

A free-boundary theory for the shape of the ideal dripping icicle

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The growth of icicles is considered as a free-boundary problem. A synthesis of atmospheric heat transfer, geometrical considerations, and thin-film fluid dynamics leads to a nonlinear ordinary differential equation for the shape of a uniformly advancing icicle, the solution to which defines a parameter-free shape which compares very favorably with that of natural icicles. Away from the tip, the solution has a power-law form identical to that recently found for the growth of stalactites by precipitation of calcium carbonate. This analysis thereby explains why stalactites and icicles are so similar in form despite the vastly different physics and chemistry of their formation. In addition, a curious link is noted between the shape so calculated and that found through consideration of only the thin coating water layer. © 2006 American Institute of Physics. [DOI: 10.1063/1.2335152]

The formation of patterns in snow and ice has been a source of fascination since antiquity. As early as 1611, Johannes Kepler¹ sought a physical explanation for the beautiful forms of snowflakes. While attention has been lavished upon snowflakes ever since,² their wintry cousins, icicles, have remained largely ignored. The basic mechanisms of icicle growth are well known,^{3–5} but there are few mathematical analyses describing their long, slender forms, most notably those of Makkonen³ and of Szilder and Lozowski.⁴ Icicle surfaces are typically covered with ripples a few centimeters in wavelength, but only recently^{6–8} has theoretical work begun to address the underlying dynamic instability that produces them. On a more basic level, the growth of dripping icicles has not been studied from the perspective of a true free-boundary approach.

As one can see in Fig. 1, icicles and stalactites—the iconic structures found in limestone caves⁹—can bear a striking resemblance, particularly insofar as they evince a slightly convex carrot-like form that is distinct from a cone. Of course visual similarity does not imply mechanistic similarity, but there is reason to think that a common mathematical structure might link the two phenomena.¹⁰ In each case, the evolving solid structure is enveloped by a thin flowing layer of fluid which regulates the rate of growth. For stalactites, this is the coating water film flowing down the surface in which carbon dioxide is produced and through which it diffuses. In icicles there is a similar water layer, but the controlling fluid is the upward flowing natural convection

boundary layer in the surrounding air through which latent heat is transported by diffusion and convection.

Recent work^{11,12} examining stalactite growth as a free boundary problem established a novel geometrical growth law based on the coupling of thin-film fluid dynamics and calcium carbonate chemistry.^{13–15} Numerical studies showed an attractor in the space of shapes whose analytical form was determined and found to compare very favorably with that of natural stalactites. Is there an analogous ideal shape for icicles? It is tempting to view icicle growth as a classic Stefan problem, as explored extensively for solidification from the melt.¹⁶ There, growth is controlled by a quasistatic diffusive field and the growth rate is determined by a gradient of that variable. However, such systems generally lack the previously mentioned thin layer of moving fluid (water or air) that separates the developing solid from its surroundings, and thus they do not conceptually match the conditions of growth. Exceptions occur, for instance, in the presence of surface premelting.¹⁷ One context in which progress has been made is the formation of “ice stalactites,” hollow tubular structures formed below sea ice as salt is rejected during solidification,^{18,19} but these formations are quite distinct from typical icicles. Here, we suggest an approach to the problem of icicle growth which synthesizes geometrical principles, heat flow in the water and atmosphere, and thin-film fluid dynamics, to arrive at the existence of an ideal growing shape for icicles. This approach can be viewed as a generalization of the important works mentioned above^{3,4} to a true free-boundary formulation. The ideal growing shape found here compares well with observations. Interestingly, the shape far from the tip has the same mathematical form as that recently derived^{11,12} for the growth of stalactites.

