Math 6338 : Roadmap for Notions of Convergence (Extra Credit)

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Roadmap of Notions of Convergence

Precise definitions of convergence

Let $(X, \mathcal{A}, \mu)$ denote a fixed measure space which we will always assume is complete. We say that a sequence of measurable functions $\{f_n\}_{n\geq 1}$ *converges pointwise a.e.* (PWC) to a function f on X if

$$\mu\left[\left\{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\right)\right] = 0,$$

and write $f_n \to f$ a.e. as a shorthand for this type of convergence. We say that a sequence of measurable functions $\{f_n\}$ *converges in measure* (CIM) to f provided that given any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu\left[\{x : |f_n - f|(x) \geq \varepsilon\}\right] = 0,$$

and write $f_n \xrightarrow{m} f$ to denote when this happens. The notion of *almost uniform convergence* (AUC) is stated as follows: $\{f_n\}$ converges almost uniformly (to f) on X if and only if for all $\varepsilon > 0$, there is a measurable $E_\varepsilon \subseteq X$ such that $\mu(E_\varepsilon) < \varepsilon$ and $f_n \xrightarrow{\textbf{unif}} f$ on $X \setminus E_\varepsilon$, i.e., f_n converges uniformly to f on X except possible at points in measurable sets of arbitrarily small size.

The next two notions of convergence are somewhat separate in that they define convergence of functions by convergence of sequences of integrals. We say that f_n *converges to f in mean* (CMEAN), i.e., in the L^1 sense, provided that

$$\int_X |f_n - f| d\mu \to 0,$$

as $n \to \infty$. For $1 \leq p < \infty$, we (approximately) define the space of L^p functions to be the real-valued functions g such that

$$||g||_p := \left(\int_X |g|^p d\mu\right)^{1/p} < \infty.$$

We say that f_n *converges to f in the L^p norm* (LPC) provided that $||f_n - f||_p, ||f_n - f||_p^p \to 0$ as $n \to \infty$. In the special case where $p := \infty$, we define the L^∞ norm of g to be

$$||g||_\infty := \sup_{x\in X} |g(x)|,$$

and use similar notions of convergence of sequences of functions in L^∞ as we did in the finite cases.

A Roadmap Table of Contents

The next table (Table 1.1 on page 3) provides an index of the labelled sections below where we will compare each of our notions of convergence to one another. We will consider convergence in L^∞ as a subpart of of treatments with respect to L^p convergence.

1. Comparison of PWC and CIM

▶ PWC implies CIM

Proof. Suppose that $f_n \to f$ pointwise a.e. on X. Set

$$Z_2 := \{x \in X : f_n(x) \nrightarrow f(x)\}.$$

By our hypothesis $\mu(Z_2) = 0$. Now let $\varepsilon > 0$ and for $n \geq 1$ define the sets

$$E_{n,\varepsilon} := \{x : |f_n - f|(x) \geq \varepsilon\}.$$

Notions of Convergence

Notions of Convergence		PWC	CIM	AUC	CMEAN	LPC
Pointwise Convergence	(PWC)	X	**1**	**2**	**3**	**4**
Convergence in Measure	(CIM)	X	X	**5**	**6**	**7**
Almost Uniform	(AUC)	X	X	X	**8**	9
Convergence in Mean	(CMEAN)	X	X	X	X	**10**
Convergence in L^p	(LPC)	X	X	X	X	**11**

Table 1.1: *An index by numbered subsection of our approach to the roadmap.*

Then for sufficiently large $n \geq N(\varepsilon)$ we must have that $E_{n,\varepsilon} \subseteq Z_2$. By monotonicity we then have that

$$\lim_{n\to\infty} \mu(E_{n,\varepsilon}) \leq \mu(Z_2) = 0.$$

Hence we conclude that $f_n \xrightarrow{\mathbf{m}} f$. □

► CIM noes not imply PWC

Proof. We will give a counterexample by providing a sequence that converges in measure to zero but which does not converge pointwise a.e. to zero. For $n \geq 1$ and $1 \leq j \leq n$, define the intervals

