Factorization theorems for generating special sums

Maxie Dion Schmidt

Georgia Institute of Technology
School of Mathematics

mschmidt34@gatech.edu

http://people.math.gatech.edu/~mschmidt34/

August 14, 2020

Ph.D. Oral Examination Presentation

Committee Members:

- Professor Michael Lacey (Adviser)
- Professor Christopher Heil
- Professor Plamen Iliev
- Professor Josephine Yu

Overview and goals of the talk

- Motivate why generating function approaches to enumerating special sums is useful in applications
- Identify a general method for expanding out the generating functions of special sums (many examples)
- Published work on these so-called *factorization theorems* and new approaches moving forward
- Questions about the most natural ways of forming the generating-function-based expansions of these sums.

Generating functions of sequences

- Recall that an *ordinary generating function* (or OGF) of a sequence $\{f_n\}_{n\geq 0}$ (or arithmetic function) is defined by $F(z) := \sum_{n\geq 0} f_n \cdot z^n$.
- We write the series coefficient extraction operator in the notation $[z^n]F(z) \equiv f_n$ for $n \geq 0$.
- We can treat $F(z)$ as a formal power series object that enumerates, or "*pins up*" the terms of the sequence in powers of z like clothes on a clothesline.
- Alternately, we can view $F(z)$ as an analytic function within its radius of convergence to justify properties like asymptotic expansions of the f_n.
- Compare to the analogy given in Wilf's classic *Generatingfunctionology* book.

Motivating examples

Motivating examples

Lambert series factorization theorems

- In some sense, the first and prototypical example
- The idea was based on some combinatorially themed work with these generating functions I published in *Acta Arithmetica* around 2017. Grew into collaborative work with M. Merca.
- Relates multiplicative number theoretic functions to more additive constructions in the theory of partitions.
  (A detailed acount of why this relation is rare, special and interesting is found in the *AMM* article.)

Review: Multiplicative functions in number theory

- Recall that an arithmetic function f is called *multiplicative* if $f(ab) = f(a) \cdot f(b)$ for all integers $a, b \geq 1$ such that $(a, b) = 1$.
- Typical examples of multiplicative (or sometimes completely multiplicative) functions include the following:
  1. The *Möbius function* $\mu(n)$ is the signed indicator function of the squarefree integers.
  2. Euler's classical *totient function* $\phi(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} 1$.
  3. The generalized *sum-of-divisors functions* $\sigma_\alpha(n) := \sum_{d|n} d^\alpha$ with the special cases $d(n) = \sigma_0(n)$ and $\sigma(n) = \sigma_1(n)$.
  4. The completely multiplicative *Liouville lambda function* $\lambda(n) = (-1)^{\Omega(n)}$ (definition).

What is a Lambert series generating function?

- Typically a more classical generating function construction in number theory to generate series enumerating special multiplicative functions via divisor sums of the function convolved with one.
- Formally, given an arithmetic function $f : \mathbb{Z}^+ \rightarrow \mathbb{C}$ we define its *Lambert series generating function* (or LGF) to be

$$L_f(q) := \sum_{n \geq 1} \frac{f(n) q^n}{1 - q^n} = \sum_{m \geq 1} \left(\sum_{d|m} f(d) \right) q^m, |q| < 1.$$

Examples of Lambert series generating functions

$$\sum_{n\geq 1} \frac{\mu(n)q^n}{1-q^n} = q, \tag{1a}$$

$$\sum_{n\geq 1} \frac{\phi(n)q^n}{1-q^n} = \frac{q}{(1-q)^2}, \tag{1b}$$

$$\sum_{n\geq 1} \frac{n^\alpha q^n}{1-q^n} = \sum_{m\geq 1} \sigma_\alpha(n)q^n, \tag{1c}$$

$$\sum_{n\geq 1} \frac{\lambda(n)q^n}{1-q^n} = \sum_{m\geq 1} q^{m^2}, \tag{1d}$$

$$\sum_{n\geq 1} \frac{\mu^2(n)q^n}{1-q^n} = \sum_{m\geq 1} 2^{\omega(m)}q^m. \tag{1e}$$

