Jacobi-type continued fractions for generating functions of combinatorial sequences

Computational aspects and combinatorial properties of J-fractions motivated by examples from recently published works.

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Talk Overview

Introduction to J-Fractions

- Continued fractions which converge to (or can be treated formally as) an ordinary generating function (OGF) for a sequence.
- Most generally we can expand these J-fractions formally using an adaptation of Flajolet's notation:

$$J_\infty(z) = \cfrac{1}{1 - c_1 z - \cfrac{\mathrm{ab}_2\, z^2}{1 - c_2 z - \cfrac{\mathrm{ab}_3\, z^2}{\cdots}}}.$$

- This approach allows us to obtain new properties of combinatorial sequences whose OGF does not converge for any $z \neq 0$ by considering a specially formed sequence of rational functions which enumerate the sequence of interest up to any desired accuracy.
- We will enumerate the flavor of some characteristic formal expansions which arise in special cases next.

An OGF for the Stirling numbers of the first kind

$$\sum_{n,k\geq 0} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix} z^n w^k = \cfrac{1}{1 - wz - \cfrac{wz^2}{1-(w+1)z-\cfrac{2wz^2}{\cdots}}}$$

An OGF for single factorial function

$$\sum_{n,k\geq 0} n! z^n = \frac{1}{1 - z - \frac{1^2 z^2}{1-3z-\frac{2^2 z^2}{\cdots}}}$$

An OGF for for the rising factorial polynomials

$$\sum_{n,k\geq 0} (r)_n z^n = \cfrac{1}{1 - rz - \cfrac{1rz^2}{1-(r+2)z-\cfrac{2(r+1)z^2}{\cdots}}}$$

Convergents to the Infinite J-Fractions

- The convergents to these infinite continued fraction expansions are always *rational functions* of the OGF series variable z – and, morover, these rational h^{th} convergent functions are $2h$-order accurate in enumerating the correct prescribed sequence of terms.
- The h^{th} convergents are defined by $\mathrm{Conv}_h(z) := P_h(z)/Q_h(z)$ where the numerator and denominator component sequences are each polynomials in z satisfying:

$$
\begin{aligned}
P_h(z) &= (1 - c_h z) P_{h-1}(z) - \mathrm{ab}_h\, z^2 P_{h-2}(z) + [h = 1]_\delta \\
Q_h(z) &= (1 - c_h z) Q_{h-1}(z) - \mathrm{ab}_h\, z^2 Q_{h-2}(z) \\
&\qquad + (1 - c_1 z)\, [h = 1]_\delta + [h = 0]_\delta\,.
\end{aligned}
$$

Congruences for Convergent Functions

- Let $M_h := \mathrm{ab}_2\,\mathrm{ab}_3 \cdots \mathrm{ab}_{h+1}$.
- Then
$$J_\infty(z) \equiv \mathrm{Conv}_h(z) \pmod{M_h}.$$
- This provides us with some useful new congruence properties satisfied by the convergent functions of special combinatorial sequences, especially if we have that (say) $h | M_h$, or something similar.

A Summary of Topics in This Talk

- The speaker's work on enumerating these sequence types through the method of J-fraction approximations and convergent-based order-h accurate OGFs has been published over the last few years in the *Journal of Integer Sequences*, the *Journal of Number Theory*, the *Ramanujan Journal*, and most recently (in 2018) in *INTEGERS*.
- We identify a plethora of "nice" special case examples which motivate our exploration into further properties of these J-fraction forms.
- The speaker's discovery of new continued fractions for these sequence types has been mostly computationally motivated for parameterized symbolic sequence forms (from which the resulting expansions inherit a richer, and easier to empirically describe, structure). The corresponding correctness and/or convergence properties of the associated J-fractions require additional work and proof methods.