$$S_{n,j} := \left[\frac{j-1}{n}, \frac{j}{n}\right] \subseteq [0,1],$$

and for any fixed j in the range above define the corresponding sequence of

$$f_n(x) := \chi_{S_{n,j}}(x).$$

We claim that in the Lebesgue measure that $f_n \xrightarrow{\mathbf{m}} 0$. Let $\varepsilon > 0$ and observe that

$$\ell\left(\{x \in [0,1] : f_n(x) > \varepsilon\}\right) = \ell\left(\{x \in [0,1] : f_n(x) = 1\}\right) = \frac{j}{n} - \frac{j-1}{n} = \frac{1}{n} \to 0,$$

as $n \to \infty$. However, $f_n \nrightarrow 0$ pointwise a.e. since given any $x \in [0,1]$ there are infinitely many n such that $x \in S_{n,j}$. This implies that there is a subsequence $\{f_{n_k}\}_{k\geq 1}$ such that $f_{n_k} \to 1$ as $k \to \infty$ $\perp$. □

However, CIM of f_n to f does imply that there is a subsequence $\{f_{n_k}\}_{k\geq 1}$ converging pointwise to f:

Proof. Now we claim that convergence in measure implies that we can find a subsequence converging pointwise a.e. to f. Since $f_n \to f$ in measure, given $\varepsilon, \eta > 0$ $\exists L$ such that $k \geq L$ implies that

$$\mu\left(\{x : |f_k - f|(x) > \varepsilon\}\right) < \eta.$$

For $j \geq 1$, choose $\varepsilon := 1/j$, $\eta := 2^{-j}$, and a corresponding $L_j \in \mathbb{N}$ such that $k \geq L_j$ implies that

$$\mu\left(\{x : |f_k - f|(x) > 1/j\}\right) < 2^{-j}.$$

We may assume here that $L_1 < L_2 < L_3 < \cdots$. For $j \geq 1$, let

$$E_j := \{x : |f_{L_j} - f|(x) > 1/j\},$$

where by construction $\mu(E_j) < 2^{-j}$. Next, let

$$Z := \limsup_{j\to\infty} E_j = \bigcap_{m\geq 1}\bigcup_{j\geq m} E_j.$$

Then for $m \geq 1$ we have that

$$\begin{aligned}
\mu(Z) &\leq \mu\left(\bigcup_{j\geq m} E_j\right) && \text{(monotonicity)} \\
&\leq \sum_{j\geq m} \mu(E_j) && \text{(subadditivity)} \\
&< \sum_{j\geq m} \frac{1}{2^j} = \frac{1}{2^{m-1}} \to 0,
\end{aligned}$$

as $m \to \infty$. So we conclude that $\mu(Z) = 0$. Now if $x \in X \setminus Z$, then $x \notin \cup_{j\geq m} E_j$ for some m. This implies that $x \notin E_j$ for $j \geq m$

$$\begin{aligned}
&\implies |f_{L_j}(x) - f(x)| \leq \frac{1}{j}, \quad \text{for } j \geq m \\
&\implies \lim_{j\to\infty} f_{L_j}(x) = f(x), \quad \text{in } X \setminus Z.
\end{aligned}$$

That is to say that $f_{L_j}(x) \to f(x)$ as $j \to \infty$ in $X \setminus Z$, and so pointwise a.e. in X. □

2. Comparison of PWC and AUC

▶ AUC implies PWC

Proof. Suppose that f_n converges almost uniformly to f on X. Then given any $\varepsilon > 0$ $\exists E_\varepsilon \subseteq X$ such that $\mu(E_\varepsilon) < \varepsilon$ and where $f_n \overset{\textbf{unif}}{\longrightarrow} f$ on $X \setminus E_\varepsilon$. This also implies that $f_n \to f$ on $X \setminus E_\varepsilon$. Take

$$E := \bigcap_{n\geq 1} E_{1/n}.$$

Notice that since it is the countable intersection of sets in $\mathcal{A}$, it is also in $\mathcal{A}$. Also, for all $n \geq 0$ we know by monotonicity that

$$\mu(E) \leq \mu(E_\varepsilon) < 1/n.$$

So we can conclude that $\mu(E) = 0$. If $x \in X \setminus E$, then $x \notin E_{1/k}$ for some $k \geq 1$, so that $f_n \to f$ pointwise on $X \setminus E$. Then since $\mu(E) = 0$, $f_n \to f$ a.e. on X. □

