Partition Identities Related to Stanley's Theorem

Maxie D. Schmidt

Georgia Institute of Technology

maxieds@gmail.com

mschmidt34@gatech.edu

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Overview

Agenda Today

- Discuss the definition of partitions and various partition functions
- Connect restricted partitions with the functions $\phi(n)$ and $\mu(n)$ from multiplicative number theory
- Get a working understanding of Stanley's theorem and its variants
- Prove new identities for Euler's partition function $p(n)$ involving unusual connections to special multiplicative functions

Why are these new results important?

- **So why are these identities special?**
  These results are significant and special in form given their unusual new connections with the functions of multiplicative number theory and the more additive nature of the theory of partitions.
- Formulas and identities connecting the at times seemingly disparate branches of additive and multiplicative number theory are rare, with only a few good examples of these types known in the literature.
- This talk is primarily based on joint work with Mircea Merca accepted to appear in the *American Mathematical Monthly* in 2018.

The Partition Function $p(n)$

Partitions of n (Euler's Partition Function)

- A partition of a positive integer n is a sequence of positive integers whose sum is n
- More formally, we may write a partition of n as a sum of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$
- We denote the total number of partitions of n by $p(n)$
- For example, $p(5) = 7$ since the partitions of 5 are given as:

$$5,\ 4+1,\ 3+2,\ 3+1+1,\ 2+2+1,\ 2+1+1+1,\ 1+1+1+1+1.$$

Partitions of n (Generating Function)

- By a combinatorial argument, the generating function for $p(n)$ is

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(q;q)_\infty} = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots},$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ denotes the q-Pochhammer symbol

- **Why (you may ask)?**

Partitions of n (Generating Function)

- By the representation of the partitions of 5 given above, we may write the number of partitions of n as the number of positive integer solutions to the equation $x_1 + 2x_2 + 3x_3 + \cdots + nx_n = n$
- Moreover, by expanding geometric series we can count by expanding the generating function products and taking the coefficient of q^n

$$\sum_{n\geq 0} p(n)q^n = \frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}$$
$$= \prod_{k=1}^{\infty}\left(1 + q^k + q^{2k} + q^{3k} + \cdots\right)$$
$$= \left(1 + \cdots + q^{x_1} + \cdots\right)\left(1 + \cdots + q^{2\cdot x_2} + \cdots\right)\left(1 + \cdots + q^{3\cdot x_3} + \cdots\right)\cdots$$

Other Special Partition Functions

Other Special Partition Functions

- $q(n) := [q^n](q;q^2)_\infty^{-1}$ is another special common partition function which denotes the number of partitions of n into *distinct* parts, or equivalently into *odd* parts (see A000009)
- Other special partition functions are commonly defined by their q-series generating functions as in the following table

Generating Function	**The number of partitions of n into parts which are**
$\prod_{m=1}^{\infty} \frac{1}{1-q^{2m-1}}$	odd
$\prod_{m=1}^{\infty} \frac{1}{1-q^{2m}}$	even
$\prod_{m=1}^{\infty} \frac{1}{1-q^{m^2}}$	squares

Table: Partition Functions by Their Generating Functions

Other Special Partition Functions (cont.)

Generating Function	The number of partitions of n into parts which are
$\prod_{p \text{ prime}} \frac{1}{1-q^p}$	primes
$\prod_{m=1}^{\infty} (1+q^m)$	unequal
$\prod_{m=1}^{\infty} (1+q^{2m-1})$	odd and unequal
$\prod_{m=1}^{\infty} (1+q^{2m})$	even and unequal
$\prod_{m=1}^{\infty} (1+q^{m^2})$	distinct squares
$\prod_{p \text{ prime}} (1+q^p)$	distinct primes

Table: Partition Functions by Their Generating Functions

Restricted Partition Functions

- Applying the counting argument above by generating functions, we can generate the number of partitions of n with parts at least r by $(1-q)(1-q^2)\cdots(1-q^{r-1})/(q;q)_\infty$
- Similarly we may define the sequence $S_{n,k}^{(r)}$ for $n,k,r \geq 1$ to be the number of k's in the partitions of n with smallest part at least r
- The sequence of restricted partition functions, $S_{n,k}^{(r)}$, is primarily motivated by its combinatorial interpretation of the generating function (as we shall see):

$$\sum_{n\geq 0} S_{n,k}^{(r)} q^n = \frac{(1-q)(1-q^2)\cdots(1-q^{r-1})}{(q;q)_\infty} \cdot \frac{q^k}{1-q^k}$$

Stanley's Theorem

Stanley's Theorem

Theorem (Stanley)

The number of 1's in the partitions of n is equal to the number of parts that appear at least once in a given partition of n, summed over all partitions of n.