Generalized Factorial Functions

- We considered generalized factorial functions of the form

$$p_n(\alpha, R) = R(R+\alpha)(R+2\alpha)\cdots(R+(n-1)\alpha)\,[n \geq 1]_\delta + [n=0]_\delta\,.$$

- This embodies many specialized generalizations of multiple and integer-valued factorial function sequences.
- Our J-fractions in this case correspond to the parameter sequences: $c_n := R + 2n\alpha$ and $\mathrm{ab}_n := n(R+n\alpha)$.
- The denominator sequences, $Q_h(z)$, correspond to confluent hypergeometric functions (U).

Generalized Factorial Functions (Cont.)

- Examples of the new results obtained through these methods include:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} \equiv \frac{2^n}{4} [n \geq 2]_\delta + [n = 1]_\delta \pmod{2}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} \equiv \frac{3 \cdot 2^n}{16} (n-1) [n \geq 3]_\delta + [n = 2]_\delta \pmod{2}$$

$$\begin{bmatrix} n \\ 3 \end{bmatrix} \equiv 2^{n-7} (9n - 20)(n-1) [n \geq 4]_\delta + [n = 3]_\delta \pmod{2}$$

$$\begin{bmatrix} n \\ 4 \end{bmatrix} \equiv 2^{n-9} (3n - 10)(3n-7)(n-1) [n \geq 5]_\delta + [n = 4]_\delta \pmod{2}.$$

Generalized Factorial Functions (Cont.)

- Examples of the new results obtained through these methods include:

$$\begin{bmatrix} n \\ 1 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{36}\left(9-5j\sqrt{3}\right)\times\left(3+j\sqrt{3}\right)^n [n\geq 2]_\delta + [n=1]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{216}\left((44n-41)-(25n-24)\cdot j\sqrt{3}\right)\times\left(3+j\sqrt{3}\right)^n [n\geq 3]_\delta + [n=2]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 3 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{15552}\left((1299n^2-3837n+2412)-(745n^2-2217n+1418)\cdot j\sqrt{3}\right)\times$$
$$\times\left(3+j\sqrt{3}\right)^n [n\geq 4]_\delta + [n=3]_\delta \pmod{3}$$

$$\begin{bmatrix} n \\ 4 \end{bmatrix} \equiv \sum_{j=\pm 1} \tfrac{1}{179936}\Big((6409n^3-383778n^2+70901n-37092)$$
$$-(3690n^3-22374n^2+41088n-21708)\cdot j\sqrt{3}\Big)\times$$
$$\times\left(3+j\sqrt{3}\right)^n [n\geq 5]_\delta + [n=4]_\delta \pmod{3}.$$

Lambert Series Generating Functions

- The generalized sum-of-divisors functions are defined as $\sigma_\alpha(n) = \sum_{d|n} d^\alpha$ for $\alpha \in \mathbb{C}$.
- The divisor function is the special case where $d(n) \equiv \sigma_0(n)$.
- The divisor function has the Lambert series generating function

$$\sum_{n\geq 0} \frac{q^n}{1-q^n} = \sum_{m\geq 1} d(m) q^m.$$

More generally,

$$\sum_{n\geq 0} \frac{n^\alpha q^n}{1-q^n} = \sum_{m\geq 1} \sigma_\alpha(m) q^m.$$

Lambert Series Generating Functions (Cont.)

- The idea here is to "generate a generating function" with our J-fraction expansions by defining that

$$J_\infty(z) = \sum_{n\geq 1} \frac{q^n z^n}{1-q^n}.$$

  Then by differentiation with respect to z, we can generate Lambert series generating functions for the $\sigma_\alpha(n)$ whenever $\alpha \in \mathbb{N}$.
- We look at a special case of the q-Pochhammer symbol ratios: $[z^n]J_\infty(z) := (a;q)_n/(b;q)_n$.
- Plug in $(a,b) := (q,q^2)$ to obtain the terms of interest.

Lambert Series Generating Functions (Cont.)