Useful properties

- If the function f has an *ordinary generating function* (or OGF), $F(z) := \sum_{n\geq 1} f(n)z^n$, then it is related to the function's LGF by
$$L_f(q) = \sum_{n\geq 1} F(q^n).$$
- If $F(x) := \sum_{n\leq x} f(n)$ is the *summatory function* of f, then we have that
$$\sum_{n\geq 1} F(n)q^n = \sum_{n\geq 1} \mu(n)\frac{L_f(q^n)}{1-q}.$$
- If $A \subseteq \mathbb{Z}^+$ and its indicator function is denoted by $\chi_A(n)$, then we have that
$$\sum_{n\geq 1} \mu(n)L_{\chi_A}(q^n) = \sum_{a\in A} q^a.$$

Review: The partition function $p(n)$

- A partition of a positive integer n is a (finite) sequence of positive integers whose sum is n.
- More formally, we may partition n as a sum of integers $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_k \geq 1$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$.
- We denote the total number of partitions of n by $p(n)$
- For example, $p(5) = 7$ since the distinct partitions of 5 are given by

$$5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1.$$

- By a combinatorial argument with products of geometric series, we have that the OGF for $p(n)$ is given by

$$p(n) = [q^n](q;q)_\infty^{-1} = [q^n]\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots}.$$

Work on Lambert series factorization theorems

- Let $s_e(n,k)$ and $s_0(n,k)$ respectively denote the the number of k's in all partitions of n into an even (odd) number of distinct parts.
- Let $(a;q)_\infty = \prod_{m\geq 1}(1-aq^{m-1})$ denote the infinite q-Pochhammer symbol.
- Then we have (*cf.* work with Merca in the references):

$$\sum_{n\geq 1}\frac{f(n)q^n}{1\pm q^n} = \frac{1}{(\mp q;q)_\infty}\sum_{n\geq 1}\left(\sum_{k=1}^{n}\left(s_o(n,k)\pm s_e(n,k)\right)f(k)\right)q^n,$$

Lambert series factorization theorems (expressions by invertible matrices)

- Re-write the previous factorization theorem statement as

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{(q;q)_\infty} \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k} f(k) \right) q^n.$$

- The matrices formed by the lower triangular sequence of $s_{n,k}$ are invertible.
- Moreover, they are generated by the decidely partition theoretic series

$$s_{n,k} = [q^n] \frac{q^k}{1-q^k} \cdot (q;q)_\infty, 1 \leq k \leq n.$$

- It appears by inspection that the inverse matrix, $s_{n,k}^{-1}$, is also related to partition functions (see next table).

A table of the first few rows of the inverse matrices

1																												
	1																											
1	1	1																										
2	1	1	1																									
4	3	2	1	1																								
5	3	2	2	1	1																							
10	7	5	3	2	1	1																						
12	9	6	4	3	2	1	1																					
20	14	10	7	5	3	2	1	1																				
25	18	13	10	6	5	3	2	1	1																			
41	30	22	15	11	7	5	3	2	1	1																		
47	36	26	19	14	10	7	5	3	2	1	1																	
76	56	42	30	22	15	11	7	5	3	2	1	1																
90	69	51	39	28	21	14	11	7	5	3	2	1	1															
129	97	74	55	41	30	22	15	11	7	5	3	2	1	1														
161	124	94	72	53	40	29	21	15	11	7	5	3	2	1	1													
230	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1												
270	212	163	126	95	73	54	41	29	22	15	11	7	5	3	2	1	1											
384	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1										
458	362	281	219	169	130	98	75	55	41	30	22	15	11	7	5	3	2	1	1									
615	482	379	294	229	175	134	101	77	56	42	30	22	15	11	7	5	3	2	1	1								
750	596	468	370	286	224	171	132	99	76	55	42	30	22	15	11	7	5	3	2	1	1							
1001	792	627	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1						
1187	951	756	601	472	372	289	225	173	133	100	76	56	42	30	22	15	11	7	5	3	2	1	1					
1570	1252	1000	791	626	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1				
1881	1518	1213	972	770	612	479	378	292	228	174	134	100	77	56	42	30	22	15	11	7	5	3	2	1	1			
2414	1943	1564	1248	997	789	625	489	384	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1		
2907	2358	1901	1532	1225	980	777	616	483	380	294	229	175	134	101	77	56	42	30	22	15	11	7	5	3	2	1	1	
3717	3010	2436	1958	1575	1255	1002	792	627	490	385	297	231	176	135	101	77	56	42	30	22	15	11	7	5	3	2	1	1