An Example of Stanley's Theorem

Example (MathWorld)

- We check the theorem when $n = 5$ by an example.
- Written in set notation, the partitions of 5 are given as follows: $\{5\}, \{4, 1\}, \{3, 2\}, \{3, 1, 1\}, \{2, 2, 1\}, \{2, 1, 1, 1\}, \{1, 1, 1, 1, 1\}$.
- Then there are a total of $0 + 1 + 0 + 2 + 1 + 3 + 5 = 12$ distinct 1's in all of the partitions of n, and moreover,
- The number of unique terms in each partition of 5 is given by $1 + 2 + 2 + 2 + 2 + 2 + 1 = 12$.
- Thus the theorem holds when $n = 5$ (see A000070).

Proof of Stanley's Theorem

We do not give an explicit proof of Stanley's theorem here. However, we do prove the next restatement of it in terms of restricted partition functions.

Another Form of Stanley's Theorem

Euler's Totient Function $\phi(n)$

Definition and Generating Functions

Euler's *totient function*, $\phi(n)$, is defined for all $n \geq 1$ as

$$\phi(n) = \sum_{d:(d,n)=1} 1 \mapsto \{1, 1, 2, 2, 4, 2, 6, 4, 6, 4, 10, 4, 12, \ldots\}$$

(see sequence A000010).
Euler's totient function is multiplicative, i.e., $\phi(mn) = \phi(m)\phi(n)$ whenever $(m, n) = 1$, and since $n = \sum_{d|n} \phi(d)$ for all $n \geq 1$, it has a "*nice*" *Lambert series* generating function given by

$$\sum_{n \geq 1} \frac{\phi(n) q^n}{1 - q^n} = \frac{q}{(1-q)^2}.$$

A Restatement of Stanley's Theorem

Theorem

The number of 1's in the partitions of n is equal to

$$S_{n,1}^{(1)} = \sum_{k=2}^{n+1} \phi(k) S_{n+1,k}^{(2)},$$

where the restricted partition function $S_{n,k}^{(2)}$ counting the number of ks in the partitions of n with smallest part greater than 1 is generated by

$$\begin{aligned} S_{n,k}^{(2)} &= [q^n] \frac{(1-q)}{(q;q)_\infty} \cdot \frac{q^k}{1-q^k} \\ &= \sum_{i=1}^{\left\lfloor \frac{n}{k} \right\rfloor} \left(p(n - i \cdot k) - p(n - 1 - i \cdot k) \right). \end{aligned}$$

A Table of the $S_{n,k}^{(2)}$ $(n, k \geq 2)$

1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
4	2	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0
3	2	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0
8	3	3	1	1	0	1	0	0	0	0	0	0	0	0	0	0
7	6	2	2	1	1	0	1	0	0	0	0	0	0	0	0	0
15	6	5	3	2	1	1	0	1	0	0	0	0	0	0	0	0
15	10	5	4	2	2	1	1	0	1	0	0	0	0	0	0	0
27	14	10	5	5	2	2	1	1	0	1	0	0	0	0	0	0
29	18	10	8	4	4	2	2	1	1	0	1	0	0	0	0	0
48	24	17	10	8	5	4	2	2	1	1	0	1	0	0	0	0

A Table of the $S_{n,k}^{(2)}$ Inverse Matrix ($n, k \geq 2$)

1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	0	0	0	0
−2	0	1	0	0	0	0	0	0	0	0	0	0	0
−1	−1	0	1	0	0	0	0	0	0	0	0	0	0
−2	−2	−1	0	1	0	0	0	0	0	0	0	0	0
0	−1	−1	−1	0	1	0	0	0	0	0	0	0	0
1	0	−2	−1	−1	0	1	0	0	0	0	0	0	0
1	−1	0	−1	−1	−1	0	1	0	0	0	0	0	0
1	2	0	−1	−1	−1	−1	0	1	0	0	0	0	0
1	1	1	0	0	−1	−1	−1	0	1	0	0	0	0
4	2	1	1	−1	0	−1	−1	−1	0	1	0	0	0
1	1	1	1	1	0	0	−1	−1	−1	0	1	0	0
−1	2	2	2	1	0	0	0	−1	−1	−1	0	1	0
1	0	1	0	1	1	1	0	0	−1	−1	−1	0	1