- The resulting J-fraction expansions are then given by:

$$\mathrm{ab}_n := \frac{q^{2n-4}(1-bq^{n-3})(1-aq^{n-2})(a-bq^{n-2})(1-q^{n-1})}{(1-bq^{2n-5})(1-bq^{2n-4})(1-bq^{2n-3})}, \ n \geq 2.$$

  and

$$c_n := \begin{cases} \frac{q^{n-2}(q+abq^{2n-3}+a(1-q^{n-1}-q^n)+b(-1-q-q^n))}{(1-bq^{2n-4})(1-bq^{2n-2})}, & n \geq 2; \\ \frac{a-1}{b-1}, & n = 1; \\ 0, & \text{otherwise.} \end{cases}$$

- The proof techniques utilitzed in this article, including proofs of convergence, are more technical perhaps than in the other examples.

Lambert Series Generating Functions (Cont.)

A table of some other related q-series expansions which we can generate with J-fractions of this type include those experimentally obtained in the following table:

$[z^n]J_\infty(z)$	c_1	c_h for $h \geq 2$
$(a;q)_n$	$1-a$	$q^{h-1} - aq^{h-2}\left(q^h + q^{h-1} - 1\right)$
$\frac{1}{(q;q)_n}$	$\frac{1}{1-q}$	$\frac{q^{h-1}\left(q^{h-1}\left[{h-1 \atop 1}\right]_q - \left[{h-2 \atop 1}\right]_q\right)}{\left[{2h-3 \atop 1}\right]_q (q^{2h-1}-1)}$
$\frac{q^{\binom{n}{2}}}{(q;q)_n}$	$\frac{1}{1-q}$	$\frac{q^{h-2}(1-q)\left(q^h\left[{h-2 \atop 1}\right]_q - \left[{h-1 \atop 1}\right]_q\right)}{(1-q^{2h-3})(1-q^{2h-1})}$
$(zq^{-n};q)_n$	$\frac{q-z}{q}$	$\frac{q^h - z - qz + q^h z}{q^{2h-1}}$
$\frac{1}{(zq^{-n};q)_n}$	$\frac{q}{q-z}$	$\frac{q^{h-1}\left(q^{2h-2} + z + q^{h-1}z - q^h z\right)}{(q^{2h-3}-z)(q^{2h-1}-z)}$
$\frac{(a;q)_n}{(b;q)_n}$	$\frac{1-a}{1-b}$	$\frac{q^{i-2}\left(q + abq^{2i-3} + a\left(1-q^{i-1}-q^i\right) + b\left(-1-q+q^i\right)\right)}{(1-bq^{2i-4})(1-bq^{2i-2})}$

Lambert Series Generating Functions (Cont.)

A table of some other related q-series expansions which we can generate with J-fractions of this type include those experimentally obtained in the following table:

$[z^n]J_\infty(z)$	ab_h
$(a;q)_n$	$aq^{2h-4}(aq^{h-2}-1)(q^{h-1}-1)$
$\frac{1}{(q;q)_n}$	$-\frac{q^{3h-5}}{(q^{2h-3}-1)^2(1+q^{h-2}+q^{h-1}+q^{2h-3})}$
$\frac{q^{\binom{n}{2}}}{(q;q)_n}$	$\begin{cases} -\frac{1}{(1-q)^2(1+q)}, & \text{if } h=2; \\ -\frac{q^{\binom{h}{2}}}{(1-q^h)^2(1+q^{h-2}(1+q)+q^{2h-3})}, & \text{if } h\geq 3 \end{cases}$
$(zq^{-n};q)_n$	$\frac{(q^{h-1}-1)(q^{h-1}-z)\cdot z}{q^{4h-5}}$
$\frac{1}{(zq^{-n};q)_n}$	$\begin{bmatrix} h-1 \\ 1 \end{bmatrix}_q \cdot \frac{q^{3h-4}(1-q)(q^{h-2}-z)\cdot z}{(q^{2h-4}-z)(q^{2h-3}-z)^2(q^{2h-2}-z)}$
$\frac{(a;q)_n}{(b;q)_n}$	$\frac{q^{2i-4}(1-bq^{i-3})(1-aq^{i-2})(a-bq^{i-2})(1-q^{i-1})}{(1-bq^{2i-5})(1-bq^{2i-4})^2(1-bq^{2i-3})}$

