* is a measure on the Borel σ-algebra. The *Lebesgue measure* is defined on $\mathcal{U} = \mathbb{R}$ with $\ell(a, b) = |(a, b)| = b - a$. The Lebesgue measure in invariant under translation: $|A| = |A + x|$.

Definition 1.3 (Outer measures)**.** For $X \subset \cup_i X_i$, we define the *outer measure* by

$$\mu^*(X) = |X|_e = \inf \sum_{i=1}^{\infty} \mu(X_i).$$

Outer measures have the following properties:

(i) (*Monotonicity*) If $X \subset Y$ then $\mu^*(X) \leq \mu^*(Y)$;

(ii) (*Domain*) $\mu^* : 2^{\mathbb{R}} \to \mathbb{R}$;

(iii) (*Non-negativity*) $\mu^* \geq 0$;

(iv) (*Null Sets*) $\mu^*(\varnothing) = 0$;

(v) (*Countable Subadditivity*) $\mu^*(\cup_i X_i) \leq \sum_i \mu^*(X_i)$.

By *Caratheodory*, we see that E is *measurable* <u>iff</u> for *ALL* subsets A (i.e., not just the measurable ones) we have that

$$\mu^*(A) = \mu^*(E \cap A) + \mu^*(E^c \cap A).$$

Every Borel measurable set is Lebesgue measurable, but not vice versa. Open sets are measurable.

Proposition 1.4 (Approximation of outer measure by open sets)**.** *For all $E \subseteq \mathbb{R}^n$ and $\forall \varepsilon > 0$, there exists an open set H such that $E \subseteq H$ and $\mu(H) < \mu^*(E) + \varepsilon$. One can alternately choose a compact (closed) set F such that $F \subseteq E$ with $\mu(F) > \mu^*(E) - \varepsilon$. Note that we can form any open set by a countable disjoint union of boxes in $\mathbb{R}^n$.*

Definition 1.5 (Borel sets)**.** A G_δ *set* is a countable *intersection* of open sets. A F_σ *set* is a countable *union* of closed sets.

Theorem 1.6 (E is measurable iff it differs from a Borel set of zero measure)**.** *Let $E \subseteq \mathbb{R}^n$. Then*

(a) E is measurable <u>iff</u> we can find a G_δ set H and a set Z of measure zero such that $E = H \setminus Z$.

(b) E is measurable <u>iff</u> we can find a F_σ set F and Z with $|Z| = 0$ such that $E = F \cup Z$.

Proof of (a). For sufficiency, suppose that $E = H \setminus Z$. Here H is G_δ, so measurable, and $|Z| = 0$ is measurable. By a theorem E is measurable. For necessity, Suppose that E is measurable. By a theorem, there exists a G_δ set H with $E \subseteq H$ and such that $|E|_e = |H|_e$. Set $Z := H \setminus E$ so that Z is measurable as above and $|Z| = |H| - |E| = 0$. □

Proof of (b). For sufficiency, suppose that $E = F \cup Z$ which is a union of measurable sets and is hence itself measurable. To show necessity, suppose that E is measurable. Then E^c is measurable, so by part (a) $E^c = H \setminus Z$ for some H, Z with $|Z| = 0$ and $H = \cap_{k \geq 1} E_k$ for E_k all open. Then

$$E = C(E^c) = C(H \setminus Z) = C(H \cap CZ) - CH \cup Z = \cup_{k \geq 1} CE_k \cup Z.$$

Here each CE_k is closed, so $F = \sup_{k \geq 1} CE_k$ is a F_σ set. □

1.2 Sets and sequences of sets

Definition 1.7 (Limsups and liminfs of sets)**.** Given a sequence of sets $\{E_n\}$ define

$$\limsup_{n \to \infty} E_n = \bigcap_{m \geq 1} \bigcup_{n \geq m} E_n \qquad \text{(In } \infty\text{-many of the sets } E_n\text{)}$$

$$\liminf_{n \to \infty} E_n = \bigcup_{m \geq 1} \bigcap_{n \geq m} E_n \qquad \text{(In all but finitely-many of the } E_n\text{)}$$

Observe that for any $N \geq 1$:

$$\limsup_{n \to \infty} E_n \subseteq \bigcup_{i=N}^{\infty} E_i.$$

Definition 1.8 (Monotone sequences of sets)**.** We use the notation $E_k \nearrow E$ if $E_1 \subseteq E_2 \subseteq \cdots$ and $E = \cup_{k \geq 1} E_k$. Similarly, we say that $E_k \searrow E$ if $E_1 \supseteq E_2 \supseteq \cdots$ and $E = \cap_{k \geq 1} E_k$. We get continuity properties of limits of sets for monotone sequences of sets (see below).

Theorem 1.9 (Continuity of the Lebesgue measure on monotone sets)**.** *Let $\{E_k\}$ be a sequence of measurable sets in $\mathbb{R}^n$.*

(1) If $E_k \nearrow E$, then $\lim_{k \to \infty} |E_k| = |E|$;

(2) If $E_k \searrow E$ and $|E_k| < \infty$ for some k, then $\lim_{k\to\infty} |E_k| = |E|$.

Proof of 1. If for some $j \geq 1$, $|E_j| = \infty$ then $|E_k| = \infty$ for all $k \geq j$, and hence $|E| = \infty$. Now we can assume that $|E_k| < \infty$ for all $k \geq 1$. We write

$$E = E_1 \cup (E_2 \setminus E_1) \cup \cdots = E_1 \cup \bigcup_{k\geq 1} (E_{k+1} \setminus E_k).$$

Then

$$|E| = |E_1| + \sum_{k\geq 1} |E_{k+1} \setminus E_k| = |E_1| + \lim_{n\to\infty} \sum_{k=1}^{n} (|E_{k+1}| - |E_k|) = \lim_{n\to\infty} |E_{n+1}|. \qquad \square$$

Proof of 2. We can assume that $|E_1| < \infty$ since otherwise we can truncate the sequence and relabel to obtain the same limit. Moreover, we can write E_1 as a disjoint union of measurable sets:

$$E_1 = E \cup \bigcup_{k\geq 1} (E_k \setminus E_{k+1}).$$

Now we have that

$$\begin{aligned} |E_1| &= |E| + \sum_{k\geq 1} |E_k \setminus E_{k+1}| \\ &= |E| + \lim_{n\to\infty} \sum_{k=1}^{n} (|E_k| - |E_{k+1}|) \\ &= |E| + \lim_{n\to\infty} (|E_1| - |E_{n+1}|), \end{aligned}$$

which implies the result. $\square$

Example 1.10 (Spring 2018, #6)**.** Prove each of the following:

i. Let $E_k \subset \mathbb{R}^n$ for $k \geq 1$ be sets such that $E_1 \subseteq E_2 \subseteq E_3 \subseteq \cdots$, and denote $E := \cup_{k\geq 1} E_k$. Assume that $|E|_e < \infty$. Prove that
$$|E|_e = \lim_{k\to\infty} |E_k|_e.$$

ii. Let E be a set in $\mathbb{R}^n$ with $0 < |E|_e < \infty$. Let $0 < \vartheta < 1$. Show that there is a set $E_\vartheta \subset E$ with $|E_\vartheta|_e = \vartheta \cdot |E|_e$.

1.3 Sample problems

Example 1.11 (Spring 2017, #6)**.** Given $A \subseteq [0, 1]$, prove that A is Lebesgue measurable iff

$$|A|_e + |[0, 1] \setminus A|_e = 1.$$

Example 1.12 (Fall 2016, #2)**.** Show that for $A \subset \mathbb{R}^d$, A is Lebesgue measurable iff for every $\varepsilon > 0$ there is a Lebesgue measurable set $E \subset \mathbb{R}^d$ such that $|A \Delta E|_e < \varepsilon$.

Example 1.13 (Spring 2016, #4)**.** Assume $E \subset \mathbb{R}^d$ is measurable such that $|E| < \infty$.

(a) Suppose that $f : E \to [-\infty, \infty]$ is measurable and finite a.e. Given $\varepsilon > 0$, prove that there is a closed set $F \subseteq E$ such that $|E \setminus F| < \varepsilon$ and f is bounded on F.

(b) For each $n \geq 1$ let f_n be a measurable function on E, and suppose that

$$\forall x \in E : M_x = \sup_{n \geq 1} |f_n(x)| < \infty.$$

Prove that for each $\varepsilon > 0$ there is a closed set $F \subseteq E$ and a finite constant M such that $|E \setminus F| < \varepsilon$ and $|f_n(x)| \leq M$ for all $x \in F$ and $n \geq 1$.

Example 1.14 (Spring 2014, #2)**.** Let A be any subset of $\mathbb{R}^d$. Prove that there exists a measurable set $H \supseteq A$ that satisfies $|A \cap E|_e = |H \cap E|$ for every measurable set $E \subseteq \mathbb{R}^d$.

Example 1.15 (Fall 2012, #1)**.** Given a set $E \subseteq \mathbb{R}^d$ with $|E|_e < \infty$, show that E is Lebesgue measurable <u>iff</u> for each $\varepsilon > 0$ we can write $E = (S \cup A) \setminus B$ where S is a union of finitely many non-overlapping boxes and $|A|_e, |B|_e < \varepsilon$.

Example 1.16 (Spring 2012, #3)**.** Suppose that $0 < \theta < 1$, $E \subset \mathbb{R}^n$, and $0 < |E| < \infty$. Prove that there is a cube Q such that $\theta \cdot |Q| < |E \cap Q|$.

Example 1.17 (Spring 2008, #2 $*$)**.** Let $|\cdot|_e$ denote the exterior Lebesgue measure on $\mathbb{R}$. Suppose that E is a subset of $\mathbb{R}$ with $|E|_e < \infty$. Show that E is Lebesgue measurable <u>iff</u> for every $\varepsilon > 0$ we can write $E = (S \cup N_1) \setminus N_2$, where S is a finite union of non-overlapping intervals and $|N_1|_e, |N_2|_e < \varepsilon$.

2 DIFFERENTIATION

2.1 Key results and theorems

Theorem 2.1 (LDT). *Let f be measurable and defined on the domain A. Then for a.e. $x \in A$ we have that*

$$\lim_{r\to 0} \frac{1}{2r} \int_{x-r}^{x+r} f(y)dy = f(x),$$

i.e., if the previous equation holds for all $x \in B$, then $|A \setminus B| = 0$. Lebesgue points satisfy this relation, and almost every $x \in [a, b]$ is a Lebesgue point. We can also express the LDT in the special case of $f := \chi_E$, the characteristic function of E, in the form of

$$\left| \frac{1}{2r} \int_{x-r}^{x+r} \chi_E(y)dy - \chi_E(x) \right| \to 0, \text{ a.e. as } r \to 0.$$

A few useful facts:

- By the LDT, if $|A| > 0$, $\exists a \in A$, $\delta_0 > 0$ such that $|A \cap (a - \delta, a + \delta)| > \frac{3\delta}{2}$ for all $0 < \delta < \delta_0$.
- Consider $f(x) := \chi_E(x)$ for $|E| < \infty$. Then $f \in L^1(\mathbb{R})$ and so a.e. $x \in \mathbb{R}$ is a Lebesgue point of f.
- By the LDT, a.e. point in $[0, 1]$ is a density point of $\chi_A(x)$ in that for a.e. $x \in A$:

$$\lim_{r\to 0} \frac{|A \cap [x - r, x + r]|}{|[x - r, x + r]|} = 1.$$

Lemma 2.2 (The Riemann-Lebesgue lemma). *For all $f \in L^1$, we have that*

$$\lim_{\lambda\to 0} \int_{-\infty}^{\infty} f(x)e^{\imath\lambda x}dx = 0.$$

Note that the exponential function in the previous integral can effectively be replaced by any other function that integrates to zero to obtain an analogous result.

Proof Sketch. We use a *dense class argument* to approximate $f(x)$ on $\mathbb{R}$ within some small $\varepsilon > 0$. Namely, we know that the functions $g \in C_0^1$, or the compactly upported continuous functions with one continuous derivative, are *dense* in L^1. Then we can complete the proof using IBP where we pick up a factor of $1/\lambda$ and *give ourselves a little room*, i.e., show that the limit is less than ε for every $\varepsilon > 0$. □

The *mean value theorem* (MVT for derivatives) states that if f is defined and continous on $[a, b]$ and differentiable on (a, b), then $\exists c \in (a, b)$ such that $(b - a)f'(c) = f(b) - f(a)$. The *intermediate value theorem* (IVT) states that if f is continuous on $[a, b]$, then for all c in the range between $f(a)$ and $f(b)$, $\exists x \in (a, b)$ such that $f(x) = c$. The *MVT for integrals* states that if f is continuous on $[a, b]$ then $\exists c \in (a, b)$ such that

$$f(c) = \frac{1}{b - a} \int_a^b f(t)dt.$$

Lemma 2.3 (Growth lemma II). *Suppose that $f : [a, b] \to \mathbb{R}$ and $E \subset [a, b]$ be measurable. Then if f is differentiable on E, $|f(E)|_e \leq \int_E |f'|$.*

2.2 Past exam problems using the LDT

Example 2.4 (Spring 2018, #1)**.** Let $A \subseteq \mathbb{R}$ be a measurable set. For $x \in \mathbb{R}$ denote $A + x = \{a + x : a \in A\}$. Prove that if A satisfies

$$|A \setminus (A + x)| = 0, \ \forall x \in \mathbb{R},$$

then either $|A| = 0$ or $|\mathbb{R} \setminus A| = 0$.

Example 2.5 (Spring 2017, #4)**.** Let A be a measurable subset of $[0, 1]$.

(a) Prove that if $|A| > 2/3$, then A contains an arithmetic progression of length 3, that is, prove that there are $a, d \in \mathbb{R}$ such that $a, a + d, a + 2d \in A$;

(b) Use part (a) to prove that if $|A| > 0$, then A contains an arithmetic progression of length 3.

Example 2.6 (Fall 2016, #1)**.** Let $E \subseteq \mathbb{R}$ be a Lebesgue measurable set with $0 < |E| < \infty$.

