1. Consider a paging algorithm that, on a page fault, just throws out a random page from the cache. Prove that the competitive ratio of this algorithm is $\Omega(k)$.

2. Suppose you are standing next to a long fence that extends as far as you can see in either direction. You want to cross the fence and you know that somewhere it has a hole in it, but you don’t know whether the hole is to your left or to your right, or how far away the hole is.

We can model the problem as follows: you are initially located at the origin on the real line. The hole is at some unknown positive or negative integer coordinate $h$ (why can we assume $h$ is an integer?). You can move left or right at cost equal to the distance moved, and the game continues until you reach $h$. We will consider the goal of minimizing the competitive ratio: i.e., the ratio of the distance traveled and $|h|$.

a) Give a deterministic strategy with competitive ratio 9. (Hint: Consider walking 1 unit to the right, 2 units the left, 4 to the right, etc.)

b) Describe an randomized algorithm whose competitive ratio is at most 7. (I.e., Show that for any $h$, the expected cost of the algorithm is at most $7|h|$.)

c) Give a randomized algorithm that is better than 7 competitive. It does not have to be optimal – even $(7 - \epsilon)$-competitive will suffice.

3. In the ski-rental problem, skis cost $r$ to rent and $b$ to buy, and our goal was to minimize the amount of money spent on skis for various ski trips. Unfortunately, we don’t know ahead of time how many trips we will take, so the best we could guarantee with a deterministic algorithm is 2-competitive (by buying just before trip number $b/r$).

a) Suppose a couple is going skiing together. At any time, they can purchase skis for $b$ each. If the couple does not own any skis, then they pay $2r$ to rent two pairs. If they own one, then they pay $r$ to rent the other pair. If they own two pairs, then they don’t have to rent at all. We will assume that this couple always goes skiing together. Show that the best offline strategy remains to buy two pairs of skis if and only if the number of trips is at least $b/r$, and otherwise to rent both pairs.

b) Design a deterministic online algorithm that is better than 2-competitive for the couple in question.

4. Given $n$ points in the plane, construct a simple (i.e., non-self-intersecting) polygon having these points as its vertices. Devise an $O(n \log n)$ algorithm and show that this is optimal.

5. Given two arbitrary convex polygons $P$ and $Q$ with $n$ vertices each, compute the convex hull of $P \cup Q$. Note that the two polygons can intersect (any number of times), they can be disjoint, or one can be contained within the other.
a) Specify as efficient an algorithm as you can to solve this problem, and analyze its complexity and prove its correctness.

b) Use this algorithm to generate a divide-and-conquer algorithm without presorting that finds the convex hull of an arbitrary set of \( n \) points.

6. Consider the two (seemingly unrelated) definitions. A triangulation is \textit{acute} if the angles of all its triangles are less than 90 degrees. A Voronoi diagram is said to be \textit{medial} if each edge of the diagram contains in its interior the midpoint of the two defining sites for the edge.

a) Prove that an acute triangulation of a set of points is always Delaunay. (Hint: recall the empty circle property.)

b) Prove that if a Voronoi diagram is medial, then the corresponding Delaunay triangulation is acute.