▶ PWC sometimes implies AUC

We have seen basic examples in elementary real analysis classes showing that the functions $f_n(x) := x^n$ do not converge uniformly on $X := [0, 1]$, though they do converge pointwise on this interval. However, suppose that $\mu(X) < \infty$ and that f_n is measurable for all $n \geq 1$ and $f_n \to f$ pointwise a.e. on X for some measurable f. If $|f_n| \leq g$ for some g integrable on X, then $f_n \to f$ almost uniformly.

Proof. Let the set

$$N := \{x : f_n(x) \nrightarrow f(x)\}.$$

By hypothesis $\mu(N) = 0$ and for $x \in X \setminus N$, we have that $|f| \leq g$. Since g is integrable it is finite a.e. and hence on $X \setminus N$. So by this inequality for $|f|$, we can conclude that f is finite a.e. as well. So we have satisfied the hypotheses of Egorov's theorem (since we have assumed $\mu(X) < \infty$). Now given $\varepsilon > 0$, there is a closed set $F \subset X$ such that the set $A := X \setminus F$ satisfies $\mu(A) < \varepsilon$ and so that f_n converges uniformly to f on $X \setminus A = F$. So f_n converges almost uniformly to f on X. □

Alternate Proof. We can give an alternate proof of the result that does not require that $\mu(X) < \infty$ using Lebesgue's dominated convergence theorem. □

3. Comparison of PWC and CMEAN

▶ CMEAN does not imply PWC

This is a special case of the more general example showing that LPC $\not\Rightarrow$ PWC for $1 \leq p < \infty$. This more general result will be established in a later section (see the roadmap table), so we do not give it here.

▶ PWC and monotone convergence implies CMEAN

Proof. We will only handle the increasing sequence case in this proof, though we do note that it is similarly easy to prove this holds in the decreasing sequence case as well. Suppose that $f_n \nearrow f$ on X, i.e., $f_{n+1}(x) \geq f_n(x)$ for all n, x and where $f_n \to f$ a.e. on X. Further suppose that there is an integrable ϕ on X such that $f_n \geq \phi$ a.e. in X for all $n \geq 1$. Then by the monotone convergence theorem,

$$\lim_{n\to\infty} \int f_n = \int f.$$

In other words, for any $\varepsilon > 0$ $\exists N$ such that $n \geq N$ implies that

$$\varepsilon > \left| \int (f_n - f) \right| = \int |f_n - f|,$$

where the last equality on the right-hand-side of the previous equation follows from our assumption that f_n increases to f on X. Hence $||f_n - f||_1 \to 0$ as $n \to \infty$. □

4. Comparison of PWC and LPC

▶ PWC does not imply LPC

Proof. For any $1 \leq p \leq \infty$, let

$$f_n(x) := n \cdot \chi_{\left[0, \frac{1}{n}\right]}(x),\ n \geq 1.$$

Then 1) $f_n \in L^p$ for all n (see calculation of the integral below); and 2) $f_n(x) \to f(x) \equiv 0$. But in L^p with the Lebesgue measure we get that $\forall x$

$$\begin{aligned}
||f_n(x) - f(x)||_p^p = ||f_n(x)||_p^p &= \int \left| n \cdot \chi_{\left[0, \frac{1}{n}\right]}(x) \right|^p d\ell(x) \\
&= n^p \times \int \chi_{\left[0, \frac{1}{n}\right]}(x) d\ell(x) \\
&= n^p \times \ell\left(\left[0, \frac{1}{n}\right]\right) = n^{p-1} \not\to 0,
\end{aligned}$$