(1) For each $x \in \mathbb{R}$ and $r > 0$ define $I_r(x) := [x - r/2, x + r/2]$ and $h_r(x) := |E \cap I_r(x)|$. Prove that for a fixed $r > 0$, the function $h_r(x)$ is continuous at every $x \in \mathbb{R}$.

(2) Prove that there exists $r_0 > 0$ such that for each $0 < r < r_0$ there exists a closed interval $I \subset \mathbb{R}$ which satisfies $|I| = r$ and $|E \cap I| = r/2$.

Example 2.7 (Spring 2016, #3)**.** Suppose that $E \subseteq [0, 1]$ is measurable and that there exists $\delta > 0$ such that

$$|E \cap [x - r, x + r]| \geq \delta r,$$

for all $x \in (0, 1)$ and $r > 0$ such that $(x - r, x + r) \subseteq [0, 1]$. Prove that $|E| = 1$.

2.3 Other differentiation problems

Example 2.8 (Fall 2016, #4)**.** Prove that if $f(x), xf(x) \in L^1(\mathbb{R})$ then the function

$$F(w) := \int_{\mathbb{R}} f(x) \sin(wx) dx,$$

is defined, continuous, and differentiable at every point $w \in \mathbb{R}$. (HINT: Use the identity that $\sin(\alpha) - \sin(\beta) = 2 \sin(\frac{\alpha-\beta}{2}) \cos(\frac{\alpha+\beta}{2})$.)
$**$ **NOTE: Continuity proof uses the DCT.** $**$

Example 2.9 (Fall 2007, #4)**.** Prove that if f is integrable on $[a, b]$, and

$$\int_a^x f(t) dt = 0,$$

for all $x \in [a, b]$, then $f(t) = 0$ a.e. in $[a, b]$.

Example 2.10 (Spring 2008, #1)**.** Consider a sequence of functions $f_n \in L^1(\mathbb{R}^d)$ for $n \geq 0$ with

$$C := \sup_{n \geq 0} \int_{\mathbb{R}^d} |f_n| dx < \infty.$$

Suppose that the following assumptions are satisfied:

(i) $f_n \to f_0$ in measure as $n \to \infty$;

(ii) For all $\varepsilon > 0$, $\exists \delta > 0$ such that

$$|A| \leq \delta \implies \sup_{n \geq 0} \int_A |f_n| dx \leq \varepsilon;$$

(iii) For all $\varepsilon > 0$ there exists a set $K \subseteq \mathbb{R}^d$ with $|K| < \infty$ such that

$$\sup_{n \geq 0} \int_{\mathbb{R}^d \setminus K} |f_n| dx \leq \varepsilon.$$

Prove that $f_n \to f_0$ strongly (i.e., in norm) in $L^1(\mathbb{R}^d)$. Also show that all three conditions are necessary by constructing three counterexamples, each of which satisfies exactly two of the three hypotheses, and for which f_n does not converge strongly to f_0.

3 INTEGRATION

3.1 Absolute Continuity, bounded variation, and related characterizations

Definition 3.1 (Absolute continuity)**.** Let I be an interval in $\mathbb{R}$. Then we say that $f : I \to \mathbb{R}$ is *absolutely continuous on* I if $\forall \varepsilon > 0\ \exists \delta > 0$ such that for $(x_k, y_k) \subset I$ whenever $\sum_k (y_k - x_k) < \delta$ then $\sum_k |f(x_k) - f(y_k)| < \varepsilon$. Equivalently, we have the characterization that f is absolutely continuous on $[a, b]$ when f has a derivative f' a.e., its derivative is Lebesgue integrable, and

$$f(x) = f(a) + \int_a^x f'(t)dt,\ \forall x \in [a, b].$$

In other words, if f is absolutely continuous then f' exists and is integrable a.e., and absolutely continuous functions satisfy the FTC. Note that if f_n is absolutely continuous on $[0, 1]$ then $\exists f_n' \in L^1([0, 1])$ such that $f_n(x) = \int_0^x f_n'(t)dt$.

NOTE: Ordinarily, we only can show that $\int_a^b f'(t)dt \leq f(b) - f(a)$ (*See previous exam problem on* $[0, 1]$.).
Proof strategies: To prove absolute continuity of a function: Use Banach-Zaretsky, can show f is Lipschitz, use the FTC, etc.

Theorem 3.2 (Banach-Zaretsky)**.** *Let* $f : [a, b] \to \mathbb{R}$. *Then TFAE:*

(1) f *is absolutely continuous.*

(2) f *is continuous,* $f \in \mathrm{BV}[a, b]$, *and satisfies Luzin's condition: if* $|A| = 0$ *then* $|f(A)| = 0$.

(3) f *is continuous,* f *is differentiable a.e.,* f *satisfies Luzin's condition,* $f' \in L^1[a, b]$.

Definition 3.3 (Functions of bounded variation)**.** Let $\Gamma = \{a = x_0 < x_1 < \ldots < x_m = b\}$ be a *partition of* $[a, b]$ and define $S_\Gamma := \sum_{i=1}^m |f(x_i) - f(x_{i-1})|$. Let the *variation of* f *over* $[a, b]$ be defined as

$$V := \sup_\Gamma \{S_\Gamma : \Gamma \text{ a partition of } [a, b]\}.$$

If $V < \infty$, then we say that f is of *bounded variation* on $[a, b]$. If f has bounded variation on $[a, b]$, then we can express $f = g - h$ where g, h are both bounded and monotone increasing on $[a, b]$ (***Jordan decomposition theorem***). Note that these functions can be extended to be monotone increasing on all of $\mathbb{R}$ as well.

Proof strategies: A typical trick used whenever sup is used in problems is applied in this special case. Namely, $V(f) = \sup_\Gamma S_\Gamma$ should be interpreted as: $\forall \varepsilon > 0$, there exists a partition $\Gamma_\varepsilon := \{a = x_0 < x_1 < \cdots < x_n = b\}$ such that

$$V(f) \leq \sum_{k=1}^n |f(x_k) - f(x_{k-1})| + \varepsilon.$$

This version of the definition is much easier to manipulate. Also, if we let $V(x) = V[f;a,x]$ denote the *total variation* of f on $[a, a+x]$, then we see that V is monotone increasing and hence differentiable a.e. Moreover, the total variation of f on $[x, x+h]$ dominates $|f(x+h) - f(x)|$:

$$|f(x+h) - f(x)| \leq V[f;x,x+h] = V(x+h) - V(x).$$

Then it follows that $|f'| \leq V'$ so that, for example,

$$\int_a^b |f'(t)|dt \leq \int_a^b V' \leq V(a) - V(b) = V[f;a,b].$$

Notice that f ***monotone increasing*** on $[a,b] \implies f \in \mathrm{BV}[a,b]$. Also, f monotone increasing implies that f is differentiable a.e., $f' \geq 0$, and $\int_a^b f'(t)dt \leq f(b) - f(a)$ for all $a < b$.

Proposition 3.4 (Monotone increasing function properties)**.** *If f is monotone increasing, then*

(a) f has at most countably many points of discontinuity.

(b) f is differentiable a.e.

(c) f' is a measurable function.

(d) $f' \in L^1$ and we have the FTC as an upper bound: $0 \leq \int_a^b f' \leq f(b) - f(a)$.

Nested characterizations: We say that f is *Lipschitz on* $[a,b]$ if there exists a $K > 0$ (where K is independent of all $x, y \in [a,b]$) such that $|f(x) - f(y)| \leq K \cdot |x - y|$ for all $x, y \in [a,b]$. We have the following important nested inclusion of function types (C^1 denotes the set of functions with one continuous derivative):

$$C^1[a,b] \subsetneq \mathrm{Lip}[a,b] \subsetneq \mathrm{AC}[a,b] \subsetneq \mathrm{BV}[a,b] \subsetneq L^\infty[a,b].$$

Typical counter examples to these types of functions are $\sqrt{x}$ or are of the form $f(x) = x^a \sin(1/x^b)$.

3.2 Practice problems

Example 3.5 (Spring 2018, #4)**.** Let $f : [0,1] \to [0,1]$ be defined by $f(0) = 0$ and $f(x) = x^2|\sin(1/x)|$ for $x \in (0,1]$. Show that f is absolutely continuous on $[0,1]$. Give an example of a function $\phi : [0,1] \to [0,1]$ that is of bounded variation, and such that ϕ' exists in $(0,1]$, but such that $\phi \circ f$ is not absolutely continuous in $[0,1]$.
NOTE: $\sqrt{\mathbf{x}}$ is increasing, and hence of bounded variation.

Example 3.6 (Spring 2014, #4)**.** Suppose that $f \in L^1(\mathbb{R})$ is such that $f' \in L^1(\mathbb{R})$ and f is absolutely continuous on every finite interval $[a,b]$. Show that $\lim_{x\to\infty} f(x) = 0$.
NOTE: Suppose not, and then show that f is not integrable.

Example 3.7 (Spring 2017, #1)**.** Assume f is real-valued and has bounded variation on $[a,b]$. Extend f to $\mathbb{R}$ by setting $f(x) = f(a)$ for $x < a$ and $f(x) = f(b)$ for $x > b$. Prove that there exists a constant $C > 0$ such that $||T_t f - f||_1 \leq C \cdot |t|$ when $t \in \mathbb{R}$ where $T_t f(x) = f(x - t)$ denotes the translation of f by t.

Example 3.8 (Fall 2017, #3)**.** Assume that $f : \mathbb{R} \to \mathbb{R}$ is monotone increasing, and that we have $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = 1$. Prove that f is absolutely continuous on every finite interval $[a,b]$ iff $\int_{-\infty}^{\infty} f'(x)dx = 1$.

Example 3.9 (Spring 2016, #8)**.** Show that $f : [a,b] \to \mathbb{R}$ is Lipschitz iff f is absolutely continuous and $f' \in L^\infty[a,b]$.

Example 3.10 (Fall 2015, #2)**.** Suppose that f_n are absolutely continuous functions on $[0,1]$ such that $f_n(0) = 0$ and

$$\sum_{n\geq 1} \int_0^1 |f_n'(x)|dx < \infty.$$

Show that

- $\sum_{n\geq 1} f_n(x)$ converges for every x. Call this limit $f(x)$;
  (absolutely convergent series are convergent)
- f is absolutely continuous;
- For a.e. $x \in [0,1]$ we have that
$$f'(x) = \sum_{n\geq 1} f_n'(x).$$

Example 3.11 (Spring 2015, #7)**.** Assume that f has bounded variation on $[a,b]$. Letting $V[f;a,b]$ denote the total variation of f on $[a,b]$, prove that

$$\int_a^b |f'| \leq V[f;a,b].$$

Example 3.12 (Fall 2012, #2)**.** (a) Let $E \subseteq \mathbb{R}^d$ be a measurable set such that $|E| < \infty$. Let $\{f_n\}_{n\geq 1}$ be a sequence of measurable functions on E, and suppose that f_n is finite a.e. for each n. Show that if $f_n \to f$ a.e. on E, then $f_n \xrightarrow{m} f$.
(b) Show by example that part (a) can fail if $|E| = \infty$.

Example 3.13 (Fall 2012, #3)**.** Suppose that f is a bounded, real-valued, measurable function on $[0,1]$ such that $\int_0^1 x^n f(x)dx = 0$ for all $n \geq 0$. Show that $f(x) = 0$ a.e.

Example 3.14 (Spring 2008, #4)**.** Suppose that $f_n \in C^1[0,1]$ for $n \geq 1$, and we have:

(a) $f_n(0) = 0$;

(b) $|f_n'(x)| \leq \frac{1}{\sqrt{x}}$ a.e.; and

(c) There exists a measurable function h such that $f_n'(x) \to h(x)$ for every $x \in [0,1]$.

Prove that there exists an absolutely continuous function f such that $f_n \to f$ uniformly as $n \to \infty$.

4 NOTIONS OF CONVERGENCE

4.1 Modes of convergence

Pointwise a.e. Convergence: We say that $f_n(x) \to f(x)$ *pointwise a.e. on* E if $f_n \to f$ for all $x \in E \setminus Z$ where $|Z| = 0$. Equvalently, $\mu(\{x : f_n(x) \nrightarrow f(x)\}) = 0$.

Convergence in Measure: We write $f_n \xrightarrow{\mu} f$ if $\forall \varepsilon > 0\ \exists N$ such that $\mu(\{|f_n - f| > \varepsilon\}) < \varepsilon$ (or $< \eta$) for all $n \geq N$. Note that a sequence being convergent in measure is the same as it being ***Cauchy in measure***.