We first consider the water layer flowing down the surface of a growing icicle to set some initial scales. The volu-

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FIG. 1. (Color online) Icicles and stalactites. (a) A collection of icicles (Ref. 23). (b) Stalactites in Kartchner Caverns, Benson, AZ.

metric flow rate Q over icicles is typically^{3,20} on the order of tens of milliliters per hour ($\sim 0.01 \text{ cm}^3/\text{s}$), and icicle radii are usually in the range of 1–10 cm. To understand the essential features of the flow, consider a cylindrical icicle of radius r , over the surface of which flows an aqueous film of thickness h (Fig. 2). Since $h \ll r$ over nearly the entire icicle surface, the velocity profile in the layer may be determined as that flowing on a flat surface. Furthermore, we expect the Reynolds number to be low enough that the Stokes approximation is valid. If y is a coordinate normal to the surface and θ is the angle that the tangent vector \hat{t} makes with respect to the horizontal, then the Stokes equation for gravity-driven flow is $\nu_w d^2 u / dy^2 = g \sin \theta$, where g is the gravitational acceleration and $\nu_w = 0.01 \text{ cm}^2/\text{s}$ is the kinematic viscosity of water. Enforcing no-slip and stress-free boundary conditions at the solid-liquid and liquid-air interfaces, the thickness is

$$h = \left(\frac{3Q\nu_w}{2\pi g r \sin \theta} \right)^{1/3}. \quad (1)$$

Using typical flow rates and radii, we deduce a layer thickness that is tens of microns and surface velocities

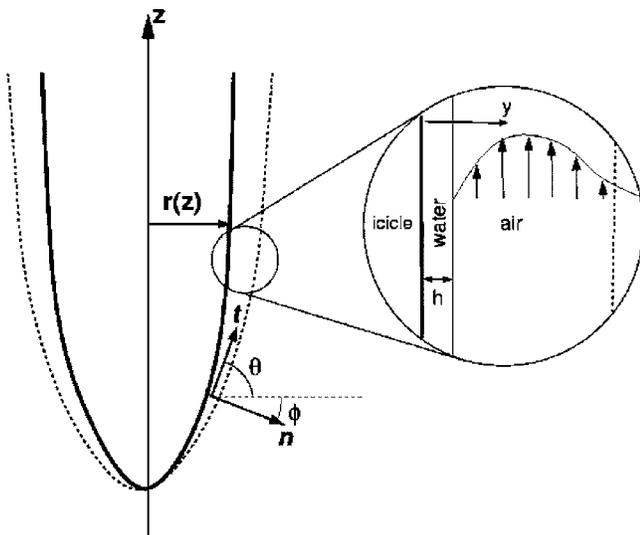


FIG. 2. Features of a hanging axisymmetric shape used in development of the theory. The flowing water layer, not to scale, is much thinner than the rising thermal boundary layer.

$u_s \approx (gh^2/2\nu_w)\sin \theta$ below several mm/s, consistent with known values,^{3,20} yielding $\text{Re} = 0.01\text{--}0.1$, well in the laminar regime as anticipated. At distances from the icicle tip comparable to the capillary length (several millimeters), the complex physics of pendant drop detachment takes over and the thickness law (1) ceases to hold.

Of course, if the icicle is growing, the volumetric flux Q must vary along the arc length s of the icicle as water is converted to ice. With the icicle profile described by $r(z)$ (Fig. 2) and the growth velocity normal to the ice at any point being v_g , Q varies along the surface as

$$\frac{dQ}{ds} = 2\pi r v_g, \quad (2)$$

the positive sign on the right-hand side reflecting the choice of origin at the tip, with s increasing upward. We seek to find a final answer in the form of a uniformly translating shape,^{11,12} for which every point on the icicle must grow at a rate such that $v_g = v_t \cos \theta$, where v_t is the growth velocity of the tip, usually millimeters per hour ($\sim 10^{-4} \text{ cm/s}$)^{3,20} (given the complexities of droplet detachment,³ the tip velocity here will be considered a parameter of the theory). Therefore, we substitute this rule into (2), using $dr = ds \cos \theta$, and find that an exact integration may be performed, yielding

$$Q = Q_t + \pi r^2 v_t, \quad (3)$$

where Q_t is the flow rate at the icicle's tip. This result, which neglects evaporation, conforms to the obvious fact that, for a given Q , Q_t will eventually approach zero as the icicle becomes so long as to allow all of the feeding water to freeze before it reaches the tip. For further analysis, we will only consider the growth of icicles up to this point, and not beyond, and only consider growth into a calm environment.