Proof of Theorem Theorem 1

We can expand the series

$$\frac{1-q}{(q;q)_\infty}\sum_{k=1}^{\infty}\frac{\phi(k)q^k}{1-q^k}=\frac{q}{(q;q)_\infty}+\sum_{n=2}^{\infty}\sum_{k=2}^{\infty}\phi(k)S_{n,k}^{(2)}q^n.$$

On the other hand, considering the well-known Lambert series generating function [HardyWright, Theorem 309]

$$\sum_{k=1}^{\infty}\frac{\phi(k)q^k}{1-q^k}=\frac{q}{(1-q)^2},$$

we obtain

$$\begin{aligned}\frac{1-q}{(q;q)_\infty}\sum_{k=1}^{\infty}\frac{\phi(k)q^k}{1-q^k}&=\frac{q}{(q;q)_\infty}+\frac{q}{1-q}\cdot\frac{q}{(q;q)_\infty}\\&=\frac{q}{(q;q)_\infty}+\sum_{n=1}^{\infty}S_{n,1}^{(1)}q^{n+1}.\end{aligned}$$

Proof of Theorem Theorem 1 (cont'd)

Thus, we deduce the identity

$$S_{n,1}^{(1)} = \sum_{k=2}^{n+1} \phi(k) S_{n+1,k}^{(2)},$$

for any $n \geq 1$, which completes the proof of the theorem.

New Formulas for $p(n)$

A New Formula for the Partition Function

Theorem

For $n \geq 0$*,*

$$p(n) = \sum_{k=3}^{n+3} P_{\phi}(k) S_{n+3,k}^{(3)},$$

where $P_{\phi}(k) := \phi(k)/2$*.*

The Möbius Function $\mu(n)$

Definition

The *Möbius function*, $\mu(n)$, is the multiplicative arithmetic function defined by

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1; \\ (-1)^{\nu(n)}, & \text{if } a_1 = a_2 = \cdots = a_{\nu(n)} = 1; \\ 0, \text{ otherwise,} & \end{cases}$$

when the factorization of n into distinct prime factors is given by $n = p_1^{a_1} p_2^{a_2} \cdots p_{\nu(n)}^{a_{\nu(n)}}$ and where $\nu(n)$ denotes the number of distinct prime factors dividing n.

The Lambert series generating function for $\mu(n)$ is given by [HardyWright, Theorem 308]

$$\sum_{n \geq 1} \frac{\mu(n) q^n}{1 - q^n} = q.$$

A Related Formula for $p(n)$

Corollary (Expansion by the Möbius Function)

For all $n \geq 1$, we have the identity that

$$p(n) = -\sum_{j=0}^{n} \sum_{k=0}^{j} \mu(k+3) S_{j+3,k+3}^{(3)}.$$

Wrapping Up: Other Generalized Formulas for $p(n)$ Involving the Möbius Function

Since for $\alpha \geq 2$ prime we can expand the Lambert series for $\mu(\alpha n)$ as

$$\sum_{n \geq 1} \frac{\mu(\alpha n) q^n}{1-q^n} = -\sum_{n \geq 0} q^{\alpha^n},$$

we similarly can prove the following two special case identities providing even further new recurrence relations for $p(n)$:

$$\sum_{k=1}^{n+1} S_{n+1,k}^{(1)} \mu(\alpha k) = -\sum_{k \geqslant 0} p(n+1-\alpha^k)$$

$$\sum_{k=2}^{n+2} S_{n+2,k}^{(2)} \mu(\alpha k) = \sum_{k \geqslant 0} \left(p(n+1-\alpha^k) - p(n+2-\alpha^{k+1})\right).$$

Concluding Remarks

Summary

- We have connected Euler's totient function with a new decomposition of Stanley's theorem.
- Along the way, we have derived a new formula for Euler's partition function $p(n)$ expanded in terms of our additive restricted partition functions $S_{n,k}^{(r)}$ when $r = 3$ and duplicated the proof of this result to obtain new analogous formulations for $p(n)$ involving the Möbius function.
- We have defined and computed several generalized forms of both identities for $p(n)$ involving the restricted partition functions and $\phi(n)$ and $\mu(n)$, respectively.

References

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The End.
Thank you for listening.
Questions or comments?