Theorem 4.1 (Pointwise convergence of a subsequence)**.** *In general, convergence in measure does not imply pointwise convergence a.e. However, if $f_n \xrightarrow{\mu} f$ what we can say is that there exists a subsequence $\{f_{n_k}\}$ which does converge pointwise to f.*

Counter Example. We will demonstrate a sequence that converges in measure to zero on $[0, 1]$, but which does not converge pointwise a.e. to zero. For $n \geq 1$ and $1 \leq j \leq n$, define

$$S_{n,j} := \left[\frac{j-1}{n}, \frac{j}{n}\right] \subseteq [0, 1],$$

and set $f_n(x) := \chi_{S_{n,j}}(x)$. Let $\varepsilon > 0$ and observe that

$$\ell(\{x \in [0, 1] : f_n(x) > \varepsilon\}) = \ell(\{x \in [0, 1] : f_n(x) = 1\}) = |S_{n,j}| = \frac{1}{n} \to 0,$$

as $n \to \infty$. So $f_n \xrightarrow{m} 0$. However, given any $x \in [0, 1]$ there are infinitely-many $n \in \mathbb{N}$ such that $x \in S_{n,j}$. This implies that there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to 1$ and hence $f_n \nrightarrow 0$ pointwise a.e. on $[0, 1]$. □

Proof. Suppose that $f_n \xrightarrow{\mu} f$. We need to find a subsequence $\{f_{n_k}\}_{k \geq 1}$ such that $f_{n_k} \to f$ a.e. on X as $k \to \infty$. For $j \geq 1$, choose L_j such that for all $k \geq L_j$

$$\mu(\{x : |f_k - f|(x) \geq 1/j\}) < \frac{1}{2^j}.$$

We can just as well assume that $L_1 < L_2 < L_3 < \cdots$. For $j \geq 1$, we define

$$E_j := \{x : |f_{L_j} - f|(x) \geq 1/j\},$$

and set

$$Z := \limsup_{j \to \infty} E_j = \bigcap_{m \geq 1} \bigcup_{j \geq m} E_j.$$

Now for $m \geq 1$, we can see that

$$\mu(Z) \leq \mu\left(\bigcup_{j \geq m} E_j\right) \leq \sum_{j \geq m} \mu(E_j) < \sum_{j \geq m} 2^{-j} = 2^{1-m} \to 0,$$

as $m \to \infty$. So we conclude that $\mu(Z) = 0$. Next, if $x \in X \setminus Z$, then $x \notin \cup_{j \geq m} E_j$ for some $m \geq 1$. This implies that $x \notin E_j$ for all $j \geq m$. Thus $|f_{L_j} - f|(x) \leq 1/j$ for $j \geq m$, which implies that $\lim_{j\to\infty} f_{L_j}(x) = f(x)$ for all $x \in X \setminus Z$. So it suffices to take $f_{n_j} := f_{L_j}$ so that $f_{n_k} \to f$ pointwise a.e. in X. $\square$

Almost Uniform Convergence: We have *almost uniform convergence* when $\forall \varepsilon > 0\ \exists E \subset X$ such that $\mu(E) < \varepsilon$ and $f_n \xrightarrow{\text{unif}} f$ for all $x \in X \setminus E$, or write $f_n\Big|_{E^c} \to f\Big|_{E^c}$. Almost uniform convergence implies both pointwise a.e. convergence and convergence in measure: $\forall \varepsilon > 0$ there is $E_n \subset X$ such that $\mu(E_n) < \varepsilon/2^n$.

Theorem 4.2 (Egorov)**.** *Let $E \subseteq \mathbb{R}^n$ be measurable with $|E| < \infty$ and let $f_k, f : E \to \mathbb{R}$ for $k \geq 1$. Furthermore, assume that $\lim_{k\to\infty} f_k(x) = f(x)$ a.e. in E and that f is finite a.e. in E. Let $\varepsilon > 0$. Then there exists a closed $F \subset E$ such that $|E \setminus F| < \varepsilon$ and $\{f_k\}$ converges uniformly to f on F. That is,*

$$\sup_{x\in F} |f_k(x) - f(x)| \to 0,$$

as $k \to \infty$. This convergence is almost uniform on E.

Note that to apply Egorov on the exam, one should state all of the necessary conditions which must hold before we can apply it: $\mu(X) < \infty$, f_n are all measurable, and $f_n \to f$ a.e.; The following statement of *Lusin's theorem* follows from Egorov and a dense classs argument for L^1 functions: If $f : [a, b] \to \mathbb{C}$, or equivalently if f is bounded, then $\forall \varepsilon > 0\ \exists$ a compact set $E \subset [a, b]$ such that $\mu(E) < \varepsilon$ and $f\Big|_{E^c}$ is continuous.

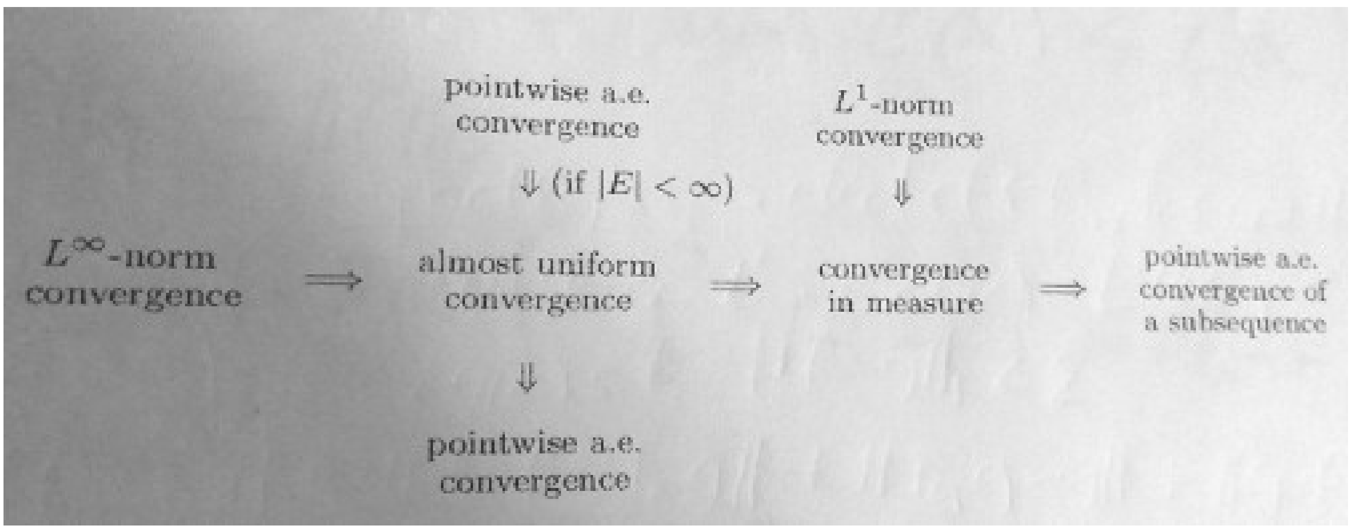
L^p-Norm Convergence: $f_n \to f$ in L^p if $||f_n - f||_p \to 0$ as $n \to \infty$. L^1, L^p-norm convergence of a sequence imply convergence in measure of the same sequence: By *Chebyshev* we have that

$$\mu\left(\{|f_n - f| \geq \varepsilon\}\right) \leq \frac{1}{\varepsilon^p} \int |f_n - f|^p = \frac{1}{\varepsilon^p} ||f_n - f||_p^p.$$

Weak Convergence in a Hilbert Space: We say that x_n converges weakly to $x \in H$, written $x_n \xrightarrow{w} x$, if $\langle x_n, y\rangle_H \to \langle x, y\rangle_H$ for all $y \in H$.

4.2 Flowchart of modes of convergence

The following diagram is a useful summary of modes of convergence (*From Dr. Heil's notes*):



4.3 Convergence theorems

Theorem 4.3 (Lebesgue's BCT)**.** *Let f_n be integrable $\forall n \geq 1$ such that (a) $f_n \xrightarrow{m} f$; or (b) $f_n \to f$ a.e. for some measurable function f. If $|f_n(x)| \leq g(x)$ a.e. for all n where g is integrable, then f is integrable and*

$$\lim_{n\to\infty} \int |f_n - f| d\mu = 0.$$

Theorem 4.4. *If f, g are measurable such that $|f| \leq g$ and g is integrable, then f is integrable.*

Theorem 4.5 (Monotone convergence theorem)**.** *Let $\{f_k\}_{k\geq 1}$ be a sequence of measurable functions on E. Then*

(1) If $f_k \nearrow f$ a.e. on E and $\exists \varphi$ integrable such that $f_k \geq \varphi$ a.e. in E for all $k \geq 1$, then

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

(2) If $f_k \searrow f$ a.e. on E and $\exists \varphi$ integrable such that $f_k \leq \varphi$ $\forall k \geq 1$, then

$$\lim_{k\to\infty} \int_E f_k = \int_E f.$$

Theorem 4.6 (General form of LDCT)**.** *Let f_k be measurable for all $k \geq 1$. Suppose that (a) $\lim_{k\to\infty} f_k = f$ a.e. in E; and (b) $\exists \varphi$ integrable such that for all $k \geq 1$, $|f_k| \leq \varphi$ a.e. in E. Then*

$$\lim_{k\to\infty} \int f_k = \int f.$$

Sketch. For a proof sketch of the DCT, we require that the $f_k \geq 0$ and then use Fatou's lemma. □

Lemma 4.7 (Fatou)**.** *Suppose that $E \subseteq \mathbb{R}^n$ is measurable and let $f_k \geq 0$ be measurable for all $k \geq 1$. Then*

$$\liminf_{k\to\infty} \int_E f_k \geq \int_E \left(\liminf_{k\to\infty} f_k\right).$$

We can obtain the same conclusion if we instead assume that the $f_k \geq \varphi$ for all $k \geq 1$ when φ is integrable on E. Notice that the statement of Fatou's lemma makes no a priori assumptions on the convergence of the sequence $\{f_k\}_{k\geq 1}$.

Theorem 4.8 (Uniform convergence theorems)**.** *We have the following variants of uniform convergence theorems:*

- *$f_n \to f$ uniformly on $[a, b]$ with f_n all continuous $\implies$ f is continuous.*
- *If f_n is differentiable on $[a, b]$ and $\lim_{n\to\infty} f_n(x_0)$ exists for some $x_0 \in [a, b]$ and f_n' converge uniformly on $[a, b]$, then $f_n \to f$ uniformly on $[a, b]$ and $f'(x) = \lim_{n\to\infty} f_n'(x)$ for all $x \in [a, b]$.*

** *If f_n is integrable on $[a, b]$ and $f_n \to f$ uniformly, then f is integrable and*

$$\int_{[a,b]} f = \lim_{n\to\infty} \int_{[a,b]} f_n.$$

5 FUBINI AND TONELLI

5.1 Statements of key theorems

Theorem 5.1 (Fubini)**.** *Let $f(x,y)$ be integrable over $I = I_1 \times I_2$. Then*

i. For a.e. $x \in I_1$, f_x is measurable and integrable on I_1. That is,

$$\int_{I_2} |f_x(y)|dy = \int_{I_2} |f(x,y)|dy < \infty.$$

ii. The function $F(x) = \int_{I_2} f(x,y)dy$ is measurable and integrable on I_1. That is,

$$\int_{I_1} |F(x)|dx = \int_{I_1} \left| \int_{I_2} f(x,y)dy \right| dx < \infty.$$

Moreover, $\int \int_I f(x,y)dxdy = \int_{I_1} \left[\int_{I_2} f(x,y)dy \right] dx$.

The necessary conditions to apply Fubini's theorem can often be obtained by first applying Tonelli's theorem stated below. *Note that to apply Tonelli, we must require that $f \geq 0$ is non-negative!*

Theorem 5.2 (Tonelli)**.** *Let $E = A \times B$ where A, B are respectively measurable sets in $\mathbb{R}^n$ and $\mathbb{R}^m$. Let $f : E \to [0, \infty]$ be measurable. Then*

i. For a.e. $x \in A$, $f_x : B \to [0, \infty]$ is measaurable.

ii. $F(x) = \int_B f(x,y)dy$ is a measurable function of x.

iii. $\int \int_E f = \int_A \left[\int_B f(x,y)dy \right] dx$.

Proof strategy: Generally speaking, one applies Tonelli on $|f|$ to prove that $f \in L^1$, then we can apply the Fubini theorem to help with reversing the order of integration. Checking the conditions of Fubini's theorem, we see that this step is actually necessary, unless it is trivial that $f \in L^1$.

5.2 List of previous exam problems

Example 5.3 (Spring 2017, #2)**.** Let $W(x) := \max(1 - |x|, 0)$ be the *hat function* on $[-1, 1]$. Given $f \in L^1(\mathbb{R})$, let

$$g(y) = \int_{-\infty}^{\infty} f(t)e^{-2\pi \imath yt}dt.$$

Prove that g is bounded on $\mathbb{R}$ and that for a.e. x we have that

$$\int_{-\infty}^{\infty} f(y) \left(\frac{\sin \pi(x-y)}{\pi(x-y)} \right)^2 dy = \int_{-1}^{1} g(t)(1 - |t|)e^{2\pi \imath tx}dt.$$

HINT: Use the fact that $\int_{-\infty}^{\infty} W(t)e^{2\pi \imath yt}dt = \left(\frac{\sin \pi y}{\pi y} \right)^2$.

Example 5.4 (Fall 2016, #5)**.** Let $f \in L^p(\mathbb{R})$ for $1 \le p < \infty$. Given $y > 0$ denote $A_y := \{x \in \mathbb{R} : |f(x)| > y\}$. Prove that

$$\int_{\mathbb{R}} |f(x)|^p dx = p \int_0^\infty y^{p-1} |A_y| dy.$$

Example 5.5 (Spring 2016, #6)**.** Given $f \in L^1(\mathbb{R})$, define

$$g(x) = \int_{-\infty}^{x} f(t)dt, \ x \in \mathbb{R}.$$

Given $c > 0$, prove that $g(x+c) - g(x)$ is an integrable function of x, and show that

$$\int_{-\infty}^{\infty} (g(x+c) - g(x))\, dx = c \int_{-\infty}^{\infty} f(t)dt.$$

Example 5.6 (Spring 2015, #5)**.** Show that

$$\int_0^\infty \frac{\sin^2 x}{x} e^{-sx} dx = \frac{\ln(1 + 4/s^2)}{4},$$

for $s > 0$ by applying Fubini's theorem to the function $f(x,y) = e^{-sx}\sin(2xy)$ on $E = (0,\infty) \times (0,1)$.

Example 5.7 (Comp Prep Exam #6)**.** Given $f \in L^1[0,1]$, define $g(x) = \int_x^1 \frac{f(t)}{t} dt$ for $0 < x \le 1$. Show that g is defined a.e. on $[0,1]$, that $g \in L^1[0,1]$, and that $\int_0^1 g(x)dx = \int_0^1 f(x)dx$.

Example 5.8 (Fall 2008, #3)**.** Let f, g be absolutely continuous functions on $[0,1]$. Show that for $x \in [0,1]$ we have

$$\int_0^x f(t)g'(t)dt = f(x)g(x) - f(0)g(0) - \int_0^x f'(t)g(t)dt.$$

(HINT: Consider the integral $\int\int_E f'(t)g'(t)dt$ over the set $E := \{(s,t) \in [0,x]^2 : s \le t\}$.)