Square Series Generating Functions

- We seek to generate $J_\infty(z) := \sum_{n\geq 0} q^{n^2} z^n$, for any fixed $0 < |q| < 1$.
- The J-fraction expansions turn out to be defined in this case by:

$$\mathrm{ab}_n := q^{6n-10}(q^{2n-2} - 1), \ \ n \geq 2,$$

and

$$c_n := \begin{cases} q^{2n-3}(q^{2n} + q^{2n-2} - 1), & n \geq 2; \\ q, & n = 1; \\ 1, & n = 0. \end{cases}$$

Square Series Generating Functions (Cont.)

A sampling of the new results obtained with these expansions includes:

- $$\sum_{n\geq 0} q^{n^2} z^n = \sum_{i=1}^{\infty} \frac{(-1)^{i-1} q^{(3i-4)(i-1)} (q^2;q^2)_{i-1} z^{2i-2}}{\sum_{0\leq j\leq n<2i} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2} \begin{bmatrix} i-1 \\ n-j \end{bmatrix}_{q^2} q^{2j} (-q^{2i-3}z)^n}.$$

- $$r_p(n) = [q^n] \left(1 + \sum_{i\geq 1} \frac{2q(-1)^{i-1} q^{3i(i-1)} (q^2;q^2)_{i-1}}{\sum_{0\leq n<2i} \left(\sum_{0\leq j\leq n} \begin{bmatrix} i \\ j \end{bmatrix}_{q^2} \begin{bmatrix} i-1 \\ n-j \end{bmatrix}_{q^2} q^{2j}\right) (-q^{2i-1})^n}\right)^p$$

- $$f(a,b) = 1 + \sum_{c\in\{a,b\}} \sum_{i\geq 1} \frac{c(-1)^{i-1} (ab)^{(3i-2)(i-1)} (ab;ab)_{i-1} c^{2i-2}}{\sum_{0\leq j\leq n<2i} \begin{bmatrix} i \\ j \end{bmatrix}_{ab} \begin{bmatrix} i-1 \\ n-j \end{bmatrix}_{ab} (ab)^j (-(ab)^{i-1}c)^n}.$$

Symbolic Forms of Binomial Coefficients

- Perform the same J-fraction expansions for the sequence $\binom{x+n}{n}$:

$$\mathrm{ab}_i := \begin{cases} -\frac{1}{4(2i-3)^2}(x-i+2)(x+i-1) & \text{if } i \geq 3 \\ -\frac{1}{2}x(x+1) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$c_i := -\frac{1}{(2i-1)(2i-3)}(1+2(i-2)i-x).$$

Symbolic Forms of Binomial Coefficients (Cont.)

- Perform the same J-fraction expansions for the sequence $\binom{x}{n}$:

$$\mathrm{ab}_i := \begin{cases} -\frac{1}{4(2i-3)^2}(x-i+2)(x+i-1) & \text{if } i \geq 3 \\ -\frac{1}{2}x(x+1) & \text{if } i = 2 \\ 0 & \text{otherwise.} \end{cases}$$

$$c_i := -\frac{1}{(2i-1)(2i-3)}\left(x+2(i-1)^2\right).$$

- Notice how similar these are to the first variation of the sequence. In fact, the sequences ab_n are the same!

Symbolic Forms of Binomial Coefficients (Cont.)