whenever $p \geq 1$. □

▶ LPC does not imply PWC if $p < \infty$

Proof. For $n \geq 1$ and $2^k \leq n < 2^{k+1}$, we define

$$f_n := \chi_{\left[\frac{n-2^k}{2^k}, \frac{n+1-2^k}{2^k}\right]}.$$

Then as we can see for any x $||f_n||_p^p \to 0$ in the Lebesgue measure:

$$\begin{aligned}
\int |f_n|^p d\ell &= \int \chi_{\left[\frac{n-2^k}{2^k}, \frac{n+1-2^k}{2^k}\right]} d\ell \\
&= \left(\frac{n-2^k}{2^k}\right) - \left(\frac{n+1-2^k}{2^k}\right) = \frac{2}{2^{k+1}} \\
&< \frac{2}{2^{\log_2(n)}} = \frac{2}{n} \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

However, as we can see that $f_n \not\to 0$ pointwise a.e. since for $x \in (0, 1)$ we can always choose n sufficiently large so that

$$\frac{n-2^k}{2^k} \leq x \leq \frac{n+1-2^k}{2^k},$$

and then the characteristic function evaluates to 1 (not 0). □

However, if f_n is a sequence in L^p which converges to f (in L^p), then we can find a subsequence converging pointwise a.e. to f:

Proof. As we have seen in part #5.2 of homework 5, $||f_n - f||_p \to 0$ in L^p implies that $f_n \to f$ in measure. As we proved above (in Section 1), CIM of f_n to f implies that we can find a subsequence of the $\{f_n\}$ that converges pointwise a.e. to f. We will not repeat the details to that proof given above again here. $\square$

▶ L^∞ convergence implies PWC

Proof. For all $n \geq 1$ and $x \in X$, we have that

$$|f_n - f|(x) \leq ||f_n - f||_\infty = \sup_{x \in X} |f_n - f|(x).$$

Since $||f_n - f||_\infty \to 0$ as $n \to \infty$, the right-hand-side of the previous equation goes to zero as $n \to \infty$ which gives us PWC of f_n to f on X. $\square$

5. Comparison of CIM and AUC

▶ AUC implies CIM

Proof. Suppose that $f_n \to f$ almost uniformly. The given $\varepsilon, \eta > 0$, $\exists D \subseteq X$ such that $\mu(D) < \eta$ and $|f_n(x) - f(x)| < \varepsilon$ for all $x \in X \setminus D$. Let

$$D_\varepsilon := \{x : |f_n - f|(x) \geq \varepsilon\} \subseteq D.$$

So since $D_\varepsilon \subseteq D$, $\mu(D_\varepsilon) \leq \mu(D) < \eta$. Letting $\eta \to 0$, we see that $f_n \xrightarrow{\mathbf{m}} f$. $\square$

▶ CIM does not imply AUC

We adapt the counterexample given in the lecture notes found at http://www.tau.ac.il/~tsirel/Courses/RealAnal/lect5.pdf for our proof.

Proof. Let $X := [0, 1)$ and $\mu := \ell$ denote the Lebesgue measure. For each $n \geq 1$, we claim that we can partition X into n subintervals each with measure $1/n$. Indeed, for a fixed n and $1 \leq k \leq n$ let $I_{n,k} := \left[\frac{k-1}{n}, \frac{k}{n}\right)$ so that 1) $\ell(I_{n,k}) = \frac{1}{n}$ and

$$\bigcup_{k=1}^{n} I_{n,k} = \left[0, \frac{1}{n}\right) \cup \left[\frac{1}{n}, \frac{2}{n}\right) \cup \cdots \cup \left[\frac{n-1}{n}, 1\right) = [0, 1).$$