6 L^P–SPACES

6.1 Key results and inequalities

If $p <\leq \infty$ then we define the L^p-norm to be

$$||f||_p := \left(\int |f|^p d\mu\right)^{1/p},$$

and when $p = \infty$ we set $||f||_\infty = \sup_{x\in E} |f(x)|$. Alternately we could define it as the essential sup which takes $x \in E \setminus Z$ for some $|Z| = 0$:

$$\text{essup}_{x\in E} |f(x)| = \sup\{\alpha : \mu(\{|f| \leq \alpha\}) > 0\}.$$

The sequence analog to this space is

$$\ell^p = \left\{(a_i)_{i\geq 1} : \left(\sum_i |a_i|^p\right)^{1/p} < \infty\right\},\ p < \infty,$$

and where we set $||(a_i)||_{\ell^\infty} = \sup_n |a_n|$.

Fact: If $\mu(X) < \infty$, then $L^q(\mu) \subsetneq L^p(\mu)$ for any $0 < p < q \leq \infty$. This can be proved by applying Hölder with the conjugate exponents $\boxed{p_0 := q/p \text{ and } q_0 := q/(q-p)}$.

Proposition 6.1 (Cauchy-Schwarz and Hölder inequalities)**.** *Note that we assume that the functions f, g are both square integrable, i.e., $|f|^2, |g|^2$ are both integrable. If this is the case then,*

$$\int_0^1 |f| \leq \left(\int_0^1 |f|^2\right)^{1/2}$$
$$\left|\int fg\right| \leq \left(\int |f|^2\right)^{1/2} \left(\int |g|^2\right)^{1/2}.$$

Note that the latter equation above is the special case of the more general cases in Hölder's inequality for $\frac{1}{p} + \frac{1}{q} = 1 \iff p + q = pq$ when $p = q = 2$:

$$\left|\int fg\right| \leq ||f||_p \cdot ||g||_q.$$

Dual (conjugate) exponents: $\boxed{\textit{If } p + q = pq, \textit{ then } p = q/(q-1).}$

Note that *Minkowski's theorem* states that $||f + g||_p \leq ||f||_p + ||g||_p$. Minkowski also implies that

$$\left\|\int f\right\|_p \leq \int ||f||_p.$$

*Proof of Minkowski (**Know This Proof Technique**).* We let p be arbitrary with $f, g \in L^p$ and observe that:

$$\begin{aligned}||f+g||_p^p &= \int |f+g|^p = \int |f+g|^{p-1}|f+g| \\ &\leq \int |f+g|^{p-1}(|f|+|g|)d\mu, \quad \text{then we apply Hölder} \\ &\leq (||f||_p + ||g||_p)\left(\int |f+g|^{(p-1)\frac{p}{p-1}}\right)^{1-1/p} \\ &= (||f||_p + ||g||_p)\frac{||f+g||_p^p}{||f+g||_p}.\end{aligned}$$

□

If $\alpha > 0$, we can obtain that

$$\mu\left(\{x : |f(x)| > \alpha\}\right) \leq \left(\frac{||f||_p}{\alpha}\right)^p.$$

Another important result which is non-trivial to prove and due to Riesz is that for $1 \leq q < \infty$ and $g \in L^1$:

$$||g||_q = \sup\left\{\left|\int fg\right| : ||f||_p = 1\right\}, \quad \text{where } \frac{1}{p} + \frac{1}{q} = 1.$$

Proposition 6.2 (Completeness of normed vector spaces)**.** *A normed vector space is complete <u>iff</u> any Cauchy sequence converges (in E). Also, if any absolutely convergent series converges. In other words, if $||f_n - f_m|| \to 0$ as $n, m \to \infty$, then $\exists f \in E$ such that $||f_n - f|| \to 0$ as $n \to \infty$. Also, $\sum_{n\geq 1} ||f_n|| < \infty \implies \exists f \in E$ such that $\sum_{n\geq 1} f_n = f$, or equivalently*

$$\left\|\sum_{n=1}^{N} f_n - f\right\| \to 0, \quad \text{as } N \to \infty.$$

To prove these completeness properties, we can appeal to the completeness of $\mathbb{R}^n$ or other known complete spaces. For example, L^p is complete for $1 \leq p \leq \infty$.

6.2 Dense class arguments

Compactly supported continous functions are dense in L^1: $C_c[a,b] = L^1[a,b]$. Other dense functions in L^p are the simple (i.e., staircase) functions, and polynomials. If $f \in L^p$ and C is *dense* in L^p then $\forall \varepsilon > 0$, $\exists g \in C$ such that $||f - g||_p < \varepsilon$.

Example 6.3 (Invariance under translation in L^1)**.** Given $f \in L^1$, let $T_a f(x) = f(x-a)$ denote *translation of f by a.* Prove that $T_a f \to f$ in the L^1-norm as $a \to 0$, i.e., $\lim_{a\to 0} ||T_a f - f||_1 = 0$.

Proof. We use a density argument noting that compactly supported continuous functions on $\mathbb{R}$ are dense in L^1: $C_c(\mathbb{R}) = L^1(\mathbb{R})$. Then for all $\varepsilon > 0$, $\exists g \in C_c(\mathbb{R})$ such that $||f - g||_1 < \varepsilon/2$. So

$$||T_a f - f||_1 \leq ||T_a f - T_a g||_1 + ||T_a g - g||_1 + ||g - f||_1 \to \frac{\varepsilon}{2} + 0 + \frac{\varepsilon}{2},$$

where $||T_a g - g||_1 \to 0$ since g is continuous. □

6.3 Convex functions and Jensen's inequality

Definition 6.4 (Convexity)**.** The function $\varphi : \mathbb{R} \to \mathbb{R}$ is *convex* if

$$\varphi(\vartheta t + (1-\vartheta)u) \le \vartheta\varphi(t) + (1-\vartheta)\varphi(u),$$

for all $t, u \in \mathbb{R}$ and $\vartheta \in [0,1]$. We note an important special case which is that the function $\varphi(t) := t^p$ is convex for all $p > 1$.

Theorem 6.5 (Jensen's inequality)**.** *Suppose that φ is convex on X. Then for all functions $f : X \to \mathbb{R}$ which are measurable and integrable on X:*

$$\int_X \varphi(f(x))dm(x) \ge \varphi\left(\int f(x)dm(x)\right).$$

If φ is convex and increasing, then

$$\varphi^{-1}\left(\int(\varphi \circ f)dm\right) \ge \int f dm.$$

For simple functions of the form $f := \sum_{i=1}^n c_i \chi_{E_i}$, i.e., where $\varphi \circ f = \sum_{i=1}^n \varphi(c_i)\chi_{E_i}$, we obtain the corollary that

$$\int \phi \circ f = \sum_{i=1}^n \varphi(c_i)m(E_i) \ge \varphi\left(\sum_{i=1}^n c_i m(E_i)\right).$$

6.4 Proofs of standard results from Folland

Theorem 6.6. *If $0 < p < q < r \le \infty$, then $L^q \subset L^p + L^r$.*

Proof. We first note that $f = f\chi_{\{f\le 1\}} + f\chi_{\{f>1\}}$. Let $f \in L^q$. Define $E := \{x : |f(x)| > 1\}$ and set $g := f\chi_E$, $h := f\chi_{E^c}$. Then

$$\begin{aligned} |g|^p &= |f|^p\chi_E \le |f|^q\chi_E \implies g \in L^p \\ |h|^r &= |f|^r\chi_{E^c} \le |f|^q\chi_{E^c} \implies h \in L^r, \end{aligned}$$

with $f = g + h$. □

Theorem 6.7. *If $0 < p < q < r \le \infty$, then $L^p \cap L^r \subset L^q$ and*

$$||f||_q \le ||f||_p^\lambda ||f||_r^{1-\lambda},$$

where $\lambda = (q^{-1} - r^{-1})/(p^{-1} - r^{-1})$, i.e., where $\lambda \in (0,1)$ is defined by $\frac{1}{q} = \lambda \cdot \frac{1}{p} + (1-\lambda)\frac{1}{r}$.

Proof. If $r = \infty$, $|f|^q \le ||f||_\infty^{q-p}|f|^p$ with $\lambda = p/q$ so that

$$||f||_q \le ||f||_p^{p/q} \cdot ||f||_\infty^{1-p/q} = ||f||_p^\lambda ||f||_\infty^{1-\lambda}.$$

If $r < \infty$, then we can use Hölder's inequality with the conjugate exponents $\frac{pq}{\lambda}$ and $\frac{rq}{1-\lambda}$:

$$\begin{aligned} \int |f|^q = \int |f|^{\lambda q}|f|^{(1-\lambda)q} &\le ||\ |f|^{\lambda q}\ ||_{p/\lambda q} ||f^{(1-\lambda)q}||_{rq/(1-\lambda)} \\ &= \left(\int |f|^p\right)^{\lambda q/p}\left(\int |f|^r\right)^{(1-\lambda)q/r} \\ &= ||f||_p^{\lambda q} \cdot ||f||_r^{(1-\lambda)q}. \end{aligned}$$

Taking q^{th} roots implies the result in this case as well. □

Theorem 6.8. *If $0 < p < q \le \infty$ and A is any set, then $\ell^p(A) \subset \ell^q(A)$ and $||f||_q \le ||f||_p$ (in ℓ^*)*

Proof. First, $||f||_\infty^p = \sup_\alpha |f(\alpha)|^p \le \sum_\alpha |f(\alpha)|^p \implies ||f||_\infty \le ||f||_p$ (in ℓ^*). The case of $q < \infty$ follows from the previous if $\lambda = p/q$:

$$||f||_q \le ||f||_p^\lambda ||f||_\infty^{1-\lambda} \le ||f||_p.$$ □

Theorem 6.9. *If $\boxed{\mu(X) < \infty}$ and $0 < p < q \le \infty$, then $L^q \subset L^p$ and $||f||_p \le ||f||_q \mu(X)^{1/p-1/q}$.*

Proof. If $q = \infty$ then the result is obvious:

$$||f||_p^p = \int |f|^p \le ||f||_\infty^p \int 1 = ||f||_\infty^p \mu(X).$$

If $q < \infty$, then we can use Hölder's inequality with the conjugate exponents q/p and $q/(q-p)$:

$$||f||_p^p = \int |f|^p \cdot 1 \le ||\ |f|^p\ ||_{q/p} \cdot ||1||_{q/(q-p)} = ||f||_q^p \mu(X)^{(q-p)/q}.$$ □

6.5 Previous exam problems

Example 6.10 (Fall 2016, # 4)**.** Let $1 \le p < q \le \infty$ and $f \in L^p \cap L^q$. Show that $f \in L^r$ for all $p \le r \le q$.

Proof. First, suppose that $1 \le p < r < q < \infty$ and write $r = tp + (1-t)q$ for $0 < t < 1$. Set the conjugate exponents $u = 1/t$ and $u' = u/(u-1) = 1/(1-t)$. Then this implies that

$$\begin{aligned}\int |f|^r = \int |f|^{tp}|f|^{(1-t)q} &\le \left(\int |f|^{tpu}\right)^{1/u} \left(\int |f|^{(1-t)qu'}\right)^{1/u'} \\ &= \left(\int |f|^p\right)^t \left(\int |f|^q\right)^{1-t} < \infty.\end{aligned}$$

Next, for $1 \le p < r < q = \infty$:

$$\int |f|^r = \int |f|^p |f|^{r-p} \le ||f||_\infty^{r-p} \int |f|^p < \infty.$$

The cases where $r = p$ or $r = q$ are trivial. □

Example 6.11 (Fall 2007, #2)**.** Let (X, M, μ) be a measure space, let μ be a positive measure, and let $f_n, f \in L^1(X, M, \mu)$ for $n \ge 1$. Assume that:

(1) $f_n(x) \to f(x)$ for a.e. $x \in X$;

(2) $||f_n||_1 \to ||f||_1$.

Prove that $||f_n - f||_1 \to 0$ as $n \to \infty$.
(HINT: Fatou's lemma.)

Example 6.12 (Spring 2010, #5)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f, f_n \in L^p$ where $1 \le p < \infty$. Prove that if $f_n \to f$ a.e., then $||f_n - f||_p \to 0$ iff $||f_n||_p \to ||f||_p$.