Turning now to heat transport, note that the curvature of the icicle surface is sufficiently small everywhere that the Gibbs-Thompson correction²¹ to the melting temperature T_m is negligible. Thus, the temperature of the water at the ice-water interface is well-approximated as T_m along the entire icicle, neglecting the tip. Furthermore, since most icicles possess an unfrozen liquid core,^{3–5} heat does not travel radially outward from the center of the icicle, as it would if the core were solid and the temperature inside were decreasing over time. Hence, any flux of heat present at the ice-water interface consists solely of latent heat being removed as the water changes phase. The issue of advective heat transport by the flowing water is addressed by considering the Peclet number $\text{Pe} = u_s h / \alpha_w$, where $\alpha_w \approx 10^{-3} \text{ cm}^2/\text{s}$ is the thermal diffusivity of water. Using our previous estimates for the flow velocity u_s and thickness, we find $\text{Pe} \approx 0.1\text{--}1$, indicating that energy transport down the icicle is generally subordinate to conduction of heat across the water layer. The heat flux across the water, then, is $F_w = \kappa_w (T_m - T_i) / h$, where κ_w is the thermal conductivity of water and T_i , the temperature at the air-water interface, is found below.

The rate-limiting, and hence, controlling, step in growth occurs once the heat has traversed the water layer and must then be transported through the air surrounding the icicle. This transport can be greatly influenced by the presence of

forced convection, as considered in previous works,^{3,4} but we shall ignore this in the present study, assuming a calm environment for growth. Instead, we will consider natural convection, such as found in the study by Makkonen.³ As is well known, objects warmer than their surroundings create rising thermal boundary layers in the adjacent atmosphere due to the buoyancy of the heated surrounding air. Similarity solutions for the coupled Navier-Stokes and heat transport equations in the Boussinesq approximation can provide the basis for understanding this boundary layer. For instance, for a flat, vertical, isothermal plate, solutions show that the rising warm air is confined to a boundary layer whose thickness δ as a function of the vertical coordinate z is²²

$$\delta = C\ell \left(\frac{z}{\ell}\right)^{1/4}, \quad \text{with } \ell \equiv \left(\frac{\nu_a^2}{g\beta\Delta T}\right)^{1/3}, \quad (4)$$

where C is a dimensionless constant that depends on the Prandtl number of air (0.68) and is of order unity, $\nu_a \approx 0.13 \text{ cm}^2/\text{s}$ is the kinematic viscosity of air, $\beta \approx 3.7 \times 10^{-3} \text{ K}^{-1}$ is the volumetric coefficient of expansion for air, and ΔT is the temperature difference between the plate and the ambient temperature T_a far away. For a temperature difference of 10 K the characteristic length scale $\ell \approx 0.01\text{--}0.1 \text{ cm}$.

To justify our future use of (4) to approximate the boundary layer thickness for our icicle, we submit the following. First, using a temperature difference of 10 K, one finds a boundary layer thickness on the order of a few millimeters to a centimeter, much greater than the thickness of the water layer on a typical icicle, but less than a typical icicle radius, so that flatness is approximated. Second, the peak velocity of the warm air in the layer is

$$u_p \approx \frac{2}{3} \sqrt{g\Delta T\beta z}, \quad (5)$$

around 5–10 cm/s, much greater than the downward water velocity, so the no slip condition used in the flat plate analysis is nearly attained. Third, the atmospheric heat flux can be written as $F_a = \kappa_a(T_i - T_a)/\delta$, where κ_a is the thermal conductivity of air, differing from the exact form only by the multiplication of an order one constant. If we equate this heat flux with that through the water layer, which we previously described, one finds that T_i is given by

$$T_i = T_m - (T_m - T_a) \frac{h\kappa_a/\delta\kappa_w}{1 + h\kappa_a/\delta\kappa_w}. \quad (6)$$

On account of the vast difference in scale between h and δ mentioned above, the ratio $h\kappa_a/\delta\kappa_w \approx 0.01$, so T_i is lower than T_m by only $10^{-3}\text{--}10^{-2} \text{ K}$. Hence, from the view of atmospheric heat transport, the icicle walls are essentially isothermal at T_m (although the tip is cooler³). Finally, we note that (4) can be used to describe a slightly nonvertical plate by simply replacing g with $g \cos \phi$ and z with $z/\cos \phi$ (the arc length along the plate). Since, barring the tip region, an icicle's surface is nearly vertical and has a very slowly varying slope, this is a valid approximation in our case; we treat the order one factor of $\cos \phi$ as a constant and fold it into the

parameter C . A more sophisticated treatment would account for the effects of wall curvature.³