Now we form a sequence $\{A_j\}_{j \geq 1}$ by taking this construction for only odd $n := 2k + 1 \geq 3$ and aligning these sets in order as $A_j := I_{\lfloor \sqrt{j} \rfloor, j - (\lfloor \sqrt{j} \rfloor)^2}$. More pictorially, what this sequence corresponds to is the construction

$$\{A_j\}_{j \geq 1} = \left\{ \underbrace{A_1, A_2, A_3}_{\substack{\text{partition of } X \\ n=3}}, \underbrace{A_4, A_5, A_6, A_7, A_8}_{\substack{\text{partition of } X \\ n=5}}, \underbrace{A_9, A_{10}, A_{11}, A_{12}, A_{13}, A_{14}, A_{15}}_{\substack{\text{partition of } X \\ n=7}}, \ldots \right\}.$$

Then by our construction above, we have that for each $k \geq 1$

$$X = A_{k^2} \cup A_{k^2+1} \cup \cdots \cup A_{(k+1)^2-1},$$

where

$$\mu(A_{k^2}) = \mu(A_{k^2+1}) = \cdots = \mu(A_{(k+1)^2-1}) = \frac{1}{2k+1}.$$

Now we take our sequence of functions $f_n(x) := \chi_{A_n}(x)$ and see plainly that since $\mu(A_n) \to 0$ as $n \to \infty$, we get $f_n \xrightarrow{\mathbf{m}} 0$. However, we do not get AUC of f_n to zero because

$$\limsup_{n \to \infty} |f_n(x)| = 1 \neq 0,$$

since for any $x \in [0, 1)$, x is contained in infinitely many A_n. $\square$

6. Comparison of CIM and CMEAN

▶ CMEAN implies CIM

Proof. Let $\varepsilon > 0$. Then if $\int |f_n - f| d\mu \to 0$ as $n \to \infty$, by Chebyshev we have that

$$\mu\left(\{x : |f_n - f|(x) \geq \varepsilon\}\right) \leq \frac{1}{\varepsilon} \times \int |f_n - f| d\mu \to 0.$$

So $f_n \to f$ in measure as well as $n \to \infty$. □

▶ CIM implies CMEAN if $\mu(X) < \infty$

Proof. Fix $\varepsilon, \eta > 0$ and define the sets

$$E_{n,\varepsilon} := \{x : |f_n - f|(x) \geq \varepsilon\},$$

where we have that $\exists N(\eta)$ such that for all $n \geq N(\eta)$ $\mu(E_{n,\varepsilon}) < 1/\eta$ by the convergence in measure of f_n to f on X. Now to show convergence in mean when $\mu(X) < \infty$, we can write

$$\begin{aligned}
\int_X |f_n - f| &= \int_{X \setminus E_{n,\varepsilon}} |f_n - f| + \int_{E_{n,\varepsilon}} |f_n - f| \\
&\leq \varepsilon \cdot \mu(X \setminus E_{n,\varepsilon}) + \int_{E_{n,\varepsilon}} |f_n - f| \\
&\longrightarrow \varepsilon \cdot \mu(X) + 0, \quad \text{as } n \to \infty, \eta \to 0 \\
&\longrightarrow 0, \quad \text{as } \varepsilon \to 0.
\end{aligned}$$

Hence we get convergence in mean here as well. □

7. Comparison of CIM and LPC

▶ LPC implies CIM if $1 \leq p < \infty$

Proof. We suppose that $||f_n - f||_p^p \to 0$ pointwise a.e. as $n \to \infty$. For $\varepsilon > 0$ and $\eta := \varepsilon^p$, we define the corresponding sets

$$X_n(\varepsilon) := \{x : |f_n - f|(x) > \varepsilon\}.$$

Then we see that

$$\begin{aligned}
||f_n - f||_p^p = \int |f_n - f|^p &\geq \int_{X_n(\varepsilon)} |f_n - f|^p \\
&> \int_{X_n(\varepsilon)} \varepsilon^p = \varepsilon^p \times \mu\left(X_n(\varepsilon)\right) \to 0,
\end{aligned}$$

as $\eta \times \mu\left(\{x : |f_n - f|^p(x) > \eta\}\right) \to 0$ as $n \to \infty$. Since the η was arbitrary, this implies that $f_n \xrightarrow{\text{m}} f$. □

