New Connections Between Partitions and Multiplicative Functions

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Talk Overview

Overview

- Connections between partitions and special classical functions in multiplicative number theory are rare
- Recently, several publications by Merca and Schmidt have made connections between multiplicative functions with special Lambert series generating functions, restricted partitions, and the partition function $p(n)$ more explicit.
- We will briefly summarize the results in the new 2017–2018 papers by Merca and Schmidt.
- We then present a generalization of these so-termed *Lambert series factorization theorems* into the form of factorization theorems for the generating functions of other special divisor and gcd-sums. (Joint work with Hamed Mousavi at GA Tech)

Recent New Connections Between the Additive and Multiplicative

Results in an Upcoming AMM Article

- In 2018 Merca and Schmidt studied connections between restricted partition functions and Euler's totient function $\phi(n)$.
- Let $S_{n,k}^{(r)}$ denote the number of k's in the partitions of n with smallest part at least r:
$$S_{n,k}^{(r)} = [q^n]\frac{q^k}{1-q^k}\frac{(1-q)(1-q^2)\cdots(1-q^{r-1})}{(q;q)_\infty}.$$
- Then we have proved that the number of 1's in all partitions of n (cf. *Stanley's theorem*)
$$S_{n,1}^{(1)} = \sum_{k=2}^{n+1}\phi(k)S_{n+1,k}^{(2)},$$
and that
$$p(n) = \sum_{k=3}^{n+3}\frac{\phi(k)}{2}S_{n+3,k}^{(3)}.$$
- Related results connecting $p(n)$, $S_{n,k}^{(r)}$, and the Möbius function $\mu(n)$ are proved in another article published in *Ramanujan J.* (2018).

Recent Work on Lambert Series Factorization Theorems

We have considered Lambert series factorization theorems of the following forms,

$$\sum_{n\geq 1} \frac{a_n q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^{n} s_{n,k} \cdot a_k \cdot q^n \tag{1}$$

$$\sum_{n\geq 1} \frac{a_n q^{\alpha n+\beta}}{1-q^{\alpha n+\beta}} = \frac{1}{C(q)} \sum_{n\geq 1} \sum_{k=1}^{n} \widetilde{s}_{n,k}(\alpha,\beta) \cdot a_k \cdot q^n, \ \alpha \in \mathbb{Z}^+, 1 \leq \beta < \alpha$$

$$\sum_{n\geq 1} \frac{(f*g)(n) q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \sum_{k=1}^{n} \bar{s}_{n,k}(g) \cdot f(k) \cdot q^n.$$

where $s_{n,k} := [q^n](q;q)_\infty \frac{q^k}{1-q^k} = s_o(n,k) - s_e(n,k)$ is the difference of the number of k's in all partitions of n into an odd (even) number of distinct parts.

Lambert Series Factorization Theorems (Cont'd)

We proved that

- $s_{n,k}^{(-1)} = \sum_{d|n} p(d-k)\mu(n/d)$,
- $a_n = \sum_{k=1}^{n} \sum_{d|n} p(d-k)\mu(n/d) \times \sum_{j \geq 0} (-1)^{\left\lceil \frac{j}{2} \right\rceil} \left(\sum_{d | k - \frac{1}{2}\left\lceil \frac{j}{2} \right\rceil \left\lceil \frac{3j+1}{2} \right\rceil} a_d \right)$,

Notable special cases are given by the known divisor sums for the functions

$$\left(a_n, \sum_{d|n} a_d\right) = (\mu(n), \delta_{n,1}), (\phi(n), n), (n^\alpha, \sigma_\alpha(n)), (\lambda(n), \chi_{\mathrm{sq}}(n)),$$

$$(\Lambda(n), \log n)(|\mu(n)|, 2^{\omega(n)}), (J_t(n), n^t).$$

Lambert Series Factorization Theorems (Cont'd)

Let $s_{\alpha,\beta}(n,k)$ denote the number of all $(\alpha n+\beta)$'s in all partitions of n and $s_{e/o}(n)$ denote the number of partitions of n into an even (odd) number of parts. Other results we proved include the following:

- $\sum_{k=1}^{n}\left(\sum_{2d-1|k} a_d\right)(s_e(n-k)-s_o(n-k)) = \sum_{k=1}^{n}(-1)^{n-1}a_k\cdot s_{2,-1}(n,k)$,
- $r_2(n) = \sum_{k=0}^{n}\sum_{j=1}^{k} 4(-1)^{j+1}(s_e(n-k)-s_o(n-k))\, s_{2,-1}(n,k)$,
- $\omega(n) = \log_2\left[\sum_{k=1}^{n}\sum_{j=1}^{k}\left(\sum_{d|k}\sum_{i=1}^{d} p(d-ji)\right) s_{n,k}\cdot|\mu(j)|\right]$.

Generalized Factorization Theorems

Generalized Factorization Theorems

- More generally, we can consider analogous factorization theorems of the form

$$\sum_{n\geq 1}\left(\sum_{\substack{k\in\mathcal{A}_n \\ \mathcal{A}_n\subseteq[1,n]}} f(k)\right) q^n = \frac{1}{(q;q)_\infty}\sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k}(\mathcal{A}) f(k)\cdot q^n.$$