At this point, we are in a position to derive a formula for the growth velocity v_g of the icicle's surface. We divide the heat flux as calculated through the atmospheric boundary layer by the latent heat of fusion per volume L of water (334 J/cm^3) to obtain the velocity

$$v_g = v_c \left(\frac{\ell}{z}\right)^{1/4}, \quad (7)$$

where v_c is a characteristic velocity given by

$$v_c \equiv \frac{\kappa_a \Delta T}{LC\ell}, \quad (8)$$

and is, with $\Delta T = 10 \text{ K}$, around 10^{-4} cm/s , which is in good agreement with the known velocities cited earlier. To find the equation governing the icicle profile, we enforce the condition for uniformly translating shapes, $v_g = v_r \sin \phi$, upon (7) and scale the variables r and z both by the factor $a = \ell(v_c/v_r)^4$, thereby defining the new dimensionless variables ρ and ζ . After rewriting trigonometric functions in terms of the slope of the profile ρ' , one finds the equation

$$\rho' = \frac{1}{\sqrt{\zeta^{1/2} - 1}}, \quad (9)$$

which can be exactly integrated to yield the final expression for our ideal icicle shape,

$$\rho = \frac{4}{3} (\zeta^{1/2} + 2) \sqrt{\zeta^{1/2} - 1}. \quad (10)$$

This shape is shown in Fig. 3(a). Note that this shape at large ζ goes as $\rho \sim \zeta^{3/4}$ and therefore the thickness of the thermal boundary layer relative to the icicle radius scales as $\delta/r \sim \zeta^{-1/2}$ and the two-dimensional boundary layer calculation becomes ever more satisfactory further up the icicle, albeit slowly.

As promised, this asymptotic power law is identical to that found in the case of stalactites,^{11,12} finally explaining their strikingly similar appearances. Furthermore, if we evaluate this asymptotic form at some point on the surface (ρ^*, ζ^*) where the aspect ratio (length/width) is $A = \zeta^*/\rho^*$, then the shape can be rewritten as $\zeta/\zeta^* \approx (\rho/\rho^*)^{4/3}$, a universal, self-similar form. Hence, we can compare our ideal shape to natural icicles by simply finding the correct aspect ratio, or, equivalently, the correct scaling factor a , that best equates the two forms. The results of such comparisons are seen in Figs. 3 and 4. Figures 3(b)–3(d) show overlays of the appropriately scaled theoretical shape with three images of natural icicles. To find the appropriate scaling, each image was passed through an edge-detection algorithm to extract the profile $r(z)$. Each profile was then compared to the ideal form through a least-squares analysis to determine the best fit a . For the more quantitative analysis of Fig. 4, we first found a for each of the eight icicle images, then scaled the image profiles by this best-fit factor. All eight of the now-dimensionless profiles were then averaged together, forming the data points and associated error bars seen in the graph in Fig. 4, which are to be compared with the theoretical shape

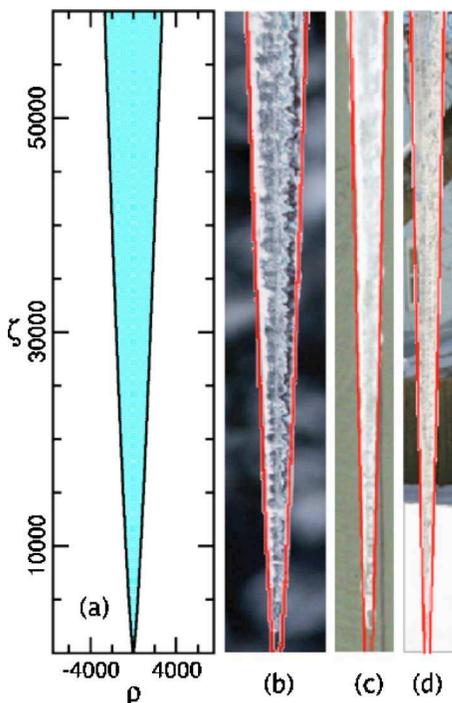


FIG. 3. (Color online) Ideal shape of an icicle and comparison with natural icicles. (a) Ideal shape in dimensionless units of radius and height. (b)–(d) A selection of natural icicles (Ref. 23) each with the appropriately scaled ideal form overlaid.