Slick Proof. The forward direction is easy by Minkowski: $0 \leq |\ ||f_n||_p - ||f||_p\ | \leq ||f_n - f||_p$. For the reverse direction, we must show that $||f_n||_p \to ||f||_p \implies ||f_n - f||_p \to 0$. We note that the function $|\cdot|^p$ is always convex when $p \geq 1$. Then

$$\begin{aligned}|f_n - f|^p = |f_n + (-f)|^p = 2^p \cdot \left|\frac{1}{2}f_n + \frac{1}{2}(-f)\right|, &\quad \text{then by convexity}\\ \leq 2^p\left(\frac{|f_n|^p}{2} + \frac{|f|^p}{2}\right).\end{aligned}$$

Now noticing that $||\cdot||^p \geq 0$ where we have that $||f_n||_p \to ||f||_p$ a.e., we will *immediately* try Fatou's lemma:

$$\begin{aligned}\int_X \lim_{n\to\infty} 2^p\left(\frac{|f_n|^p}{2} + \frac{|f|^p}{2} - |f_n - f|^p\right) d\mu &\leq \liminf_{n\to\infty} \int_X 2^p\left(\frac{|f_n|^p}{2} + \frac{|f|^p}{2} - |f_n - f|^p\right) d\mu\\ &= \liminf_{n\to\infty}\left(\int_X 2^{p-1}|f_n|^p d\mu + \int_X 2^{p-1}|f|^p d\mu\right)\\ &\quad - \limsup_{n\to\infty}\int_X |f_n - f|^p d\mu\\ &= \liminf_{n\to\infty}\left(2^{p-1}||f_n||_p^p + 2^{p-1}||f||_p^p\right) - \limsup_{n\to\infty} 2^p \cdot ||f_n - f||_p^p.\end{aligned}$$

This implies by subtraction that

$$0 \leq \limsup_{n\to\infty}\int_X |f_n - f|^p d\mu \leq 0. \qquad \square$$

Alternate "Obtainable" Proof. The forward direction is easy and uses the (reverse) triangle inequality. The reverse direction uses Egorov which can be applied in this case since informally $f \in L^p$ implies that "*f lives on a compact set*". In particular, let $\varepsilon > 0$ be fixed. We can choose a *bounded* set E such that $||f\chi_{E^c}||_p < \varepsilon$. So by Egorov, we can choose a measurable $F \subseteq E$ so that $f_n(x) \to f(x)$ uniformly on F and where $||f\chi_{F^c}||_p < 2\varepsilon$. Then we have that

$$\int_E |f_n|^p dx \to \int_E |f|^p dx \geq ||f||_p^p - 2\varepsilon.$$

Lastly, by hypothesis we must have that

$$\limsup_{n\to\infty}\int_{E^c} |f_n|^p dx < 2\varepsilon,$$

so that

$$\begin{aligned}\limsup_{n\to\infty}\int_{E^c}|f_n|^p &\leq \int |f_n|^p - \int_{E^c}|f_n|^p\\ &\to ||f||_p^p - ||f\chi_{E^c}||_p^p \leq ||f||_p^p - 2\varepsilon.\end{aligned} \qquad \square$$

Example 6.13 (Fall 2015, #7)**.** Fix a finite measure space $(X, \mathcal{M}, \mu)$ and $1 \leq p < q \leq \infty$. Show that $L^p \not\subset L^q$ iff X contains sets of arbitrarily small positive measure. (HINT: For the "if" implication: construct a disjoint sequence $\{E_n\}$ with $0 < \mu(E_n) < 2^{-n}$, and consider $f = \sum_n a_n\chi_{E_n}$ for suitable constants a_n.)

Proof. [$\implies$]: Suppose first that there exists $f \in L^p \setminus L^q$. Next, we construct the following sets for $n \geq 1$: $E_n := \{x \in X : |f(x)| > n\}$. Then by Chebyshev we see that

$$\mu(E_n) \leq \left(\frac{||f||_p}{n}\right)^p \to 0,$$

as $n \to \infty$. So it suffices to prove that $\mu(E_n) > 0$ for all $n \geq 1$. Suppose that $\mu(E_n) = 0$ for some n. Then if $|f| \leq n$ a.e., i.e., $f \in L^\infty \subseteq L^q$: $||f||_q^q = \int |f|^q \leq n^q \cdot \mu(X) < \infty$. But this contradicts the assumption that $f \notin L^q$.

[$\Longleftarrow$]: We inductively construct measurable sets F_n such that $0 < \mu(F_1) < 1/2$ and $0 < \mu(F_n) < \mu(F_{n-1})/4$ for $n \geq 2$. If we then set $E_n := F_n \setminus \cup_{j=1}^{n-1} F_j$ then the $\{E_n\}$ are disjoint and $0 < \mu(E_n) < 2^{-n}$. Suppose first that $q < \infty$ and consider $f = \sum_n a_n \chi_{E_n}$ for $a_n := 1/\mu(E_n)^{1/q}$. Then if we set $\lambda = 1 - p/q$ we have that

$$||f||_p^p = \sum_n |a_n|^p \mu(E_n) = \sum_n \mu(E_n)^\lambda < \sum_n \frac{1}{2^{\lambda n}} < \infty,$$

which shows that $f \in L^p$. On the other hand,

$$||f||_q^q = \sum_n |a_n|^q \mu(E_n) = \sum_n 1 = +\infty,$$

which shows that $f \notin L^q$. If $q = \infty$, we can define $f = \sum_n a_n \chi_{E_n}$ with $a_n := \left(\frac{1}{\mu(E_n)}\right)^{\frac{1}{2p}} \to \infty$. Then $f \in L^p$ as before, while $f = a_n$ on E_n. Since $\mu(E_n) > 0$, we see that $||f||_\infty = \infty$. □

Example 6.14 (Spring 2011, #7)**. (a)** Assume that μ is a finite measure on a measurable space $(X, \mathcal{M})$. With $q \in (1, \infty)$, let $\{f_k\}_{k\geq 1} \subset L^q(X)$ and $f \in L^q(X)$ be given. Suppose also that

- $\sup_{k\geq 1} ||f_k||_{L^q(X)} < \infty$; and
- $f_k \to f$ for μ-a.e. on X.

Prove that $f_k \to f$ in $L^p(X)$ for all $p \in [1, q)$.
(b) Is the statement in (a) still true if μ is only assumed to be σ-finite? Justify your answer.

Proof of (a). First, note that by Fatou we have that

$$0 \leq ||f||_q = \left(\int \liminf_{n\to\infty} |f_n|^q\right)^{1/q} \leq \liminf_{n\to\infty} ||f_n||_q \leq \sup_{n\geq 1} ||f_n||_q < \infty,$$

which implies (again) that $f \in L^q(X)$. So we consider the functions $g_k := f_k - f$, which are uniformly bounded and converge to zero a.e. in X. We need to show that $||g_k||_p \to 0$ for all $p \in [1, q)$. Now we take the conjugate exponents $p_0 := q/p$ and $q_0 := q/(q-p)$, i.e., so that $1/p_0 + 1/q_0 = 1$, in Hölder and apply Chebyshev in the forllowing forms for $m \geq 1$:

$$\int_X \chi_{\{|g_k|>m\}} |g_k|^p d\mu \leq \mu(\{|g_k| > m\}) \cdot ||\ |g_k|^p\ ||_{q/p} \leq \frac{1}{m^{q-p}} \left(\int_X |g_k|^q\right)^{\frac{q-p}{q}} \cdot \left(\int_X |g_k|^q\right)^{p/q} = \frac{1}{m^{p-q}} ||g_k||_q^q.$$

Next, we define

$$\widetilde{g}_k := \begin{cases} g_k, & |g_k| \leq m; \\ 0, & \text{otherwise.} \end{cases}$$

Then $|\widetilde{g}_k| \leq m$ pointwise a.e. in X and we are given that $\widetilde{g}_k \to 0$ a.e. in X. Now by the DCT and Minkowski's inequality, we obtain that

$$\begin{aligned} \lim_{k\to\infty} ||g_k||_p &= \lim_{k\to\infty} \left(\int_X \chi_{\{|g_k|>m\}} |g_k|^p + \int_X |\widetilde{g}_k|^p\right)^{1/p} \\ &\leq \lim_{k\to\infty} ||\chi_{\{|g_k|>m\}} |g_k|^p||_p + \left|\left|\lim_{k\to\infty} \widetilde{g}_k\right|\right| \\ &\leq \frac{1}{m^{p-q}} ||g_k||_q^q \longrightarrow 0, \end{aligned}$$

as each of $k, m \to \infty$. □

Example 6.15 (Spring 2014, #3)**.** Let $\{f_n\}$ be a sequence in $L^p(\Omega)$ for $1 \le p \le \infty$, and assume that $f \in L^p(\Omega)$ is such that $||f_n - f||_p \to 0$. Show that there exists a subsequence $\{f_{n_k}\}$ and a function $h \in L^p(\Omega)$ such that

(i) $f_{n_k} \to f$ a.e. on Ω; and

(ii) For every $k \ge 1$, we have that $|f_{n_k}(x)| \le h(x)$ a.e. on Ω.

Proof. The conclusion is apparently obvious when $p = \infty$, so we assume that $1 \le p < \infty$. Since $\{f_n\}$ is Cauchy in $L^p(\Omega)$ by hypothesis, we have a subsequence $\{f_{n_k}\}$ such that

$$||f_{n_{k+1}} - f_{n_k}|| \le 2^{-k}, \ \forall k \ge 1.$$

From this point on, we simplify notation by calling f_{n_k} instead by just f_k. Let

$$g_n(x) := \sum_{k=1}^{n} |f_{k+1}(x) - f_k(x)|,$$

and observe that by computation $||g_n||_p \le 1$. Now since $g_n(x) \le g_{n+1}(x)$, the MCT implies that $g_n(x)$ tends to a finite limit $g(x)$ a.e. on Ω. We also know that $g \in L^p(\Omega)$. On the other hand, for any $m \ge n \ge 2$, we have that

$$|f_m(x) - f_n(x)| \le |f_m - f_{m-1}|(x) + \cdots + |f_{n+1} - f_n|(x) \le g(x) - g_{n-1}(x) \longrightarrow 0,$$

as $n \to \infty$. Thus a.e. on Ω, $\{f_n(x)\}$ is Cauchy and therefore converges to some finite limiting $f^*(x)$. We notice that a.e. on Ω:

$$|f^*(x) - f_n(x)| \le g(x), \quad \text{for } n \ge 2.$$

In particular, this shows that $f^* \in L^p(\Omega)$. Finally, since

$$|f_n(x)| \le g(x) + |f^*(x)| =: h(x),$$

where $h \in L^p(\Omega)$ since L^p is closed under addition, the DCT implies that $f_k \to f^*$ in $L^p(\Omega)$. This implies that $f = f^*$ a.e., completing the proof. ☐

Example 6.16 (Spring 2015, #3)**.** Let $f \in L^p[0, \infty]$. Show that for $1 < p < \infty$ we have

$$\lim_{x\to\infty} \frac{1}{x^{1-1/p}} \int_0^x f(t)dt = 0.$$

6.6 Proofs of results from the comp prep homework

Example 6.17 (Problem #3)**.** Given a measurable set $E \subseteq \mathbb{R}^d$ with $|E| < \infty$, prove that for each measurable function f on E we have that $||f||_p \to ||f||_\infty$ as $p \to \infty$.

Proof. Suppose that f is bounded. Then

$$||f||_p \le ||f||_\infty |E|^{1/p} \to ||f||_\infty,$$

as $p \to \infty$. Secondly, we notice that for any $\varepsilon > 0$

$$||f||_p \ge ||f\chi_{\{|f|>||f||_\infty-\varepsilon\}}||_p \ge (||f||_\infty - \varepsilon)|\{|f| > ||f||_\infty - \varepsilon\}|^{1/p} \to ||f||_\infty - \varepsilon.$$ ☐

Note that somewhat in contrast to the result proved above, we have that if $f \in \ell^p$ and $1 < p < \infty$, then $||f||_\infty \le ||f||_p$.

Example 6.18 (Problem #4)**.** Let $1 \leq p \leq \infty$ be given. Suppose that φ is a measurable function on $\mathbb{R}$ such that $f\varphi \in L^p(\mathbb{R})$ for every $f \in L^p(\mathbb{R})$. Prove that $\varphi \in L^\infty(\mathbb{R})$, i.e., we must prove that $\varphi L^p \subseteq L^p$.

Proof. We argue by contradiction. Suppose that $\varphi \notin L^\infty$. Fix a sequence of integers k_j so that the sets

$$E_j := \{x : 2^{k_j} < \varphi(x) < 2^{k_{j+1}}\},$$

have positive measure. Set $f_j := \chi_{E_j}/|E_j|^{1/p}$. Then $||f_j||_p = 1$ and $||\varphi f_j||_p \geq 2^{k_j}$. Take $f := \sum_{j\geq 1} f_j/j^2$ where $f \in L^p$ by the triangle inequality. But we can compute that $\varphi f \notin L^p$. This yields our desired contradiction. $\square$

Example 6.19 (Problem #5)**.** Let $E \subseteq \mathbb{R}^d$ be measurable and fix $1 < p < \infty$. Assume that $f_n \in L^p(E)$, $f_n \to f$ a.e., and $\sup ||f_n||_p < \infty$. Prove that $f \in L^p(E)$, and if $g \in L^{p'}(E)$ $(1/p + 1/p' = 1)$ then

$$\lim_{n\to\infty} \int_E f_n g = \int_E fg.$$

Does the same result hold if $p = 1$?

Counter example when $p = 1$. Let $f_n := \chi_{[n,n+1)}$ so that $f_n \to 0$ a.e., but where $||f_n||_p = 1$ for all p. Let $g \equiv 1$ so that $g \in L^\infty$ when $|E| < \infty$. Then $\int f_n g = 1$, but $\int fg = 0$. Thus the hypothesis fails when $p = 1$. $\square$

Proof. Guided by two examples, namely,

(a) $f_n = \chi_{[n,n+1)} \to 0$ a.e., but $||f_n||_p = 1$ $\forall p$ $\forall n$;

(b) $f_n = n^{1/p}\chi_{[0,1/n)} \to 0$ a.e., but $||f_n||_p = 1$ for all n,

we argue that $f \in L^p$. By Fatou's lemma:

$$\left(\int |f|^p\right)^{1/p} = \left(\int |\liminf f_n|^p\right)^{1/p} \leq \liminf \left(\int |f_n|^p\right)^{1/p} \leq \sup ||f_n||_p < +\infty.$$

Take $1 < p < \infty$ and $g \in L^{p'}$. We argue that $\int f_n g \to \int fg$. The key point in our argument is the reduction to an integral over a finite-measure set. In particular, let $\varepsilon > 0$ and pick $|E_0| < \infty$ such that g is bounded on E_0 so that

$$\int_{E_0^c} |g|^{p'} < \left(\frac{\varepsilon}{\sup ||f_n||_p}\right)^{p'},$$

and on which $f_n \to f$ uniformly (by Egorov). By uniform convergence and the boundedness of g on E_0, $\int_{E_0} f_n g \to \int_{E_0} fg$, and by Hölder

$$\left|\int_{E_0^c} f_n g\right| \leq ||f_n||_p ||g\chi_{E_0^c}||_{p'} < \varepsilon.$$

The same inequality holds for f. $\square$

7 SIGNED MEASURES

7.1 Definitions and key theorems

Definition 7.1 (Signed measures). Let (X, M) be a measurable space. A *signed measure* on (X, M) is a function $v : M \to [-\infty, \infty]$ such that

(1) $v(\varnothing) = 0$;

(2) v assumes at most one of the values $\pm\infty$;

(3) If $\{E_j\}$ is a sequence of disjoint sets in M, then $v(\cup_j E_j) = \sum_j v(E_j)$, where the latter sum *converges absolutely* if $v(\cup_j E_j)$ is finite.