The first 29 rows of the function $s_{n,k}^{(-1)}$ where the values of Euler's partition function $p(n)$ are highlighted in blue:

$$\{p(n)\}_{n\geq 0} = \{1, 1, 2, 3, 5, 7, 11, 15, 22, 30, 42, 56, 77, \ldots\}.$$

A formula for the inverse matrices

- We can prove exactly how $s_{n,k}^{-1}$ is related to $p(n)$:

$$s_{n,k}^{-1} = \sum_{d|n} p(d-k)\mu\left(\frac{n}{d}\right).$$

- The matrix inverse has the LGF

$$\sum_{n\geq 1} \frac{s_{n,k}^{-1} q^n}{1-q^n} = \frac{q^k}{(q;q)_\infty}.$$

- It follows that for all $n \geq 1$, we can completely invert the construction to solve for $f(n)$ as

$$f(n) = \sum_{m=0}^{n-1} \sum_{d|n} p(d-m-1)\mu\left(\frac{n}{d}\right) \times [q^m]L_f(q) \cdot (q;q)_\infty, n \geq 1.$$

Relation of the matrices to partition functions brings up a natural question

- Here, there is a very **natural** relation of both sequences of $s_{n,k}$ and $s_{n,k}^{-1}$ to *partition theoretic functions*.
- This is unusual in so much as LGFs typically generate *multiplicative* functions, whereas partition function variants have a much more *additive* structure (e.g., they count the numbers of ways to put decompositions of n items into bins)
- Brings up a natural question: Why did partitions fit so naturally with the multiplicative function LGFs above? Is this the most natural way to expand things in the context of other special sums?

More general constructions

Generating special sums in more generality

Generating functions for a more general class of sums

- More generally, we can consider factorization theorems of the form

$$\sum_{n\geq 1}\left(\sum_{\substack{k\in\mathcal{A}_n\\ \mathcal{A}_n\subseteq[1,n]}} f(k)\right) q^n = \frac{1}{\mathcal{C}(q)}\sum_{n\geq 1}\sum_{k=1}^{n} s_{n,k}(\mathcal{A},\mathcal{C}) f(k)\cdot q^n,$$

  where $\mathcal{C}(q)$ is a representation-specific GF such that $\mathcal{C}(0)\neq 0$ (independent of f and $\mathcal{A}$).
- The matrix entries are generated by

$$s_{n,k}(\mathcal{A},\mathcal{C}) = [q^n]\mathcal{C}(q)\times\sum_{m\geq 1}[k\in\mathcal{A}_m]_{\delta}\cdot q^m.$$

- **Natural question (again):** What is a “good” choice of the generating function $\mathcal{C}$ given the definitions of the sets $\{\mathcal{A}_n\}_{n\geq 1}$ that leads to natural formulas for $s_{n,k}(\mathcal{A},\mathcal{C})$ and its inverse matrix?
- The inverse matrices are very closely tied up to generalized forms of Möbius inversion by special Möbius functions defined by the problem.

Examples of these types of special sums

- We will provide several examples of these types of special sums and inversion operations on them:
  1. A-convolutions
  2. K-convolutions
  3. B-convolutions
  4. Matrix generated convolution sums
- The proposed topics for research surround generating these sums using particular series-based constructions
- The next examples demonstrate some known special sum types we can focus on

Example: *A*-convolutions (restricted Dirichlet convolutions and divisor sums)

- For each $n \geq 1$, let $A(n) \subseteq \{d : 1 \leq d \leq n, d|n\}$ be a subset of the divisors of n.
- We can define the *A-convolution* of f and g by

$$(f *_A g)(n) := \sum_{d \in A(n)} f(d) g\left(\frac{n}{d}\right).$$

- We say that a natural number $n \geq 1$ is *A-primitive* if $A(n) = \{1, n\}$.

