New Connections Between Partitions and Multiplicative Functions

Maxie D. Schmidt

Georgia Institute of Technology
School of Mathematics

maxieds@gmail.com

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Connections between partitions and special classical functions in multiplicative number theory are rare

Recently, several publications by Merca and Schmidt have made connections between multiplicative functions with special Lambert series generating functions, restricted partitions, and the partition function $p(n)$ more explicit.

We will briefly summarize the results in the new 2017–2018 papers by Merca and Schmidt.

We then present a generalization of these so-termed *Lambert series factorization theorems* into the form of factorization theorems for the generating functions of other special divisor and gcd-sums.

(Joint work with Hamed Mousavi at GA Tech)
Recent New Connections Between the Additive and Multiplicative
In 2018 Merca and Schmidt studied connections between restricted partition functions and Euler’s totient function $\phi(n)$.

Let $S_{n,k}^{(r)}$ denote the number of $k$’s in the partitions of $n$ with smallest part at least $r$:

$$S_{n,k}^{(r)} = [q^n] \frac{q^k}{1 - q^k} \frac{(1 - q)(1 - q^2) \cdots (1 - q^{r-1})}{(q; q)_\infty}.$$ 

Then we have proved that the number of 1’s in all partitions of $n$ (cf. Stanley’s theorem)

$$S_{n,1}^{(1)} = \sum_{k=2}^{n+1} \phi(k) S_{n+1,k}^{(2)},$$

and that

$$p(n) = \sum_{k=3}^{n+3} \frac{\phi(k)}{2} S_{n+3,k}^{(3)}.$$ 

Related results connecting $p(n)$, $S_{n,k}^{(r)}$, and the Möbius function $\mu(n)$ are proved in another article published in *Ramanujan J.* (2018).
Recent Work on Lambert Series Factorization Theorems

We have considered Lambert series factorization theorems of the following forms,

\[
\sum_{n \geq 1} \frac{a_n q^n}{1 - q^n} = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k} \cdot a_k \cdot q^n \quad (1)
\]

\[
\sum_{n \geq 1} \frac{a_n q^{\alpha n + \beta}}{1 - q^{\alpha n + \beta}} = \frac{1}{C(q)} \sum_{n \geq 1} \sum_{k=1}^{n} \tilde{s}_{n,k}(\alpha, \beta) \cdot a_k \cdot q^n, \quad \alpha \in \mathbb{Z}^+, \quad 1 \leq \beta < \alpha
\]

\[
\sum_{n \geq 1} \frac{(f \ast g)(n)q^n}{1 - q^n} = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} \bar{s}_{n,k}(g) \cdot f(k) \cdot q^n.
\]

where \( s_{n,k} := [q^n](q; q)_\infty \frac{q^k}{1 - q^k} = s_o(n, k) - s_e(n, k) \) is the difference of the number of \( k \)'s in all partitions of \( n \) into an odd (even) number of distinct parts.
We proved that

\[ s_{n,k}^{(-1)} = \sum_{d|n} p(d - k) \mu(n/d), \]

\[ a_n = \sum_{k=1}^{n} \sum_{d|n} p(d - k) \mu(n/d) \times \sum_{j \geq 0} (-1)^{[\frac{j}{2}]} \left( \sum_{d|k-\frac{1}{2} \left[ \frac{j}{2} \right] \left[ \frac{3j+1}{2} \right]} a_d \right), \]

Notable special cases are given by the known divisor sums for the functions

\[ \left( a_n, \sum_{d|n} a_d \right) = (\mu(n), \delta_{n,1}), (\phi(n), n), (n^\alpha, \sigma_\alpha(n)), (\lambda(n), \chi_{sq}(n)), (\Lambda(n), \log n)(|\mu(n)|, 2^{\omega(n)}), (J_t(n), n^t). \]
Let $s_{\alpha,\beta}(n, k)$ denote the number of all $(\alpha n + \beta)$’s in all partitions of $n$ and $s_{e/o}(n)$ denote the number of partitions of $n$ into an even (odd) number of parts. Other results we proved include the following:

- $\sum_{k=1}^{n} \left( \sum_{d \mid k} a_d \right) (s_e(n - k) - s_o(n - k)) = \sum_{k=1}^{n} (-1)^{n-1} a_k \cdot s_{2,-1}(n, k)$,

- $r_2(n) = \sum_{k=0}^{n} \sum_{j=1}^{k} 4(-1)^{j+1} (s_e(n - k) - s_o(n - k)) s_{2,-1}(n, k)$,

- $\omega(n) = \log_2 \left[ \sum_{k=1}^{n} \sum_{j=1}^{k} \left( \sum_{d \mid k} \sum_{i=1}^{d} p(d - ji) \right) s_{n,k} \cdot |\mu(j)| \right]$. 

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More generally, we can consider analogous factorization theorems of the form

\[ \sum_{n \geq 1} \left( \sum_{k \in A_n \subseteq [1,n]} f(k) \right) q^n = \frac{1}{(q; q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} s_{n,k}(A) f(k) \cdot q^n. \]

We consider two primary variants of the factorization theorems above, respectively, denoted by *Type I* and *Type II* sums below.