Notation: (*Mutual signularity of measures*) We write $u \perp v$ if $\exists E, F \in M$ such that $E \cap F = \varnothing$, $E \cup F = X$ and E is *null* for u, F is *null* for v. Note that we say a set E is *NULL* for u if $u(E) = 0$.

Examples of signed measures:

- If u_1, u_2 are measures on $\mathcal{M}$ and at least one of them is finite, then $v = u_1 - u_2$ is a signed measure.
- If u is a measure on $\mathcal{M}$ and $f : X \to [-\infty, \infty]$ is a measurable function such that at least one of $\int f^{\pm} du$ is finite, then the set function defined by $v(E) = \int_E f du$ is a signed measure.

Theorem 7.2 (Hahn decomposition). *If v is a signed measure on (X, M), then $\exists P, N \subset X$ such that*

(1) $P \cup N = X$ and $P \cap N = \varnothing$;

(2) P is positive for v: $\forall E \in M$ such that $E \subseteq P$, $v(E) \geq 0$;

(3) N is negative for v: $\forall E \in N$ such that $E \subseteq N$, $v(E) \leq 0$.

Moreover, this composition is essentially unique in that if (P', N') are another such pair, then every M-measurable subset of $P \Delta P'$, $N \Delta N'$ has measure zero.

Theorem 7.3 (Jordan decomposition). *Let (P, N) be a Hahn decomposition for the signed measure u. Then u has a unique decomposition into $u^+ - u^-$ of positive measures $u^{\pm}$, at least one of which is finite, such that $u^+(E) = 0$ for all measurable $E \subseteq N$ and for all measurable $E \subseteq P$, $u^-(E) = 0$.*

Definition 7.4. The *total variation* of the signed measure u is given by $|u| = u^+ + u^-$. The signed measure u is *absolutely continuous with respect to* v, written $u \ll v$ if for every measurable A, $v(A) = 0 \implies u(A) = 0$.

Theorem 7.5 (Radon-Nikodym). *Let (X, Σ) be a measurable space on which two σ-finite measures u, v are defined (i.e., $u(X) < \infty$). If $u \ll v$ then $\exists$ measurable $f : X \to [0, \infty)$ such that for all measurable $A \subseteq X$:*

$$u(A) = \int_A f dv.$$

Theorem 7.6 (Lebesgue-Radon-Nikodym)**.** *Let v be a σ-finite signed measure and u a σ-finite positive measure on (X, M). Then there exists unique σ-finite signed measures λ, ρ on (X, M) such that $\lambda \perp \rho$, $\rho \ll u$, and $v = \lambda + \rho$. Moreover, there is an extended u-integrable function $f : X \to \mathbb{R}$ such that $d\rho = f du$, and any two such functions are equal u-a.e.*

Consequences:

- $\boxed{v \ll u \implies dv = f du}$ for f a real extended u-integrable function;
- $d|v| = |f|du$ and $\int |f|du = \int d|v| = |v|(X) = ||v|| < \infty$;
- $u \ll u + v$;
- $v^+(E) = v(E \cap P)$, $v^-(E) = -v(E \cap N)$ (the sign here is important);
- Good sets to take to find quick examples include: $E_1 := E \cap P$, $E_2 := E \cap N$.

7.2 Practice problems

Example 7.7 (Fall 2017,#1)**.** Let v be a signed Borel measure on $\mathbb{R}$. Prove that

$$|v|(E) = \sup \left\{ \left| \int_E f dv \right| : |f| \leq 1 \right\}.$$

Example 7.8 (Spring 2017,#8)**.** Let X be a set and $\mathfrak{M}$ be a sigma algebra of subsets of X. Suppose that $(X, \mathfrak{M}, u)$ and $(X, \mathfrak{M}, v)$ are two finite measure spaces with u a positive measure and v a signed measure. Prove that the following two statements are equivalent:

(a) $v \ll u$, i.e., if $E \in \mathfrak{M}$ satisfies $u(E) = 0$ then $v(E) = 0$;

(b) For every $\varepsilon > 0$ there exists some $\delta > 0$ such that if $E \in \mathfrak{M}$ satisfies $u(E) < \delta$, then $v(E) < \varepsilon$.

Notes on (b) $\implies$ (a). Fix $\varepsilon > 0$. If this were not true then $\forall \delta > 0$, $u(E) < \delta \implies v(E) \geq \varepsilon$. Now for $n \geq 1$, choose E_n so that $u(E_n) < 2^{-n}$ and define $E := \cap_{m \geq 1} \sup_{j \geq m} E_j$. Then for any fixed $m \geq 1$,

$$u(E) \leq u(\cup_{j \geq m} E_j) \leq 2^{1-m} \to 0,$$

as $m \to \infty$. So we conclude that $u(E) = 0$. On the other hand, if we define $F_m := \cup_{n \geq m} E_n$, then $F_m \supseteq F_{m+1}$ for all m, which implies by a continuity theorem that

$$v(\cap_{m \geq 1} F_m) = \lim_{m \to \infty} v(F_m) \geq \varepsilon.$$

Indeed, since for all $m \geq 1$, we have that

$$v(F_m) = v(\cup_{n \geq m} E_n) \geq v(E_m) \geq \varepsilon,$$

by assumption. We thus arrive at a contradiction. □

Example 7.9 (Fall 2016,#6)**.** Let u and v be two σ-finite positive measures on a measure space $(X, \mathfrak{M})$. Show that there exists a measurable function $f : X \to \mathbb{R}$ such that for each $E \in \mathfrak{M}$,

$$\int_E (1 - f) du = \int_E f dv.$$

Does the above statement hold for every pair of signed finite measures u, v?

Example 7.10 (Spring 2014, #7)**.** Let v be a bounded signed Borel measure on $\mathbb{R}$, and assume that $f \in L^1(\mathbb{R})$. Prove that

$$g(x) := \int_{-\infty}^{\infty} f(x-y)dv(y),$$

is defined at almost every $x \in \mathbb{R}$, and that

$$||g||_1 \leq ||f||_1 \cdot |v|(\mathbb{R}),$$

where $|v|$ is the total variation of v. Show further that if f is uniformly continuous on $\mathbb{R}$, then so is g.

Example 7.11 (Spring 2008, #6)**.** Let v be a signed Borel measure on $I = [0,1]$ such that $|v|(I) = 1$ and $v(I) = 0$. Suppose that there exists a continuous function $f : I \to \mathbb{R}$ such that $||f||_\infty \leq 1$ and

$$\int_0^1 f dv = 1.$$

Show that the Lebesgue measure on I is not absolutely continuous with respect to $|v|$.

7.3 Extra problems from Folland

Example 7.12. Suppose that v is a signed measure and that λ, μ are positive measures such that $v = \lambda - \mu$. Prove that $\lambda \geq v^+$ and $\mu \geq v^-$.

Proof. Let $X = P \cup N$ be the Hahn decomposition for v on X and suppose that $E \subseteq X$ is v-measurable. Then

$$\begin{aligned} v^+(E) &= v(E \cap P) = \lambda(E \cap P) - \mu(E \cap P) \leq \lambda(E \cap P) \leq \lambda(E) \\ v^-(E) &= -v(E \cap N) = \mu(E \cap N) - \lambda(E \cap N) \leq \mu(E). \end{aligned}$$

□

Example 7.13. Suppose that v is a signed measure on $(X, \mathcal{M})$ and that $E \in \mathcal{M}$. Prove that

$$|v|(E) = \sup \left\{ \sum_{j=1}^n |v(E_j)| : E_1, \ldots, E_n \text{ are disjoint, and } E = \bigcup_{j=1}^n E_j \right\}.$$

Example 7.14. Let u, v be signed measures. Prove the following:

(A) $v \perp u$ iff $|v| \perp u$ iff $v^+, v^- \perp u$;

(B) $v \ll u$ iff $|v| \ll u$ iff $v^+, v^- \ll u$.

8 A FEW KEY EXAMPLES

Note: *See Dr. Lacey's handout and key notes.* This section is a summary of the key examples from that write-up.

$**$ **Pointwise convergence to zero, but the norm is bounded:** Take $f_n := \chi_{[n,n+1)}$. Then $f_n \to 0$ pointwise a.e., but $||f_n||_p = 1$ for any $0 < p \leq \infty$.

$**$ **Pointwise convergence to zero, but the norm is bounded (Compact case):** Fix $1 \leq p < \infty$. Take $f_n := n^{1/p}\chi_{[0,1/n)}$. Then $f_n \to 0$ pointwise a.e., but $||f_n||_p = 1$.

$**$ **Convergence to zero in L^p, $|f_n| \leq g \in L^p$, but do not have pointwise convergence to zero:** For an integer j such that $2^j \leq n < 2^{j+1}$, define $f_n := \chi_{\left[\frac{n-2^j}{2^j}, \frac{n-2^j+1}{2^j}\right]}$ so that $f_n \xrightarrow{\mu} 0$, but $f_n \nrightarrow 0$ since $\limsup f_n = 1$ and $\liminf f_n = 0$.

- **Sequence of functions that converge weakly to zero, but do not themselves converge:** Here, for example, *converging weakly to zero* means
$$\int f_n g dx \to 0, \ g \in L^1.$$
The basic example on $[0,1]$ is $f_n := \sin(nx) \xrightarrow{w} 0 \iff \langle f_n, g\rangle \to 0 \ \forall g \in L^1$ (or for all $g \in L^2$, etc.).
- **A sequence of sets in $A_n \subset [0,1]$ with $\sum_n |A_n| = \infty$, but where $\limsup_n A_n$ has measure zero:** Take $A_n := [0, 1/n)$ and apply the *Borel-Cantelli lemma.*
- **A measurable $E \subset \mathbb{R}$ with positive measure that does not contain an interval:** Any *"fat" Cantor set* will work here. (Any fat Cantor set is nowhere dense.)
- **A uniformly continuous function on $[0, \infty)$ that is unbounded:** Take $f(x) := \sqrt{x+1}$.
- **An element of $\ell^2(\mathbb{N})$ that is not in $\ell^1(\mathbb{N})$:** Select $a_n := n^{-3/4}$, let's say. Then $||a_n||_{\ell^1} = \sum_n n^{-3/4} = \infty$, but $||a_n||_{\ell^2} = \sum_n n^{-3/2} < \infty$. Likewise, for any $0 < p < q \leq \infty$, there are sequences in $\ell^q \setminus \ell^p$.
- **A function in $L^1[0,1)$ that is not in $L^2[0,1)$:** Choose $f(x) := 1/\sqrt{x}$. Then $||f||_{L^1} = \int_0^1 \frac{dx}{\sqrt{x}} = 2$, but $||f||^2_{L^2} = \int_0^1 \frac{dx}{x} = \infty$. Likewise, for any $0 < p < q \leq \infty$, there are functions in $L^p[0,1) \setminus L^q[0,1)$.
- **Sequence of continuous functions which converge to a non-continuous function:** Consider the sequence $f_n(x) := x^n$ on $[0,1]$. Then $f_n \to \chi_{\{1\}}$, which is not a continuous function on the full compact interval.
- **A continuous function on $(0,1]$ that is not uniformly continuous:** Take $f(x) := \sin(1/x)$ as it will work well here.
- **A continuous function on $[0,1]$ that is not Lipschitz on any subinterval:** These standard examples of so-called *"pathological" continuous functions* are shown to exist by Baire, but are not easy to explicitly construct by hand.
- $f_n := n \cdot \chi_{[0,\frac{1}{n}]}$, and $f_n := \frac{1}{n}\chi_{[0,n]}$.

9 FUNCTIONAL ANALYSIS

9.1 Banach spaces

A *Banach space* is a complete normed vector space over some field $\mathbb{F}$. A *Hilbert space* is a particular type of Banach space endowed with an *inner product*.

9.2 Linear transformations

The set $Y \subset X$ is a *linear subspace* (over $\mathbb{F}$) if $\forall \lambda, \mu \in \mathbb{F}$ and $\forall x, y \in Y$, we have that $\lambda x + \mu y \in Y$. The intersection of linear subspaces is always a subspace. A *linear transformation* satisfies $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$.

Proposition 9.1 (Continuity of a linear transformation)**.** *Let $T : X \to Y$ be a linear transformation. Then TFAE:*

(1) T is continuous;

(2) T is continuous at zero;

(3) T is continuous at one point;

(4) T is bounded;

(5) T is Lipschitz.