Example: A-convolutions (inversion)

- Under a list of assumptions so that the resulting A-convolutions are *regular convolutions*, we get a generalized multiplicative Möbius function:

$$\mu_A(p^\alpha) = \begin{cases} 1, & \alpha = 0; \\ -1, & p^\alpha > 1 \text{ is } A\text{-primitive}; \\ 0, & \text{otherwise.} \end{cases}$$

- This construction leads to a generalized form of Möbius inversion between the A-convolutions.
- Note that *classical Möbius inversion* takes the typical form of

$$g(n) = \sum_{d|n} f(d) \iff f(n) = \sum_{d|n} g(d)\mu\left(\frac{n}{d}\right), n \geq 1.$$

Generating functions for a general class of K-convolutions

- Let the function $K : \mathbb{N} \times \mathbb{N} \to \mathbb{C}$ be defined on all ordered pairs (n, d) such that $n \geq 1$ and $d|n$.
- We define the K-convolution of two arithmetic functions f, g to be

$$(f \circ_K g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right) K(n, d).$$

- We can define generating function factorizations of these sums of the form

$$(f \circ_K g)(n) = [q^n] \frac{1}{\mathcal{C}(q)} \sum_{m \geq 1} \sum_{k=1}^{m} e_{n,k}(K, \mathcal{C}; g) f(k) \cdot q^m.$$

- The form of $e_{n,k}(K, \mathcal{C}; g)$ is *highly* dependent on K!

Example: B-convolutions

- Let $\nu_p(n)$ denote the maximum exponent of the prime p in the factorization of n.
- Define $B(n,d) := \prod_{p|n} \binom{\nu_p(n)}{\nu_p(d)}$ where the product runs over all prime divisors of n.
- Then we have an inversion formula of the form

$$f(n) = \sum_{d|n} g(d) B(n,d) \iff g(n) = \sum_{d|n} f(d)\lambda(d) B(n,d),$$

where $\lambda(n) = (-1)^{\Omega(n)}$ is the *Liouville lambda function*.

Matrix generated convolution sums (formalizing the definition above)

- Let $G := (g_{i,j})$ be an infinite dimensional matrix containing only elements in $\{0, 1\}$.
- Suppose that $g_{ij} = 1$ if $i = j$ and $g_{ij} = 0$ if $i > j$.
- Furthermore, let the 1's in column n of G appear in rows $n_1, n_2, \ldots, n_k$ with $n_1 < n_2 < \cdots < n_k = n$.
- We can define the convolution of two arithmetic functions f, g by

$$(f *_G g)(n) := \sum_{i=1}^{k} f(n_i) g(n_{k+1-i}).$$

Examples and special cases

- Let $D = (d_{ij})$ where $d_{ij} = [i|j]_\delta$. Then the convolution operation generated by $*_D$ is the same as the standard Dirichlet convolution of two arithmetic functions.
- Let $T = (t_{ij})$ where $t_{ij} = [(i,j) = 1]_\delta [i \leq j]_\delta$. Then we get a convolution variant of sums defined by

$$\sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} f(d).$$

Directions for future work

- Interest of mine started with GF expansions for Lambert series, and then some generalizations with another collaborator at GA Tech
- Natural DGF interpretation for multiplicative number theoretic interpretations (this gives a GF-type "clothes hanger" handle on certain series with related intent)
- New interpretations of natural expansions (like relations of multiplicative functions and divisor sums to partition functions in the LGF example case)
- Gives insight into special sums in applications from a typically combinatorially related construct of generating functions
- Binomial and Stirling-like GF transforms (another direction)

Concluding remarks and discussion

The End

Questions?

Comments?

Feedback?

Thank you for attending!

References I

- G. E. Andrews, *The Theory of partitions*, Cambridge, 1984.
- M. Merca, The Lambert series factorization theorem, *Ramanujan J.*, **44** (2017), 417–435.
- M. Merca and M. D. Schmidt, A partition identity related to Stanley's theorem, *Amer. Math. Monthly* **125** (2018), 929–933.
- M. Merca and M. D. Schmidt, Factorization Theorems for Generalized Lambert Series and Applications, *Ramanujan Journal* (2018).
- M. Merca and M. D. Schmidt, Generating Special Arithmetic Functions by Lambert Series Factorizations, *Contributions to Discrete Mathematics* (2018).
- M. Merca and M. D. Schmidt, New Factor Pairs for Factorizations of Lambert Series Generating Functions, `https://arxiv.org/abs/1706.02359` (2017).

References II

- H. Mousavi and M. D. Schmidt, Factorization Theorems for Relatively Prime Divisor Sums, GCD Sums and Generalized Ramanujan Sums , *Ramanujan Journal*, to appear (2020).
- J. Sandor and B. Crstici, *Handbook of Number Theory II*, Kluwer Adademic Publishers, 2004.
- M. D. Schmidt, A catalog of interesting and useful Lambert series identities, https://arxiv.org/abs/2004.02976 (2020).
- M. D. Schmidt, New recurrence relations and matrix equations for arithmetic functions generated by Lambert series, *Acta Arithmetica* **181**, 2017.