shown. Clearly, there is good agreement between the two, with no obvious systematic deviations present. On the far right, possible ripples can be seen as the data oscillates around the theoretical curve. Moreover, the shape is quite distinct from a conical geometry; indeed, an analogous least-squares fit of the data to a conical shape displays quite significant systematic deviations. Of course, controlled experiments on the growth of icicles are needed to check in detail various aspects of the theory, such as the assumption that a traveling shape is indeed an attractor of the dynamics.

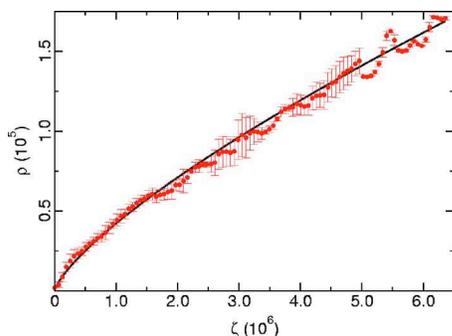


FIG. 4. (Color online) A comparison between the theoretical shape and natural icicles. The solid line represents the shape as calculated in the text, while the data points and error bars represent the averaging of the scaled profiles of eight icicles. In order to perform this averaging, each icicle's profile was scaled by the appropriate factor a through a least squares analysis. Error bars generally increase with ζ due to the fact that there are fewer icicles with appropriate aspect ratios at those points. After $\zeta = 5 \times 10^6$ only one icicle is represented, and the displayed error bars represent discretization uncertainties.

As a final interesting side note, we now calculate the ideal shape by analyzing the growth velocity using the heat flux through the thin water layer rather than the air. First, let us look at the thickness law (1) in conjunction with the depletion predicted in (3). Clearly, at large ρ , the fluid layer thickness will *grow* as

$$h \approx \left(\frac{3v_t a v_w \rho}{2g \sin \theta} \right)^{1/3}. \quad (11)$$

Using this and the ideal shape we have calculated, the ratio of h to δ in this regime must then look like

$$\frac{h}{\delta} \approx \left(\frac{2v_w a v_t^4}{g} \right)^{1/3} \frac{L}{\kappa_a \Delta T}. \quad (12)$$

So, if we substitute this ratio into (6), we see that, asymptotically, the temperature drop across the water layer goes to a fixed value of

$$\Delta T_w \rightarrow \left(\frac{2v_w a v_t^4}{g} \right)^{1/3} \frac{L}{\kappa_w}, \quad (13)$$

which is on the order of 10^{-3} K, as previously indicated. It is curious that this factor turns out as it does; for a different thickness law (1), as could be the case for a non-Newtonian fluid, the temperature drop could either approach zero or even increase at large ζ . In any case, we can now use this ΔT_w , along with the heat flux through the water layer, to find that, asymptotically, the profile ζ' should follow the scaling law

$$\zeta' \approx \left(\frac{3}{4} \rho \right)^{1/3}. \quad (14)$$

Equation (14) is another interesting result, as it shows that the shape obtained by a method that focuses on the liquid film yields the same shape as that found from the foregoing analysis of the natural convection boundary layer. We are unsure at this point whether it is mere happenstance that these two methods agree as they do, or perhaps this four-thirds scaling law has a deeper underlying significance in this class of problems.

Clearly, the scenario presented here, by which a free boundary dynamics for icicle growth is derived, contains a number of simplifications and approximations whose quantitative accuracy merits further study. Chief among these is the use of a boundary layer theory which assumes a flat and vertical surface. Both of these assumptions are justifiable only far away from the icicle's tip. A full numerical study would likely prove most illuminating. We expect the analysis presented here to serve as a basis for further understanding of ice structures, including axisymmetric perturbations such as the ripples so commonly found on icicles, as well as strongly nonaxisymmetric forms such as the sheets which are analogous to “draperies” in limestone caves. In this regard, recent work on solidification on surfaces of arbitrary curvature²⁴ may prove quite relevant.

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