Proof Sketch. For (1)–(3): T continuous at zero $\iff$ $\forall x_n \to 0$, $T(x_n) \to 0$ (where $T(0) = 0$ since T is a linear transformation). Then $\forall x^* \in X$ such that $x_n \to x^*$, $x_n - x^* \to 0 \implies T(x_n - x^*) \to 0$.
For (4): If T is continuous, we have that $\forall \varepsilon = 1\ \exists \delta > 0$ such that $||x|| < \delta \implies ||Tx|| < 1 \implies ||T|| < 1/\delta$.
For (5): Consider that $||Tx - Ty|| \leq ||t|| \cdot ||x - y||$. □

9.3 Bounded linear operators

Definition 9.2 (Bounded linear operators)**.** We say that $T : X \to Y$ is a *bounded linear operator* if $\exists K$ such that $\forall x \in X$: $||Tx|| \leq K \cdot ||x||$ (where we note that $||Tx|| \leq ||T|| \cdot ||x||$ always). In this case we write $T \in \mathcal{B}(X, Y)$. Note that

$$||T|| = \text{lub}_{x \neq 0} \frac{||Tx||}{||x||} = \text{lub}_{||x||=1} ||Tx||.$$

Proposition 9.3 (Completeness of $\mathcal{B}(X, Y)$)**.** *The space $\mathcal{B}(X, Y)$ is a Banach space endowed with the pointwise operations: $(T_1 + T_2)(x) = T_1(x) + T_2(x)$ and $(\lambda T)(x) = \lambda T(x)$.*

Proof. We need to show that $\mathcal{B}(X,Y)$ is complete when Y is itself a Banach space. To show this we use the completeness of Y. Suppose that $\{T_j\}$ is Cauchy, which implies that $||T_j - T_k||_B \to 0$ as $j,k \to \infty$. By the completeness of Y, $T_j(x) \to T(x)$. Now we see that

$$||(T_j - T)(x)|| = \lim_{k\to\infty} ||T_j x - T_k x|| \leq \limsup_{k\to\infty} ||T_j - T_k|| \cdot ||x|| \longrightarrow \infty. \qquad \square$$

The following theorem called the *Banach-Steinhaus theorem* is also refered to as the *principle of uniform boundedness.*

Theorem 9.4 (Banach-Steinhaus)**.** *Let X be a Banach space and let Y be a normed linear space. Let $\{T_\alpha\}$ be a family of bounded linear operators from X into Y. If for each $x \in X$ the set $\{T_\alpha x\}$ is bounded, then the set $\{||T_\alpha||\}$ is bounded.*

Theorem 9.5 (Open mapping theorem)**.** *Let X and Y be Banach spaces, and let $T : X \to Y$ be a bounded linear map. Then T maps open sets of X onto open sets of Y.*

Proposition 9.6. *Let X and Y be Banach spaces and let $T \in \mathcal{B}(X,Y)$ be a bijective linear map. Then T^{-1} is a bounded linear map.*

9.4 Hilbert spaces

Definition 9.7 (Hilbert spaces)**.** A *Hilbert space* is a set X satisfying:

(a) X is a vector space over some field $\mathbb{F}$ (typically $\mathbb{C}$ or $\mathbb{R}$);

(b) X possesses an *inner product*, $\langle\cdot,\cdot\rangle : X \times X \to \mathbb{F}$, that satisfies:

  i. $\langle x,y\rangle = \overline{\langle y,x\rangle}$, $\forall x,y \in X$. Note that this implies that $\langle x,x\rangle \in \mathbb{R}\ \forall x$.
  ii. $\langle x,x\rangle \geq 0$ and $\langle x,x\rangle = 0 \iff x = 0$.
  iii. For all $x,y \in X$ and $\forall\lambda \in \mathbb{F}$, $\langle \lambda x,y\rangle = \lambda\langle x,y\rangle$. Note that this implies $\langle x,\lambda y\rangle = \overline{\lambda}\langle x,y\rangle$.
  iv. For all $z,y,z,u \in X$: $\langle x,y+z\rangle = \langle x,y\rangle + \langle x,z\rangle$ and $\langle x+u,y\rangle = \langle x,y\rangle + \langle u,y\rangle$.

(c) X is complete in the sense of normed spaces.

One consequence of the above is that X is normed in the sense that $||x|| := \sqrt{\langle x,x\rangle}$ is a *norm.*

Examples of Hilbert spaces include: $L^2(X)$ *IS* a Hilbert space as with $\langle f,g\rangle := \int_X f(x)\overline{g(x)}dx$, $\langle f,f\rangle = \int_X |f|^2$ (and linearity is easy to prove). However, $L^p(X)$ for $p \neq 2$ is *NOT* a Hilbert space as it does not satisfy the polarization identity.

Lemma 9.8 (Cauchy-Schwarz)**.** *Let X be a Hilbert space. For all $x,y \in X$, $|\langle x,y\rangle| \leq ||x|| \cdot ||y||$.*

Proof (Real Case). First, we observe that

$$||x+\lambda y||^2 = ||x||^2 + 2\lambda\langle x,y\rangle + \lambda^2||y||^2.$$

Then we complete the square as

$$||x+\lambda y||^2 = \left(\lambda||y|| + \frac{\langle x,y\rangle}{||y||}\right)^2 + ||x||^2 - \frac{\langle x,y\rangle^2}{||y||^2},$$

which cannot have two roots which then implies that

$$||x||^2 - \frac{\langle x,y\rangle^2}{||y||^2} \geq 0,$$

from which the claimed result follows. $\square$

Note that we also have *Cauchy-Schwarz* in $\mathbb{R}^n$:

$$\left(\sum_{i=1}^{n} u_i v_i\right)^2 \leq \left(\sum_{i=1}^{n} u_i^2\right)\left(\sum_{i=1}^{n} v_i^2\right).$$

Proposition 9.9 (Polarization identities)**.** *We are concerned with the real case here. In particular, for H any Hilbert space over $\mathbb{R}$ we have that $\forall x, y \in H$: $||x+y||^2 + ||x-y||^2 = 2(||x||^2 + ||y||^2)$. The polarization identity also defines an inner product: $\langle x, y\rangle = \frac{1}{4}\left(||x+y||^2 - ||x-y||^2\right)$, $\forall x, y \in H$.*

Definition 9.10 (Orthogonal sets and completeness)**.** Let X be a Hilbert space. We say that $x, y \in X$ are *orthogonal* if $\langle x, y\rangle = 0$. The family $\{x_\alpha\}_{\alpha\in I}$ is an *orthogonal set* $\iff$ $\langle x_\alpha, x_\beta\rangle = 0\ \forall \alpha \neq \beta$. An *orthonormal set* is an orthogonal set such that $\langle x_\alpha, x_\alpha\rangle = 1$ for all $\alpha \in I$.

An orthogonal set is called *complete* if $\langle x_\alpha, y\rangle = 0\ \forall\alpha \in I \implies y = 0$. An example of a complete orthogonal set is the basis $\{\exp(2\pi\imath k\vartheta)\}_{k\in\mathbb{Z}}$ in $L^2_{\text{per}}(0,1)$.

Lemma 9.11 (Least squares minimization)**.** *Let $\{x_i\}_{i=1}^N$ be an orthonormal set. Then the difference $\left\|x - \sum_{i=1}^N \lambda_i x_i\right\|$ is minimized by taking $\lambda_i := \langle y, x_i\rangle$.*

Corollary 9.12 (Bessel's inequality)**.** *Let $\{x_i\}_{1\leq i\leq N}$ be an orthonormal sequence. Then*

$$\sum_{i=1}^{n} |\langle x_i, y\rangle|^2 \leq ||y||^2.$$

Theorem 9.13 (Orthogonal complement theorem)**.** *Let X be a Hilbert space and $E \subset X$ be a closed (i.e., complete) subspace. Then $\forall x \in X\ \exists x^* \in E, x^\perp$ such that $x = x^* + x^\perp$ and $\langle x^\perp, y\rangle = 0$ for all $y \in E$. Moreover, we can prove that*

$$x^* = \operatorname{argmin}_{y\in E} ||x-y||.$$

Example 9.14 (A linear space that is not closed)**.** In $\ell^2(\mathbb{N})$ consider $E := \{x : \exists N$ such that $x_j = 0 \forall j > N\}$. Then E is a linear space, but is *NOT* closed as there is no minimum to the following example: for $x := (1, 1/2, 1/3, 1/4, \ldots)$ the norm $||x - (1, 1/2, \ldots, 1/N, 0, 0, \ldots)||$ can be made as small as you wish by taking N large enough.

Proposition 9.15 (Parseval's identity)**.** *If $\{e_i\}$ is complete and orthonormal, then $\left\|x - \sum_{i=1}^N \lambda_i e_i\right\|$ can be made as small as possible by taking $\lambda_i := \langle e_i, x\rangle e_i$ and letting $N \to \infty$. In particular,*

$$||x||^2 = \lim_{N\to\infty} \sum_{i=1}^{N} |\langle e_i, x\rangle|^2.$$

9.5 Practice problems

Note that this section is intentionally underdeveloped. See the previous comprehensive exams for particular examples.

10 MISCELLANEOUS OTHER NOTES

10.1 Other problems

Example 10.1 (Dr. Lacey)**.** Let e_j be the unit basis in $\ell^2(\mathbb{N})$. Fix $v \in \ell^2(\mathbb{N})$. Consider the linear operator T defined such that $Te_j = v$ for all $j \geq 1$ extended linearly. Note that since the e_j's spab ℓ^2, this requirement completely determines T. Prove that T is bounded iff $v = 0$.

Proof 1. The norm of $e_1 + \cdots + e_n$ is $\sqrt{n}$ in ℓ^2 whereas the norm of $T(e_1 + \cdots + e_n)$ is $n \cdot ||v||$. Therefore $||T|| \geq \sqrt{n} \cdot ||v||$ for all n. This implies that if $v \neq 0$ then the norm of T is $+\infty$. □

Proof 2. Let $x := \sum_{j \geq 1} e_j/j \in \ell^2$. Then $Tx = v \sum_{j \geq 1} 1/j$. This implies that $Tx = +\infty$ whenever $v \neq 0$ where $Tx = 0$ when $v = 0$. □

Example 10.2 (Spring 2016, #5)**.** Let f be monotone increasing on $[0, 1]$. Prove that $\int_0^1 f'(x)dx \leq f(1) - f(0)$.

Proof. Let $g_n(x) := (f(x + 1/n) - f(x))/(1/n)$. Then for $\varepsilon \in (0, 1/4)$:

$$\int_{\varepsilon}^{1-\varepsilon} f'(x)dx = \int_{\varepsilon}^{1-\varepsilon} \lim_{n\to\infty} g_n(x)dx, \quad \text{and}$$

$$\begin{aligned}\int_{\varepsilon}^{1-\varepsilon} f(x + 1/n) - \int_{\varepsilon}^{1-\varepsilon} f &= \int_{\varepsilon+1/n}^{1-\varepsilon+1/n} f - \int_{\varepsilon}^{1-\varepsilon} f \\ &= \int_{\varepsilon+1/n}^{1-\varepsilon+1/n} f - \int_{\varepsilon}^{\varepsilon+1/n} f,\end{aligned}$$

which implies that

$$\int_{\varepsilon}^{1-\varepsilon} f(x + 1/n) - \int_{\varepsilon}^{1-\varepsilon} f \leq \frac{1}{n}(f(1) - f(0)).$$

Now since f is monotone and non-decreasing $g_n \geq 0$, so by Fatou:

$$f(1) - f(0) \geq \liminf_{n\to\infty} \int_{\varepsilon}^{1-\varepsilon} \frac{f(x + 1/n) - f(x)}{1/n} \geq \int_{\varepsilon}^{1-\varepsilon} f'.$$ □

Example 10.3 (Fall 2016, #3)**.** Let $E_1, \ldots, E_n$ be Lebesgue measurable subsets of $[0, 1]$ and define

$$S_q := \{x \in [0, 1] : x \text{ belongs to at least } q \text{ of the sets } E_i\}.$$

Show that for each $1 \leq q \leq n$, S_q is Lebesgue measurable, and there exists k such that

$$\frac{q|S_q|}{n} \leq |E_k|.$$

Proof. Let $f := \sum_{k=1}^{n} \chi_{E_k}$, which is a finite sum of measurable functions and hence is measurable. Then

$$S_q = \{x \in [0,1] : f(x) \geq q\},$$

and so since f is measurable, it follows that S_q is measurable for any $1 \leq q \leq n$. Moreoverm by Chebyshev's inequality, we see that

$$q \cdot |S_q| \leq \int_{[0,1]} f(x)dx = \int_{[0,1]} \sum_{k=1}^{n} \chi_{E_k}(x)dx = \sum_{k=1}^{n} |E_k| \leq n \cdot \left(\max_{1\leq k\leq n} |E_k| \right).$$

This implies that

$$\frac{q|S_q|}{n} \leq \max_{1\leq k\leq n} |E_k| =: |E_K|,$$

for some $K \in \{1, 2, \ldots, n\}$. This inequality is true for any $1 \leq q \leq n$. □

Example 10.4 (Fall 2015, #1)**.** Let $f \in L^1(\mathbb{R})$ and let $\alpha > 0$. Show that

$$\lim_{n\to\infty} f(nx)n^{-\alpha} = 0, \quad \text{for a.e. } x \in \mathbb{R}.$$

Proof. Let $f_n(x) := f(nx)n^{-\alpha}$ for $n \geq 1$, Then by a substitution, we have that

$$\int_{\mathbb{R}} |f_n(x)|dx = \frac{1}{n^{\alpha+1}} \int |f(x)|dx.$$

It follows that

$$\sum_{n=1}^{\infty} \int_{\mathbb{R}} |f_n(x)|dx = \int_{\mathbb{R}} |f(x)|dx \times \sum_{n\geq 1} \frac{1}{n^{\alpha+1}} = ||f||_1 \cdot \zeta(\alpha+1) < \infty, \tag{$*$}$$

since $f \in L^1$ and $\alpha + 1 > 1$. Now since $|f_n(x)| \geq 0$ for all n and for all $x \in \mathbb{R}$, by the monotone convergence of the sequence of partial sums on the LHS of $(*)$, we see that also

$$\int_{\mathbb{R}} \sum_{n\geq 1} |f_n(x)| < \infty \implies \sum_{n\geq 1} |f_n(x)| < \infty,$$

and so necessarily we must have that $\lim_{n\to\infty} |f_n(x)| = 0$. □

Example 10.5 (Spring 2013, #8)**.** Let $(X, \mathcal{M}, \mu)$ be a measure space, and for $n \geq 1$ let f_n, f, g_n, g be measurable complex-valued functions on X such that $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$.

(a) Show that $f_n + g_n \xrightarrow{\mu} f + g$; and

(b) Show that $f_n g_n \xrightarrow{\mu} fg$ if $\mu(X) < \infty$, but not necessarily if $\mu(X) = \infty$.