Motivation: The last sum is a composition of types in the first:

\[ \sum_{d \leq x} f(d) = \sum_{d=1}^{x} f(d) + \sum_{d \mid x \atop (d,x)=1} f(d) + \sum_{d=1}^{x} f(d) \]

\[ = \sum_{m \mid x} \sum_{k=1}^{m} f\left(\frac{kx}{m}\right). \]
Notation

- We define the interleaved *pentagonal numbers* by
  \[
  G_j := \frac{1}{2} \left\lfloor \frac{j}{2} \right\rfloor \left\lfloor \frac{3j + 1}{2} \right\rfloor \rightarrow \{0, 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, \ldots\},
  \]

- We let \([n = k]_\delta \equiv \delta_{n,k}\) denote *Iverson’s convention*,

- And define the *principal Dirichlet character* (modulo \(k\)) explicitly for all \(n \geq 1\) by
  \[
  \chi_{1,k}(n) = \begin{cases} 
  1, & \text{if } (n, k) = 1; \\
  0, & \text{if } (n, k) > 1.
  \end{cases}
  \]
Let \( T_f(n) := \sum_{d:(n,d)=1} f(d) \) for any prescribed arithmetic function \( f \).

Then we consider factorizations of the generating functions of these sequences in the form of

\[
T_f(n) = [q^n] \left( \frac{1}{(q; q)_\infty} \sum_{n \geq 2} \sum_{k=1}^{n} t_{n,k} f(k) \cdot q^n + f(1) \cdot q \right).
\]

We can prove corresponding analogous results and formulas for these sums as we did in the Lambert series factorization theorem cases.
Type I Sum Results

- Relation to Lambert series factorizations:
  \[
  t_{n,k} = \begin{cases} 
  s_{n,k}, & k = 1; \\
  \sum_{d|k} s_{n+1-k+d,d} \cdot \mu(d), & k > 1, 
  \end{cases}
  \]

- \[
  \sum_{d=1}^{n} f(d) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} p(n-j) t_{j-1,k} f(k) + f(1) \lfloor n = 1 \rfloor, 
  \]

- \[
  f(n) = \sum_{k=1}^{n} t_{n,k}^{(-1)} \left( \sum_{j \geq 0} (-1)^{\left\lfloor \frac{j}{2} \right\rfloor} T_f(k + 1 - G_j) - [q^k](q; q)_\infty \cdot f(1) \right). 
  \]

- Relation to matrix products and notation for \( t_{n,k}^{(-1)} \)
Applications of Our Type I Sum Results

- $\sum_{n < x} f(d) =$
  \[ \sum_{d \mid x} s_{x-1,n} \left[ \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} (-1)^{\left\lfloor \frac{i}{2} \right\rfloor} p(n-j) \chi_{1,k}(j-k-G_i)f\left(\frac{xk}{n}\right) \right] , \]

- $\phi(n) = \sum_{j=0}^{n} \sum_{k=1}^{j-1} \sum_{i=0}^{j} p(n-j)(-1)^{\left\lfloor \frac{i}{2} \right\rfloor} \chi_{1,k}(j-k-G_i) + [n = 1]_\delta ,$

- $d(n) = \sum_{d \mid n} \sum_{k=1}^{d} t_{d,k}^{(-1)} \left( \sum_{j \geq 0} (-1)^{\left\lfloor \frac{j}{2} \right\rfloor} \varphi(k + 1 - G_j) - [q^k](q; q)_\infty \right) ,.$
Generalized Factorization Theorems (Type II Sums)

- Let $L_{f,g,k}(n) := \sum_{d|\text{gcd}(n,k)} f(d)g(n/d)$ for any prescribed arithmetic functions $f$ and $g$.
- Then we consider factorizations of the generating functions of these sequences in the form of

$$g(x) = [q^x] \left( \frac{1}{(q;q)_\infty} \sum_{n \geq 2} \sum_{k=1}^{\infty} u_{n,k}(f, w) \left[ \sum_{m=1}^{k} L_{f,g,m}(k) w^m \right] \cdot q^n \right),$$

for a non-zero indeterminate $w \in \mathbb{C}$. (Makes the matrices invertible)
- Other variants are of course possible.
- The goal here is to identify “nice” closed-form formulas and relations for the invertible $u_{n,k}$ and $u_{n,k}^{(-1)}$ matrices.
Type II Sum Results

\[ u_{n,k}^{(-1)}(f, w) = \sum_{m=1}^{n} \left( \sum_{d \mid (m,n)} f(d) p(n/d - k) \right) w^m, \]

\[ L_{f,g,m}(n) = \sum_{k=1}^{n} \sum_{d \mid (m,n)} f(d) p\left(\frac{n}{d} - k\right) \times \sum_{\substack{j \geq 0 \\ k > G_j}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} g(k - G_j), \]

(Special case: Ramanujan sums yield many new applications to special functions)

- The formulas for the ordinary matrices are significantly more complicated and difficult to evaluate!
Using discrete Fourier transforms of functions at gcd-arguments, we have been able to prove that

\[ \sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e \left( -\frac{rd}{k} \right) \mu \left( \frac{k}{d} \right) = \sum_{d|k} \varphi(d)f(d)(k/d)^2 g(k/d), \]

There are also approaches to evaluating the ordinary matrix entries, \( u_{n,k}(f,w) \), using suitably chosen orthogonal polynomial sequences.
The End

Questions?
Comments?
Feedback?

Thank you for attending!