▶ L^∞ convergence implies CIM if $\mu(X) < \infty$

Proof. For $\varepsilon > 0$ and $n \geq 1$, define the sets

$$X_n(\varepsilon) := \{x : |f_n - f|(x) \geq \varepsilon\}.$$

By Chebyshev's inequality, we obtain that

$$\begin{aligned}
\mu(X_n(\varepsilon)) &\leq \frac{1}{\varepsilon} \int_X |f_n - f| d\mu \\
&\leq ||f_n - f||_\infty \cdot \mu(X) \\
&\longrightarrow 0,
\end{aligned}$$

as $n \to \infty$ provided that the measure space is finite. This implies that $f_n \xrightarrow{\text{m}} f$ since ε was arbitrary. □

▶ CIM does not imply LPC

Example 1.1. We employ the same example we used to show that PWC does not imply LPC. Namely, for $n \geq 1$ let
$$f_n(x) := n \cdot \chi_{\left[0,\frac{1}{n}\right]}(x).$$
We claim that f_n converges in measure to $f(x) \equiv 0$ when $\mu := \ell$ is the Lebesgue measure. For $\varepsilon > 0$, we notice that for all sufficiently large $n \geq N(\varepsilon)$ such that $\varepsilon / N(\varepsilon) < 1$, we have that
$$\mu(\{x : |f_n(x)| \geq \varepsilon\}) = \mu([1, 1/n]) = \frac{1}{n} \to 0,$$
as $n \to \infty$. However, as we have seen above $f_n \nrightarrow 0$ in L^p.

8. Comparison of AUC and CMEAN

This comparison is just a special case of the AUC versus LPC convergence conditions given in the next section when $p = 1$. See the next section #9 for details.

9. Comparison of AUC and LPC

▶ AUC does not imply convergence in L^∞

Proof. Let $f_n \to f$ almost uniformly on X such that f_n is always finite for all n, but where f assumes the values $\pm\infty$ only on a non-empty set $E \subseteq X$ of measure zero. Then we can see that
$$\begin{aligned}||f_n - f||_\infty &= \sup_{x \in (X \setminus E) \cup E} |f_n - f|(x) \\ &= \sup_{x \in E} |f_n - f|(x) = +\infty,\end{aligned}$$
for all $n \geq 1$. So we get that $||f_n - f||_\infty \nrightarrow 0$ as $n \to \infty$. □

▶ AUC implies LPC for $1 \leq p < \infty$ if $\mu(X) < \infty$

Proof. Let $\varepsilon > 0$. Then $\exists A_\varepsilon \subseteq X$ such that $\mu(A_\varepsilon) < \varepsilon$ and where
$$\lim_{n \to \infty} \sup_{x \in X \setminus A_\varepsilon} |f_n - f|(x) = 0.$$
Since $p \geq 1$ is finite we also have
$$\lim_{n \to \infty} \sup_{x \in X \setminus A_\varepsilon} |f_n - f|^p(x) = 0,$$
so that
$$\begin{aligned}||f_n - f||_p^p &= \int_{X \setminus A_\varepsilon} |f_n - f|^p + \int_{A_\varepsilon} |f_n - f|^p \\ &\leq \left(\sup_{x \in X \setminus A_\varepsilon} |f_n - f|^p\right) \times \mu\left(X \setminus A_\varepsilon \subseteq X\right) + \int_{A_\varepsilon} |f_n - f|^p \\ &\longrightarrow 0,\end{aligned}$$
as we let $\varepsilon \to 0$ and n tend to infinity. □

10. Comparison of CMEAN and LPC

▶ LPC implies CMEAN if $\mu(X) < \infty$

Proof. Let $p > 1$ and select q such that $p + q = pq$. Suppose that $||f_n - f||_p \to 0$ as $n \to \infty$ in L^p. Then if $\mu(X) < \infty$, so is $\mu(X)^{1/q} < \infty$, so that by Holder's inequality we may bound
$$\int_X |f_n - f| d\mu \leq ||f_n - f||_p \cdot ||1||_q = ||f_n - f||_p \cdot \mu(X)^{1/q} \to 0,$$
as $n \to \infty$ since f_n converges to f in L^p.