Proof of (b). Suppose that $\mu(X) < \infty$ and for the sake of contradiction that $f_n g_n \not\xrightarrow{\mu} fg$. This implies that there is some $\varepsilon > 0$ and some subsequence $\{f_{n_k} g_{n_k}\}$ such that

$$\mu(\{x : |f_{n_k} g_{n_k} - fg|(x) \geq \varepsilon\}) \geq \varepsilon, \ \forall k \geq 1.$$

Now $f_n \xrightarrow{\mu} f$ and $g_n \xrightarrow{\mu} g$ implies that $f_{n_k} \xrightarrow{\mu} f$ and $g_{n_k} \xrightarrow{\mu} g$ for any subsequence of a convergent sequence converges to the same limit. This implies further subsequences $f_{n_{k_j}} \to f$ a.e. and $g_{n_{k_j}} \to g$ a.e. on X, which implies that $f_{n_{k_j}} g_{n_{k_j}} \to fg$ a.e. on X. Now together with the fact that $\mu(X) < \infty$, and that **convergence in measure implies finiteness a.e.** on X, we can apply Egorov to see that $f_{n_{k_j}} g_{n_{k_j}} \to fg$ almost uniformly on X, which in turn implies convergence in measure of this subsequence. We have arrived at our desired contradiction.

As a counter example when $\mu(X) = \infty$, consider either of the following definitions for $n \geq 1$: *(i)* $f_n(x) = g_n(x) = x + 1/n$, $f(x) = g(x) = x$; or *(ii)* $f_n(x) = \frac{1}{n}\chi_{[0,n)}(x)$, $f(x) = 0$, $g(x) = g_n(x) = x$. □

Example 10.6 (From the Analysis II homework)**.** Prove that if $E \subset [0,1]$ has positive Lebesgue mesaure, then there exist $x, y \in E$ such that $x - y \in \mathbb{Q}$.

Claim: Let $G \subset \mathbb{R}$ be selected such that $|G| > 0$. Then there exists an open interval $I \subset \mathbb{R}$ such that $|G \cap I| \geq 3/4|I|$.

Proof. This result follows from the Vitali covering lemma. □

Proof. We prove that given $E_1, E_2 \subseteq \mathbb{R}$ such that $|E_1|, |E_2| > 0$, the difference set

$$D_{12} := \{x - y : x \in E_1, y \in E_2\},$$

contains an open interval. Here we will specify that $E_1 = E_2$ and notice that since the rationals are dense in $\mathbb{R}$, this implies the claim in the exam problem.

From the claim, we have open intervals $I, J \subseteq \mathbb{R}$ such that $|E_1 \cap I| \geq 3/4|I|$ and $|E_2 \cap J| \geq 3/4|J|$, and where WLOG we can assume that $|I| \geq |J|$. It follows that there exists a $\alpha \in \mathbb{R}$ such that $J + \alpha \subseteq I$. Let $\gamma := 1/4|J|$. Then for any $0 < |\beta| < \gamma$, we see that the intervals I, $J + \alpha + \beta$ intersect in an interval M where the length of M satisfies $|M| > 3/4|J|$. Next, let $\widetilde{E}_1 := E_1 \cap M$, $\widetilde{E}_2 := (E_2 + \alpha + \beta) \cap M$ for the fixed parameters α, β and interval M identitified above. Then we see that (1) we have

$$\begin{aligned} |E_1 \cap I| &= |\widetilde{E}_1 \cup E_1 \cap (I \setminus M)| \\ &= |\widetilde{E}_1| + |E_1 \cap (I \setminus M)| < |\widetilde{E}_1| + \frac{|J|}{4} \\ \implies |\widetilde{E}_1| &> \frac{3}{4}|I| - \frac{1}{4}|J| > \frac{3}{4}|J| - \frac{1}{4}|J| = \frac{|J|}{2}, \end{aligned} \tag{*}$$

and that (2) we have that

$$\begin{aligned} |E_2 \cap J| &= |(E_2 + \alpha + \beta) \cap (J + \alpha + \beta)| \\ &= |\widetilde{E}_2| + |(E_2 + \alpha + \beta) \cap (J + \alpha + \beta) \setminus M| \\ &< |\widetilde{E}_2| + \frac{|J|}{4} \\ \implies |\widetilde{E}_2| &> \frac{3}{4}|J| - \frac{|J|}{4} = \frac{|J|}{2}, \end{aligned} \tag{**}$$

by similar reasoning to the above. Finally, we notice that $\widetilde{E}_1 \cup \widetilde{E}_2 \subseteq M$, which shows that $|\widetilde{E}_1 \cup \widetilde{E}_2| \leq |M| \leq |J|$, where by (*) and (**) we also have that

$$|\widetilde{E}_1 \cup \widetilde{E}_2| \leq |\widetilde{E}_1| + |\widetilde{E}_2| < \frac{|J|}{2} + \frac{|J|}{2} = |J|.$$

Hence, we must have that $\widetilde{E}_1 \cap \widetilde{E}_2 \neq \varnothing$, which in turn implies that $E_1 \cap (E_2 + \alpha + \beta) \neq \varnothing$, or equivalently that there exists a $y \in E_2$ such that $y + \alpha + \beta \in E_1$. Thus we can see that $\alpha + \beta \in D_{12}$. Just to be clear, but by no means concise, as pointed out in the previous discussion we have that $\alpha + \beta \in D_{12}$ whenever $0 < |\beta| < \gamma = |J|/4$. So we obtain that

$$(\alpha - \gamma, \alpha + \gamma) = \left(\alpha - \frac{|J|}{4}, \alpha + \frac{|J|}{4}\right) \subseteq D_{12}.$$

In other words, D_{12} defined as above contains an interval – and we are done here. □

Example 10.7 (Fall 2017, #4)**.** Let ϕ be a function in $C^1(\mathbb{R}^d)$ with compact support. Show that for any $f \in L^1(\mathbb{R}^d)$ the function $\phi * f(x) := \int \phi(x - y) f(y) dy$ is differentiable. (HINT: Be careful about moving limits.)

Example 10.8 (Sprint 2017, #7)**.** Let $f_n : [0,1] \to \mathbb{R}$ be non-negative measurable functions that converge a.e. to a measurable function $f \in L^1[0,1]$.

(a) Prove that the integrals $\int_0^1 \min(f_n(x), f(x))dx$ are defined for each n, and that

$$\lim_{n\to\infty} \int_0^1 \min(f_n(x), f(x))dx = \int_0^1 f(x)dx.$$

(b) Assume that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 f(x)dx$. Use part (a) to prove that $||f_n - f||_1 \to 0$ as $n \to \infty$.

Example 10.9 (Fall 2015, #8)**.** Let $1 < p < \infty$. Show that the operator $Tf(x) = \int_0^\infty \frac{f(y)}{x+y}dy$ satisfies $||Tf||_p \le C_p \cdot ||f||_p$ where $C_p = \int_0^\infty \frac{1}{(x+1)x^{1/p}}$.

10.2 Facts and reminder notes

The following reminder list is intended to quickly refresh myself on the little facts and/or less standard tricks I have come across during the comp preparation:

- If a question contains multiple parts, it's a good guess that the later parts will use the results from the earlier ones.
- $|a-b| = \frac{\max(a,b)-\min(a,b)}{2}$.
- $x + y = \min(x,y) + \max(x,y)$.
- $\frac{1-\cos(2x)}{2x} = \frac{\sin^2(x)}{x}$ for all $x \in \mathbb{R}$.
- For all $u \in \mathbb{R}$, $|\sin(u)| \le |u|$ (follows from a truncated Taylor series expansion of the sine function).
- $\chi_{[x,x+c]}(t) = \chi_{[t-c,t]}(x)$, which can be combined with an application of Fubini or Tonelli to swap the orders of integration in a double or multiple integral.
- Whenever see convergence in measure, try to extract a subsequence which converges pointwise.
- Whenever see (i) positive g_n; and (ii) $g_n \to g$ a.e., immediately try Fatou's lemma.
- Whenever see $|E| < \infty$ and/or $f_n \to f$ a.e., think Egorov applications.
- If cannot prove convergence in measure directly, think proof by contradiction.
- Let Δ denote the *symmetric difference of sets*: $A\Delta B = (A \setminus B) \cup (B \setminus A)$. Then if I_r is an interval, E is measurable, and $0 < |E| < \infty$: $||E \cap I_r(x)| - |E \cap I_r(y)|| \le |(E \cap I_r(x))\Delta(E \cap I_r(y))| \le |I_r(x)\Delta I_r(y)|$.
- As a heuristic, convolutions tend to smooth out functions. Convolution problems may involve convergence theorems.
- All aspects of the convergence theorems on the exam are sharp.
- $E \setminus F = (E \setminus A)\dot\cup(A \setminus F)$.
- $|A \setminus E|_e < |A\Delta E|_e$.
- $f(x) = \sum_{j=1}^n \chi_{E_j}(x) \implies \int_0^1 f = \sum_{j=1}^n |E_j|$.

- Try the following special functions: $h(x,y) = py^{p-1}\chi_{|f(x)|>y}$, $f(x) = \sup_n |f_n(x)|$, $F(x,t) = f(t)\chi_{[x,x+c]}(t) = f(t)\chi_{[t-c,t]}(x)$. Try $g_n = f - \min(f_n, f)$ and $h_n = \max(f_n, f) - f$.
- An eigenvalue of U: $U(f) = \lambda f \implies ||U(f)|| = |\lambda|||f||$.
- $A_n := \{|f| > n\}$.
- $CF \setminus CE = E \setminus F$ and $E_k = F_k \cup (E_k \setminus F_k)$.
- Let $k_n := \min(f_n, f)$. To show that $\int k_n$ is well-defined: Note that k_n is measurable and non-negative a.e. provided that the f_n, f are. Therefore, $\int k_n \in [0, +\infty]$ exists.
- $F_n(x) := \int_0^x f_n(t)dt \implies |F_n(x) - F_n(y)| = |\int \chi_{[x,y]}(t)f_n(t)dt|$.
- Let $E_k^{(j)} := \{x \in E_k : j-1 \leq |x| < j\} = (E_k \cap \{x : |x| < j\}) \setminus \{x : |x| < j-1\}$. Then $E_k^{(j)}$ is *bounded* and measurable. May need to assume that $|E_k| < \infty$ for some k (the other case is easier).
- $E \setminus \cap_{m\geq 1} F_m = \cup_{m\geq 1} E \setminus F_n$
- If $|E| < \infty$, then $\chi_E(x) \in L^1(\mathbb{R})$.
- If $0 < r < s < 1$: $E \cap Q(s) \subset (E \cap Q(r)) \cup (Q(s) \setminus Q(r))$.
- As $E \cap Q(r) \subset Q(r)$ and $Q(r)$ are measurable sets:

$$\lim_{n\to\infty} |E \cap Q(1/n)| \leq \lim_{n\to\infty} |Q(1/n)| = |\varnothing| = 0.$$

- From Fall 2016 #2: $A \subseteq \mathbb{R}^d$ is Lebesgue measurable $\iff \forall \varepsilon > 0\ \exists$ a Lebesgue measurable $E \subseteq \mathbb{R}^d$ such that $|A\Delta E| < \varepsilon$. (here: $A \setminus E| < |A\Delta E| < \varepsilon/3$)
- Fall 2016 #4: Uses the DCT to show a function is continuous and differentiable.
- Continuous on $E \implies$ measurable on E.
- $\mu(\{x : |f_n - f| + |g_n - g| \geq \varepsilon\}) \leq \mu(\{x : |f_n - f| \geq \varepsilon/2\}) + \mu(\{x : |g_n - g| \geq \varepsilon/2\})$
- Counter examples: $f_n(x) = \frac{1}{n}\chi_{[0,n)}(x)$.
- $\inf_n x_n \leq \liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n \leq \sup_n x_n$ where $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} x_m = \sup_{n\geq 0} \inf_{m\geq n} x_m$ and $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m\geq n} x_m = \inf_{n\geq 0} \sup_{m\geq n} x_m$.
- $f(x) = \frac{1}{(x \log x)^{1/q}} \in L^q \notin L^p$ for $p < q$.
- Note that $\mu^*(E \cap (\cup_{j=1}^n A_j)) \leq \mu^*(E \cap (\cup_{j\geq 1} A_j)) \leq \sum_{j\geq 1} \mu^*(E \cap A_j)$.
- Let $\phi(s) := |\{x : g(x) < s\}|$. Then since $\{x : s_{k+1} \leq g(x) < s\} \subseteq \{x : s_l \leq g(x) < s\}$ and since $|[0, \sqrt{s}| < \infty$, we have that

$$\lim_{k\to\infty} |\phi(s) - \phi(s_k)| = |\cap_k \{x : s_k \leq g(x) < s\}| = 0.$$

- ϕ continuously differentiable with compact support $\implies \exists C$ constant such that

$$\sup_{h\neq 0} \left| \frac{\phi(x + e_j h - y) - \phi(x - y)}{h} \right| \leq C.$$

- If α is an irrational multiple of 2π, the numbers $n\alpha \pmod{2\pi}$ are dense on the unit circle. Moreover, any continuous function that vanishes on a dense set is identically zero.

- $|A \setminus E|_e| \leq |A \Delta E|_e$ for any A, E.
- Let $A_n := \{|f| > n\}$ and $A := \{|f| = \infty\}$. Then $A_{n+1} \subseteq A_n$ and $A = \int_{n \geq 1} A_n$ so that $A_n \searrow A$. By continuity, $\lim_{n\to\infty} |A_n| = |A| = 0$ when f is finite a.e.
- $H \supseteq [0,1] \setminus A$ open such that $|H|_e = |[0,1] \setminus A|_e \implies [0,1] \setminus H \subseteq A \subseteq G$.
- f' integrable $\implies$ for all $\varepsilon > 0$, $\exists \delta > 0$ such that for any measurable $A \subseteq \mathbb{R}$: $|A| < \delta \implies \int_A |f'| < \varepsilon$.
- Handle the cases of $|A|_e < \infty$, $|A|_e = \infty$ separately (typically, reduce the second to the first).
- First problems on Fall 2012 exam;
- $\int_E |f_n|^p = ||(f_n - f + f)\chi_E||_p^p \leq 2^p ||(f_n - f)\chi_E||_p^p + 2^p ||f\chi_E||_p^p$.
- $\liminf(s_n) + \liminf(t_n) \leq \liminf(s_n + t_n) \leq \limsup(s_n + t_n) \leq \limsup(s_n) + \limsup(t_n)$, for any sequences $\{s_n\}$ and $\{t_n\}$.