Some examples of the results obtained in the recent (2018) INTEGERS article:

- Verified (a known?) identity:

$$\binom{x+n}{n} = \sum_{i=1}^{n} \binom{x+n}{i} \binom{x+n-i}{n-i} \binom{n}{i} \binom{2n-1}{i}^{-1} (-1)^{i+1}, n > 0.$$

- Let

$$\mathcal{M}_h := \left\{ x \in \mathbb{Z} : \frac{1}{2h} \binom{x+h-1}{h-1} \binom{x}{h-1} \binom{2h-3}{h-2}^{-2} \in \mathbb{Z} \right\}.$$

Symbolic Forms of Binomial Coefficients (Cont.)

- The first few particular special cases of these restricted index sets include

$$\mathcal{M}_2 = \left\{x : \frac{x(x+1)}{4} \in \mathbb{Z}\right\} = \{x : x \equiv 0, 3 \bmod 4\}$$

$$\mathcal{M}_3 = \left\{x : \frac{(x-1)x(x+1)(x+2)}{216} \in \mathbb{Z}\right\}$$
$$= \{x : x \equiv 0, 1, 7, 10, 16, 19, 25, 26 \bmod 27\}$$

$$\mathcal{M}_4 = \left\{x : \frac{(x-1)(x-1)x(x+1)(x+2)(x+3)}{28800} \in \mathbb{Z}\right\}$$
$$= \{x : x \equiv 0, 1, 2, 7, 12, 17, 22, 23, 24 \bmod 25\}$$
$$\cap \{x : x \equiv 0, 1, 2, 13, 14, 17, 18, 29, 30, 31 \bmod 32\}$$

$$\mathcal{M}_5 = \left\{x : \frac{(x-2)(x-1)(x-1)x(x+1)(x+2)(x+3)(x+4)}{7056000} \in \mathbb{Z}\right\}$$
$$= \{x : x \equiv 0, 1, 2, 3, 21, 22, 23, 24 \bmod 25\}$$
$$\cap \{x : x \equiv 0, 1, 2, 3, 10, 17, 24, 31, 38, 45, 46, 47, 48 \bmod 49\}.$$

Symbolic Forms of Binomial Coefficients (Cont.)

$$\binom{x+n}{n} \equiv \frac{2(x+2)}{3}\binom{x+n-1}{n-1} - \frac{(x+1)(x+2)}{6}\binom{x+n-2}{n-2} \pmod{2}, \quad \text{for all } x \in \mathcal{M}_2$$

$$\begin{aligned}\binom{x+n}{n} &\equiv \frac{3(x+3)}{5}\binom{x+n-1}{n-1} - \frac{3(x+2)(x+3)}{20}\binom{x+n-2}{n-2} \\ &\quad + \frac{(x+1)(x+2)(x+3)}{60}\binom{x+n-3}{n-3} \pmod{3}, \quad \text{for all } x \in \mathcal{M}_3\end{aligned}$$

$$\begin{aligned}\binom{x+n}{n} &\equiv \frac{4(x+4)}{7}\binom{x+n-1}{n-1} - \frac{(x+3)(x+4)}{7}\binom{x+n-2}{n-2} \\ &\quad + \frac{2(x+2)(x+3)(x+4)}{105}\binom{x+n-3}{n-3} \\ &\quad - \frac{(x+1)(x+2)(x+3)(x+4)}{840}\binom{x+n-4}{n-4} \pmod{4}, \quad \text{for all } x \in \mathcal{M}_4\end{aligned}$$

$$\begin{aligned}\binom{x+n}{n} &\equiv \frac{5(x+5)}{9}\binom{x+n-1}{n-1} - \frac{5(x+4)(x+5)}{56}\binom{x+n-2}{n-2} \\ &\quad + \frac{5(x+3)(x+4)(x+5)}{252}\binom{x+n-3}{n-3} \\ &\quad - \frac{5(x+2)(x+3)(x+4)(x+5)}{3024}\binom{x+n-4}{n-4} \\ &\quad + \frac{(x+1)(x+2)(x+3)(x+4)(x+5)}{15120}\binom{x+n-5}{n-5} \pmod{5}, \quad \text{for all } x \in \mathcal{M}_5.\end{aligned}$$