- We consider two primary variants of the factorization theorems above, respectively, denoted by *Type I* and *Type II* sums below.
- Motivation: The last sum is a composition of types in the first:

$$\sum_{d\leq x} f(d) = \sum_{\substack{d=1 \\ (d,x)=1}}^{x} f(d) + \sum_{d|x} f(d) + \sum_{\substack{d=1 \\ 1<(d,x)<x}}^{x} f(d)$$

$$= \sum_{m|x}\sum_{\substack{k=1 \\ (k,m)=1}}^{m} f\left(\frac{kx}{m}\right).$$

Notation

- We define the interleaved *pentagonal numbers* by

$$G_j := \frac{1}{2}\left\lceil\frac{j}{2}\right\rceil\left\lceil\frac{3j+1}{2}\right\rceil \longmapsto \{0,1,2,5,7,12,15,22,26,35,40,51,\ldots\},$$

- We let $[n=k]_\delta \equiv \delta_{n,k}$ denote *Iverson's convention*,
- And define the *principal Dirichlet character* (modulo k) explicitly for all $n \geq 1$ by

$$\chi_{1,k}(n) = \begin{cases} 1, & \text{if } (n,k)=1; \\ 0, & \text{if } (n,k)>1. \end{cases}$$

Generalized Factorization Theorems (Type I Sums)

- Let $T_f(n) := \sum_{d:(n,d)=1} f(d)$ for any prescribed arithmetic function f.
- Then we consider factorizations of the generating functions of these sequences in the form of

$$T_f(n) = [q^n] \left(\frac{1}{(q;q)_\infty} \sum_{n \geq 2} \sum_{k=1}^{n} t_{n,k} f(k) \cdot q^n + f(1) \cdot q \right).$$

- We can prove corresponding analogous results and formulas for these sums as we did in the Lambert series factorization theorem cases.

Type I Sum Results

- Relation to Lambert series factorizations:
$$t_{n,k} = \begin{cases} s_{n,k}, & k = 1; \\ \sum_{d|k} s_{n+1-k+d,d} \cdot \mu(d), & k > 1, \end{cases}$$
- $$\sum_{\substack{d=1 \\ (d,n)=1}}^{n} f(d) = \sum_{j=1}^{n} \sum_{k=1}^{j-1} p(n-j) t_{j-1,k} f(k) + f(1) [n=1]_{\delta},$$
- $f(n) =$
$$\sum_{k=1}^{n} t_{n,k}^{(-1)} \left(\sum_{\substack{j \geq 0 \\ k+1-G_j > 0}} (-1)^{\lceil \frac{j}{2} \rceil} T_f(k+1-G_j) - [q^k](q;q)_{\infty} \cdot f(1) \right).$$
- Relation to matrix products and notation for $t_{n,k}^{(-1)}$

Applications of Our Type I Sum Results

- $$\sum_{n<x} f(d) = \sum_{d|x} s_{x-1,n}\left[\sum_{j=0}^{n}\sum_{k=1}^{j-1}\sum_{\substack{i=0 \\ j-k-G_i\geq 1}}^{j} (-1)^{\left\lceil \frac{i}{2}\right\rceil} p(n-j)\chi_{1,k}(j-k-G_i) f\left(\frac{xk}{n}\right)\right],$$
- $$\phi(n) = \sum_{j=0}^{n}\sum_{k=1}^{j-1}\sum_{\substack{i=0 \\ j-k-G_i\geq 1}}^{j} p(n-j)(-1)^{\lceil i/2\rceil}\chi_{1,k}(j-k-G_i) + [n=1]_{\delta},$$
- $$d(n) = \sum_{d|n}\sum_{k=1}^{d} t_{d,k}^{(-1)}\left(\sum_{\substack{j\geq 0 \\ k+1-G_j>0}} (-1)^{\left\lceil \frac{j}{2}\right\rceil}\varphi(k+1-G_j) - [q^k](q;q)_{\infty}\right),$$

Generalized Factorization Theorems (Type II Sums)

- Let $L_{f,g,k}(n) := \sum_{d|(n,k)} f(d)g(n/d)$ for any prescribed arithmetic functions f and g.
- Then we consider factorizations of the generating functions of these sequences in the form of

$$g(x) = [q^x]\left(\frac{1}{(q;q)_\infty}\sum_{n\geq 2}\sum_{k=1}^{n} u_{n,k}(f,w)\left[\sum_{m=1}^{k} L_{f,g,m}(k)w^m\right]\cdot q^n\right),$$

  for a non-zero indeterminate $w \in \mathbb{C}$. (Makes the matrices invertible)
- Other variants are of course possible.
- The goal here is to identify "nice" closed-form formulas and relations for the invertible $u_{n,k}$ and $u_{n,k}^{(-1)}$ matrices.

Type II Sum Results

- $u_{n,k}^{(-1)}(f,w) = \sum_{m=1}^{n} \left(\sum_{d|(m,n)} f(d)p(n/d-k) \right) w^m,$
- $L_{f,g,m}(n) = \sum_{k=1}^{n} \sum_{d|(m,n)} f(d)p\left(\frac{n}{d}-k\right) \times \sum_{\substack{j\geq 0 \\ k>G_j}} (-1)^{\left\lceil \frac{j}{2} \right\rceil} g\left(k-G_j\right),$

  (Special case: Ramanujan sums yield many new applications to special functions)
- The formulas for the ordinary matrices are significantly more complicated and difficult to evaluate!

Other Results on Type II Sums

- Using discrete Fourier transforms of functions at gcd-arguments, we have been able to prove that

$$\sum_{d|k} \sum_{r=0}^{k-1} d \cdot L_{f,g,r}(k) e\left(-\frac{rd}{k}\right) \mu(k/d) = \sum_{d|k} \varphi(d) f(d) (k/d)^2 g(k/d),$$

- There are also approaches to evaluating the ordinary matrix entries, $u_{n,k}(f,w)$, using suitably chosen orthogonal polynomial sequences.

The End

Questions?

Comments?

Feedback?

Thank you for attending!