□

▶ CMEAN and L^∞-boundedness imply LPC for $1 < p < \infty$

Proof. If we have that

$$M_\infty := \sup_{n \geq 1} ||f_n - f||_\infty < \infty$$

, i.e., that we have L^∞-boundedness of the sequence $\{f_n - f\}_{n \geq 1}$, then clearly also for any $1 < p < \infty$ we see that

$$\sup_{n \geq 1} |||f_n - f|^{p-1}||_\infty < \infty,$$

as well. So by Holder's inequality, we can bound (taking $p_0 := 1$ and $q_0 := \infty$ so that $1/p_0 + 1/q_0 = 1$)

$$\int |f_n - f|^p = \int |f_n - f|^{p-1} |f_n - f| \leq |||f_n - f|^{p-1}||_\infty \cdot ||f_n - f||_1 \to 0,$$

as $n \to \infty$ since we have assumed CMEAN convergence. □

▶ LPC does not imply CMEAN in general

We show a specific counter example. Let $f_n(x) := \frac{1}{nx}$ with $X := [1, \infty) \subseteq \mathbb{R}$ and $\mu := \ell$ the Lebesgue measure. Then for any $2 \leq p < \infty$ we see that $f_n \to 0$ in L^p:

$$\int_1^\infty \left|\frac{1}{nx}\right|^p dx = \frac{1}{(1-p) \cdot n^p} \left(\frac{1}{x^{p-1}} \Bigg|_{x=1}^{x=\infty} \right) \xrightarrow{n \to \infty} 0.$$

On the other hand, in the L^1 norm we do not obtain convergence to 0 as $n \to \infty$:

$$\int_1^\infty \left|\frac{1}{nx}\right| dx = \frac{1}{n} \left(\log(x) \Bigg|_{x=1}^{x=\infty} \right) \nrightarrow 0.$$

11. Comparison of LPC and LPC

Proposition 1.2. *Let $1 \leq q < p < \infty$ and suppose that $\mu(X) < \infty$. If f_n converges to f in L^p, then f_n converges to f in L^q.*

Proof. We can apply Holder's inequality with $p_0 := p/q$ and $q_0 := p/(p-q)$ so that $1/p_0 + 1/q_0 = 1$. Now since $||f_n - f||_p \to \infty$ and $n \to \infty$, we have that

$$\begin{aligned}
\int_X |f_n - f|^q d\mu &\leq |||f_n - f|^q||_{p/q} \cdot ||1||_{\frac{p}{p-q}} \\
&= \left(\int |f_n - f|^p \right)^{q/p} \cdot \mu(X)^{\frac{p-q}{p}} \to 0,
\end{aligned}$$

as $n \to \infty$. Hence $||f_n - f||_q \to 0$ as well. □

Proposition 1.3. *If $\mu(X) < \infty$ and $||f_n - f||_\infty \to 0$ as $n \to \infty$, then f_n converges to f in L^p for any $1 \leq p < \infty$.*

Proof. Let $\varepsilon > 0$. Then since f_n converges to f in L^∞ $\exists N(\varepsilon)$ such that for all $n \geq N(\varepsilon)$ we have that

$$\sup_{x \in X} |f_n - f|(x) < \varepsilon.$$

Now since we have assumed that the measure space is finite we can bound the L^p norm by

$$\left(\int_X |f_n - f|^p d\mu \right)^{1/p} \leq \left(\sup_{x \in X} |f_n - f|(x) \right) \times \int_x d\mu < \varepsilon \times \mu(X).$$

Since our choice of the ε was arbitrary, we may let it tend to zero as $n \to \infty$ and see that the right-hand-side of the previous equation tends to zero as well. Hence $\lim_{n \to \infty} ||f_n - f||_p = 0$ so that we get convergence in L^p. □