Symbolic Forms of Binomial Coefficients (Cont.)

$$\binom{x}{n} \equiv \frac{2(x-1)}{3}\binom{x}{n-1} - \frac{x(x-1)}{6}\binom{x}{n-2} + \frac{(n-2)(n-3)}{6}\binom{x+2}{n}[n \leq 1]_\delta \qquad \pmod{2}, \quad \text{for all } x \in \mathcal{M}_2$$

$$\begin{aligned}\binom{x}{n} \equiv \frac{3(x-2)}{5}\binom{x}{n-1} &- \frac{3(x-1)(x-2)}{20}\binom{x}{n-2} + \frac{x(x-1)(x-2)}{60}\binom{x}{n-3} \\ &- \frac{(n-3)(n-4)(n-5)}{60}\binom{x+3}{n}[n \leq 2]_\delta \qquad \pmod{3}, \quad \text{for all } x \in \mathcal{M}_3\end{aligned}$$

$$\begin{aligned}\binom{x}{n} \equiv \frac{4(x-3)}{7}\binom{x}{n-1} &- \frac{(x-2)(x-3)}{7}\binom{x}{n-2} + \frac{2(x-1)(x-2)(x-3)}{105}\binom{x}{n-3} \\ &- \frac{x(x-1)(x-2)(x-3)}{840}\binom{x}{n-4} \\ &+ \frac{(n-4)(n-5)(n-6)(n-7)}{840}\binom{x+4}{n}[n \leq 3]_\delta \qquad \pmod{4}, \quad \text{for all } x \in \mathcal{M}_4\end{aligned}$$

$$\begin{aligned}\binom{x}{n} \equiv \frac{5(x-4)}{9}\binom{x}{n-1} &- \frac{5(x-3)(x-4)}{36}\binom{x}{n-2} + \frac{5(x-2)(x-3)(x-4)}{252}\binom{x}{n-3} \\ &- \frac{5(x-1)(x-2)(x-3)(x-4)}{3024}\binom{x}{n-4} \\ &- \frac{x(x-1)(x-2)(x-3)(x-4)}{15120}\binom{x}{n-5} \\ &- \frac{(n-5)(n-6)(n-7)(n-8)(n-9)}{15120}\binom{x+5}{n}[n \leq 4]_\delta \qquad \pmod{5}, \quad \text{for all } x \in \mathcal{M}_5.\end{aligned}$$

Mathematica Package for Experimental Mathematics with These J-Fraction Expansions

```
Clear[c, ab, Phz, Qhz, Conv]
Phz[h_, z_] := Phz[h, z] =
     If[(h <= 1), KroneckerDelta[h == 1, True],
                  (1 - c[h]*z)*Phz[h - 1, z] - ab[h]*(z^2)*Phz[h - 2, z]]
Qhz[h_, z_] := Qhz[h, z] =
     If[(h <= 1), KroneckerDelta[h == 0, True] +
                  KroneckerDelta[h == 1, True] * (1 - c[1] * z),
                  (1 - c[h]*z)*Qhz[h - 1, z] - ab[h]*(z^2)*Qhz[h - 2, z]]
Conv[h_, z_] := FS[Phz[h, z]/Qhz[h, z]]

getSubsequenceValues[upper_, fnq_] := Module[{eqns, vars, cfsols},
     eqns = Table[SeriesCoefficient[Conv[upper, z], {z, 0, n}] ==
                  FunctionExpand[fnq[n]], {n, 1, upper}];
     vars = Flatten[Table[{c[i], ab[i + 2]}, {i, 0, upper}]];
     cfsols = Solve[eqns, vars][[1]] // Expand // FullSimplify;
     Return[Map[#1[cfsols]&, {FullSimplify, Factor, Apart}]];
];

Table[{Function->fnq[n], getSubSequenceValues[7, fnq]},
      {fnq, fnqFunctions}] // TableForm
```

The End

Questions?

Comments?

Feedback?

Thank you for attending the talk!