More talking points

More talking points

Focus of the remainder of the presentation

- We will provide more examples from the author's published research
- These examples provide a working corpus of examples of what is possible moving forward
- The next examples also motivate some relevance to recent work, and demonstrate variance on the GF-based factorizations we can approach

More on Lambert series factorization variants

Lambert series factorization variants

Variants of these expansions of Lambert series

- The previous construction we provided as an example in the introduction is not the only way to factorize the LGFs
- Let $\widetilde{f}_\gamma(n) := \sum_{d|n} f(d)(\gamma * 1)(n/d)$, where $(\gamma * 1)(n) = \sum_{d|n} \gamma(d)$ for any arithmetic function γ.
- Let $C(q)$ be any generating function such that $C(0) \neq 0$, and consider expanding

$$\sum_{n\geq 1} \frac{f(n)q^n}{1-q^n} = \frac{1}{C(q)} \sum_{n\geq 1} \left(\sum_{k=1}^{n} s_{n,k}(\gamma)\widetilde{f}_\gamma(k) \right) q^n,$$

- Then we have that

$$s_{n,k}^{-1}(\gamma) = \sum_{d|n} [q^{d-k}] \frac{1}{C(q)} \cdot \gamma\left(\frac{n}{d}\right).$$

Lambert series generating functions of Dirichlet convolutions

- The Dirichlet convolution of two arithmetic functions f and g is defined for $n \geq 1$ as

$$(f * g)(n) := \sum_{d|n} f(d) g\left(\frac{n}{d}\right).$$

- Consider factorizing the Lambert series generating function of $f * g$:

$$\sum_{n \geq 1} \frac{(f * g)(n) q^n}{1 - q^n} = \frac{1}{(q;q)_\infty} \sum_{n \geq 1} \sum_{k=1}^{n} \widetilde{s}_{n,k}(g) f(k) \cdot q^n.$$

- Then we can prove that $\widetilde{s}_{n,k}(g) = \sum_{j=1}^{n} s_{n,kj} \cdot g(j)$.
- The inverse matrices $\tilde{s}_{n,k}^{-1}(g)$ are related to the *Dirichlet inverse of* g when $g(1) \neq 0$.

Another class of sums

GCD and relatively prime divisor sums

Motivation

- The *summatory function* of an arithmetic function f is defined by the partial sums $F(x) := \sum_{n \leq x} f(n)$.
- The asymptotic analysis of $F(x)$ and the so-called *average order* of f, $\mathbb{E}[f(n)] \sim \frac{1}{n} \cdot F(n)$ for large n are common interesting problems in number theory
- For example, we can compute that if $\omega(n) := \sum_{p|n} 1$, then $\mathbb{E}[\omega(n)] = \log \log n + B$
- Also, the *Dirichlet divisor problem* can be summarized as finding optimal error bounds on the sums of the *divisor function*, $d(n) := \sum_{d|n} 1$:

$$\sum_{n \leq x} d(n) = x \cdot \log x + (2\gamma - 1)x + O(\sqrt{x}).$$

Motivation (continued)

- We can express $F(x)$ by a decomposition into sum types
- One of which we have Lambert series generating functions for already
- In particular, we can expand

$$\sum_{d\leq x} f(d) = \sum_{\substack{d=1 \\ (d,x)=1}}^{x} f(d) + \sum_{d|x} f(d) + \sum_{\substack{d=1 \\ 1<(d,x)<x}}^{x} f(d)$$
$$= \sum_{m|x} \sum_{\substack{k=1 \\ (k,m)=1}}^{m} f\left(\frac{kx}{m}\right).$$

- What about generating functions for the remaining sum types?

A class of Type I and Type II sums

- Have analogous factorization theorems for *type I sums* of the form

$$T_f(n) := \sum_{\substack{1 \leq d \leq n \\ (d,n)=1}} f(d).$$

- Factorization theorems for *type II sums* (Anderson-Apostol sums) of the form

$$\widehat{L}_{f,g,k}(n) := \sum_{d|(k,n)} f(d) g\left(\frac{n}{d}\right), \forall n \geq 1; \text{fixed } 1 \leq m \leq n.$$